Probability and Statistics

Question 1

Consider two independent random variables $\underline{x}_1$ and $\underline{x}_2$, distributed exponentially with $\lambda = 1$. That is,

$$f_{\underline{x}} = \begin{cases} e^{-x} & \text{for } x \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Calculate the PDF of $\underline{x}_1 + \underline{x}_2$.

Solution

First let's find the CDF of $\underline{x}_1$ (and equivalently $\underline{x}_2$). We have

$$F_{\underline{x}_1}(x) = \int_{-\infty}^{x} f_{\underline{x}_1}(t)\,dt = \int_{0}^{x} e^{-t}\,dt = \left[-e^{-t}\right]_0^x = 1 - e^{-x}. \tag{2}$$

The CDF of $\underline{y} = \underline{x}_1 + \underline{x}_2$ is now given by

$$F_{\underline{y}}(x) = P(\underline{y} \leq x) = P(\underline{x}_1 + \underline{x}_2 \leq x). \tag{3}$$

Let's say that we know $\underline{x}_2 = z$. Then this is equal to

$$F_{\underline{y}}(x) = P(\underline{x}_1 \leq x - z) = F_{\underline{x}_1}(x - z). \tag{4}$$

But of course we don't know z. In fact, z can be anything between (in this case) 0 and x. (If z is bigger than x, we can stop the integration.) So we need to integrate over all possible values of z, taking into account the probability that z occurs. Hence,

$$F_{\underline{y}}(x) = \int_0^x F_{\underline{x}_1}(x - z) f_{\underline{x}_2}(z)\,dz = \int_0^x \left(1 - e^{-(x-z)}\right) e^{-z} dz = \int_0^x \left(e^{-z} - e^{-x}\right)\,dz. \tag{5}$$

We can simply work this out and get a decent result. But by applying differentiation under the integral sign, we can go for a shortcut. (I never thought reading Feynman books would give so many awesome insights. If you don't know differentiation under the integral sign, then just work out the math and you'll get the same results. Though you should simply read some books by Feynman. And surely I'm not joking.) This results in

$$f_{\underline{y}}(x) = \frac{\partial F_{\underline{y}}(x)}{\partial x} = \int_0^x \frac{\partial\left(e^{-z} - e^{-x}\right)}{\partial x} dz = \int_0^z e^{-x}\,dz = \left[z e^{-x}\right]_0^x = x e^{-x}. \tag{6}$$

(Yeah, I know I skipped a step in the 'integration by parts' drill, but that's allowed here.) And there you go: the PDF!

Question 2

Consider an extra independent random variable $\underline{x}_3$, also distributed exponentially with $\lambda = 1$. Calculate $P(\underline{x}_1 + \underline{x}_2 > \underline{x}_3)$.

Solution

We can apply the same trick here. Let's say that $\underline{y} = \underline{x}_1 + \underline{x}_2$ is some value z. Then

$$P(\underline{x}_1 + \underline{x}_2 > \underline{x}_3) = P(\underline{x}_3 < z) = F_{\underline{x}_3}(z). \tag{7}$$

But of course we don't know z. It can be anything from 0 to ∞, and it's distributed according to the PDF (6). So we again need to integrate, taking into account the chance that a certain value of z occurs. Hence,

$$P(\underline{x}_1 + \underline{x}_2 > \underline{x}_3) = \int_0^\infty F_{\underline{x}_3}(z) f_{\underline{y}}(z)\, dz = \int_0^\infty (1 - e^{-z})(ze^{-z})\, dz = \int_0^\infty ze^{-z} - ze^{-2z}\, dz. \tag{8}$$

We need to do integration by parts to solve this. This should get us

$$\begin{aligned} P(\underline{x}_1 + \underline{x}_2 > \underline{x}_3) &= \left[-ze^{-z}\right]_0^\infty + \int_0^\infty e^{-z}\, dz + \left[\frac{1}{2}ze^{-2z}\right]_0^\infty - \int_0^\infty \frac{1}{2}e^{-2z}\, dz & (9) \\ &= 0 + 1 + 0 - \left[-\frac{1}{4}e^{-2z}\right]_0^\infty = \frac{3}{4}. & (10) \end{aligned}$$

Well, that was fun.