Vibrating Strings

In this chapter we will examine a vibrating string. And, surprisingly, we can also model this with a partial differential equation! Let's find out how.

1 What is the wave equation?

1.1 Deriving the wave equation

Let's suppose we have a string of length L. Its deviation from a certain position is given by $u(x,t)$ $[m]$. Here x $[m]$ denotes the horizontal position on the string, and t $[s]$ denotes time. Also, we define $\rho(x)$ $[kg/m]$ to be the mass per unit length of the string.

To derive a PDE for u, we look at a very small piece of string. This piece has length Δx and thus weight $\rho(x)\Delta x$. Now we examine all the vertical forces acting on this piece of string. First of all, there is the **tension** $T(x,t)$ $[N]$ in the string. It can be shown that T causes a vertical force on our particle of magnitude

$$F_T = \frac{\partial}{\partial x}\left(T\frac{\partial u}{\partial x}\right). \tag{1.1}$$

By the way, the above equation is only valid for small deviations u. It lacks accuracy if the deviations/slopes of the string become very high.

Besides tension, there can also be external forces. Let's denote the **body force** $Q(x,t)$ $[m/s^2]$ as the vertical force per unit mass acting on the string. This body force then causes a force $Q(x,t)\rho(x)$ on our piece of string.

Now we have examined all the forces. The sum of the forces should of course equal to ma, or, equivalently, to $m\partial^2 u/\partial t^2$. This gives us the equation

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = T_0\frac{\partial^2 u}{\partial x^2} + Q(x,t)\rho(x). \tag{1.2}$$

This equation is still rather difficult to solve. To make things easier, we assume that the string is **perfectly elastic**. This implies that $T(x,t)$ is actually constant for the entire string. We therefore now denote it as T_0. We also assume that there are no body forces. ($Q(x,t) = 0$.) And if we then also define c $[m/s]$ such that $c^2 = T_0/\rho(x)$, then our PDE turns into

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}. \tag{1.3}$$

The above equation is called the **one-dimensional wave equation**.

1.2 Boundary conditions

What kind of boundary conditions can we apply to a string? Of course, we can give the edges of the string a certain deflection $u(0,t) = f(t)$. (Or, similarly, $u(L,t) = f(t)$.) We can also give the edges a fixed slope $\partial u/\partial x(0,t) = f(t)$. In fact, if we attach the edge of the string to a (frictionless) vertical slider, then we have $\partial u/\partial x(0,t) = 0$.

We could make the situation even more complicated. Let's suppose we attach the edge of the string to a mass (with weight m $[kg]$) attached to a vertical spring (having stiffness k $[N/m]$). Let's examine the forces acting on this mass. There is of course the force of the spring. There is also the tension T caused by the cable. Furthermore, there can be an external force $g(t)$. The governing equation of the mass then becomes

$$m\frac{d^2u}{dt^2}(0,t) = -k(u(0,t) - u_E(t)) + T_0\frac{\partial u}{\partial x}(0,t) + g(t). \tag{1.4}$$

By the way, $u_E(t)$ is the equilibrium position of the spring. The above equation may look very complicated. But usually $g(t) = 0$. If we also assume the situation to be steady (everything stands still), then also $d^2u/dt^2(0,t) = 0$. We remain with

$$k(u(0,t) - u_E(t)) = T_0\frac{\partial u}{\partial x}(0,t). \tag{1.5}$$

This is quite interesting. If k equals zero, then we again deal with a horizontal slider. If, however, $k \to \infty$, then we have simply given the edge of the string a fixed position $u(0,t) = u_E(t)$.

1.3 Initial conditions

Next to boundary conditions, we also need initial conditions. The wave equation contains a second derivative w.r.t. time. So we need to initial conditions. Usually both the initial position and velocity are prescribed. So,

$$u(x,0) = f(x) \qquad \text{and} \qquad \frac{\partial u}{\partial t}(x,0) = g(x). \tag{1.6}$$

2 Solving the wave equation using separation of variables

We will now solve the wave equation. We do this using the method of separation of variables, which we are familiar with.

2.1 The solving method

You might have noticed that the wave equation kind of looks like the Laplace equation we have seen earlier. Solving it also goes similar. We assume that we can write $u(x,t) = X(x)T(t)$. If we insert this into the heat equation, then we can derive that

$$\frac{d^2X}{dx^2} = -\lambda X \qquad \text{and} \qquad \frac{d^2T}{dt^2} = -\lambda c^2 T. \tag{2.1}$$

First we examine the left part. Together with the boundary conditions, we can find eigenfunctions $X_n(x)$. After that, we find a solution for the right part. We can combine it with the initial conditions to find our final solution for $u(x,t)$.

Well, that's easier said than done. So we demonstrate this solving method with an example.

