Fourier series

It is often convenient to express a function as its Fourier series. But can you do this for all functions? And can you differentiate/integrate Fourier series? That's what we will examine in this chapter.

1 Basic concepts

1.1 Definitions

Before we examine Fourier series, we must examine some definitions.

- Simply said, a function $f(x)$ is **continuous** if it has no jumps, nor any places where $f(x) \to \pm\infty$ or $df/dx \to \pm\infty$.
- A function $f(x)$ is **piecewise continuous** if it can be split up into pieces, which are all continuous. This means that so-called **jump dicontinuities** are allowed for piecewise continuous functions.
- A function $f(x)$ is **smooth** if it is continuous, and its derivative df/dx is also continuous.
- A function $f(x)$ is **piecewise smooth** if it can be split up into pieces, which are all smooth.

1.2 Odd and even functions

A function $g(x)$ is **odd** if it satisfies $g(-x) = -g(x)$. In other words, if you rotate the graph of $g(x)$ by $180°$ about the origin and wind up with the same graph, then $g(x)$ is odd. Similarly, a function $h(x)$ is **even** if it satisfies $h(x) = h(-x)$. In other words, if you mirror the graph of $h(x)$ about the y-axis and wind up with the same graph, then $h(x)$ is even.

1.3 Odd and even extensions and parts

Suppose we have a function $f(x)$. Let's examine the right side of its graph (for $x > 0$). We can extend this part to the left side, such that we wind up with an odd function. As discussed before, we need to rotate this part about the origin by $180°$. This new function is called the **odd extension** of $f(x)$. Its definition is

$$f_{odd,ext}(x) = \begin{cases} f(x) & \text{if } x > 0, \\ -f(-x) & \text{if } x < 0, \\ 0 & \text{if } x = 0. \end{cases} \tag{1.1}$$

Note that this function satisfies the definition of odd functions. Similarly, we can find the even extension of $f(x)$, being

$$f_{even,ext}(x) = \begin{cases} f(x) & \text{if } x \geq 0, \\ f(-x) & \text{if } x < 0. \end{cases} \tag{1.2}$$

But we can do more with a function $f(x)$. We can also split it up in parts. The odd and even parts of a function $f(x)$ are defined as

$$f_o(x) = \frac{f(x) - f(-x)}{2} \qquad \text{and} \qquad f_e(x) = \frac{f(x) + f(-