2.2 Vibrating string example problem

Let's suppose we have a vibrating string with initial conditions given by equation (1.6). The boundary conditions are $u(0,t) = 0$ and $u(L,t) = 0$. We assume that $u(x,t) = X(x)T(t)$. It follows that $X(0) = 0$ and $X(L) = 0$. Using this, we can find that the eigenfunctions are

$$X_n(x) = \sin\sqrt{\lambda_n}x = \sin\frac{n\pi x}{L}, \qquad \text{with corresponding eigenvalues} \qquad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \tag{2.2}$$

with $n = 1, 2, 3, \ldots$. We can now use the values of λ_n to find the general solution for $T_n(t)$. This is

$$T_n(t) = A_n\cos\sqrt{\lambda_n}ct + B_n\sin\sqrt{\lambda_n}ct = A_n\cos\frac{n\pi ct}{L} + B_n\sin\frac{n\pi ct}{L}. \tag{2.3}$$

Our general solution for $u(x,t)$ thus becomes

$$u(x,t) = \sum_{n=1}^{\infty} X_n(x)T_n(t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L} + B_n \sin\frac{n\pi x}{L}\sin\frac{n\pi ct}{L}. \tag{2.4}$$

We only haven't applied the initial conditions yet. However, we can use the property of orthogonality for that. We can then find the coefficients A_n and B_n. They are

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx, \tag{2.5}$$

$$B_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\frac{n\pi x}{L}dx. \tag{2.6}$$

And this concludes our solution.

2.3 Interpreting the vibrating string solution

There are several things we can learn from the solution that we just found. So we take a closer look at it. Our solution consisted of a summation of terms

$$\sin\frac{n\pi x}{L}\left(A_n\cos\frac{n\pi ct}{L} + B_n\sin\frac{n\pi ct}{L}\right). \tag{2.7}$$

Every term represents a **normal mode of vibration**. It has its own **natural circular frequency**, given by

$$f_n = \frac{n\pi c}{L}. \tag{2.8}$$

This circular frequency is the amount of oscillations in 2π seconds. It also has its own **amplitude**, being

$$\text{Amplitude} = \sqrt{A_n^2 + B_n^2}. \tag{2.9}$$

Together, all these modes of vibration form the actual vibration of the string.

3 The method of characteristics

There is another way to solve the wave equation. It is called the method of characteristics. Let's examine this method.

3.1 Characteristics

Let's examine the wave equation. We can rewrite this equation in two ways, being

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x}\right) = 0 \qquad \text{and} \qquad \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}\right) = 0. \tag{3.1}$$

To write this in a more simple way, we define w and v as

$$w = \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} \qquad \text{and} \qquad v = \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}. \tag{3.2}$$

We can now rewrite the wave equation as two first-order partial differential equations. These equation are

$$\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = 0 \qquad \text{and} \qquad \frac{\partial v}{\partial t} - c\frac{\partial v}{\partial x} = 0. \tag{3.3}$$

That's great news, but we're not satisfied yet. We want to transform those equations into ordinary differential equations as well. To do that, we examine these equations along the lines $x(t) = x_0 + ct$ and $x(t) = x_0 - ct$, respectively. Now why would we do that? To find that out, we consider the derivative of $w(x, t)$ w.r.t. t along these lines. By using the chain rule, we can find that

$$\frac{d}{dt}w(x(t), t) = \frac{\partial w}{\partial t} + \frac{dx}{dt}\frac{\partial w}{\partial x} = \frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = 0. \tag{3.4}$$

What does this mean? It means that $dw/dt = 0$ along the lines $x = x_0 + ct$. Similarly, we can find that $dv/dt = 0$ along the lines $x = x_0 - ct$. So we have reduced the PDEs to ODEs. Any curve that reduces a PDE to an ODE is called a **characteristic**. Thus the lines we just examined are characteristics of the corresponding PDEs.

But, what's the use of all this? Well, now we know we can write $w(x, t)$ and $v(x, t)$ as

$$w(x, t) = P(x_0) = P(x - ct) \qquad \text{and} \qquad v(x, t) = Q(x_0) = Q(x + ct), \tag{3.5}$$

with $P(x)$ and $Q(x)$ any function. We have thus found the general solution to equation (3.3).

3.2 The solution of the wave equation

How can we use the things we just found, to solve the wave equation? That's an interesting question. To answer it, we define two new functions $F(x)$ and $G(x)$ as

$$F(x) = -\frac{1}{2c}\int P(x)dx \qquad \text{and} \qquad G(x) = \frac{1}{2c}\int Q(x)dx. \tag{3.6}$$

We can combine these definitions with equation (3.2). If we do this, we will find the general solution for $u(x, t)$. This solution is

$$u(x, t) = F(x - ct) + G(x + ct). \tag{3.7}$$

This holds for all functions $F(x)$ and $G(x)$.

So, what does this mean? It means that we can split the solution to $u(x, t)$ up in two parts, being $F(x-ct)$ and $G(x + ct)$. Let's examine the part $F(x - ct)$. This function is constant as $x - ct$ is constant. Now let's plot $F(x - ct)$ versus x for different times t. If both the time t and the position x increase, then the function $F(x - ct)$ remains constant. In other words, the graph simply slides to the right (the positive x-direction). And it does this with a velocity c. Similarly, we can find that the graph of $G(x + ct)$ moves to the left. It also does this with a velocity c.

So, what can we conclude from this? It means that $u(x, t)$ consists of two separate 'waves'. One wave moves to the left, while the other moves to the right with. Both do this with a velocity c.

3.3 Initial conditions

Now let's add initial conditions to our problem. Let's suppose that

$$u(x, 0) = f(x) \qquad \text{and} \qquad \frac{\partial u}{\partial t} = g(x). \tag{3.8}$$

We can insert this into equation (3.7). By working things out, we can then find that

$$F(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(\bar{x})d\bar{x} \qquad \text{and} \qquad G(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(\bar{x})d\bar{x}. \tag{3.9}$$

So, by using the initial conditions, we can find $F(x)$ and $G(x)$. How can we derive the solution $u(x, t)$ from this? Well, one way is to simply find $F(x-ct)$ and $G(x+ct)$, and then add them up. But (depending on the situation) there might be an easier way.

First you can simply plot $F(x-ct)$ and $G(x+ct)$ for $t=0$. (You thus actually plot $F(x)$ and $G(x)$.) Now what if you want to find the graphs of $F(x-ct)$ and $G(x+ct)$ at some time t? Well, in this case you simply shift the graph of $F(x)$ a distance ct to the right. Similarly, you shift the graph of $g(x)$ a distance ct to the left. Finally, you should add up both graphs to find the graph for $u(x,t)$.

3.4 Boundary conditions at $x = 0$

Previously we have not considered boundary conditions. In other words, we just assumed that our string was infintely long. This is, of course, not the case. Now let's add a boundary condition at $x=0$. We then only examine the string to the right of this boundary (with $x>0$).

We now have a slight problem. The string is only present at $x>0$. So, also the initial conditions $f(x)$ and $g(x)$ are defined only for $x>0$. This means that also $F(x)$ and $G(x)$ are defined for $x>0$. In other words, we may not insert negative variables in the functions $F(x)$ and $G(x)$. For $G(x)$, this isn't a very big problem. (We only use $G(x+ct)$. And we have $x>0$, $c>0$ and $t>0$.) However, if $x<ct$, then $F(x-ct)$ is not defined. This means that we have a problem.

To solve it, we need to use the boundary condition at $x=0$. Let's suppose we give the string a fixed position. So, $u(0,t)=0$. We can insert this into our general solution. We then find that

$$u(0,t) = F(-ct) + G(ct) = 0 \qquad \text{which implies that} \qquad F(z) = -G(-z) \qquad \text{for } z<0. \tag{3.10}$$

So, we have now defined the right-moving wave $F(z)$ for $z<0$. This means that our problem is solved.

But what is the physical meaning of this? It means that, once the left-moving wave $G(x+ct)$ reaches the left end, it is reflected back. The new **reflected wave** takes the shape of $-G(x)$ and moves to the right. (Note the minus sign.)

You may wonder, what would happen if the boundary condition was different? For example, let's suppose that $\partial u(0,t)/\partial t = 0$. This time we can find that

$$\frac{\partial u(0,t)}{\partial t} = -c\frac{dF}{dx}(-ct) + c\frac{dG}{dx}(ct) = 0 \qquad \text{which implies that} \qquad \frac{dF}{dx}(z) = \frac{dG}{dx}(-z) \qquad \text{for } z<0. \tag{3.11}$$

If we integrate the result, we can find that $F(z) = G(-z) + k$ for $z<0$, with k a constant. It can be shown that this constant is zero, which implies that $F(z) = G(-z)$, for $z<0$.

Again, we examine the physical meaning of this. Once the left-moving wave $G(x+ct)$ reaches the left end, it is reflected back. This time, the reflected wave takes the shape of $G(x)$ and moves to the right. (Note that the minus sign is gone.)

3.5 Other boundary conditions

Of course we can also put boundary conditions at other positions. What happens if we put a boundary condition at a right edge? Physically, exactly the same happens as when the boundary condition was at the left edge.

Let's suppose the boundary condition is $u(L,t)=0$. Once a right-moving wave $F(x-ct)$ encounters this boundary, it is reflected back to the left. Its new shape is that of the function $-F(x)$. (Note the minus sign.) Things are similar if the boundary condition is $\partial u(L,t)/\partial t = 0$. But this time the reflected wave has the shape of $F(x)$. (The minus sign is gone.)

So, you only need to remember the following. A fixed position at the edge reverses the wave (with a minus sign) when bouncing it back. A fixed slope at the edge just bounces the wave back.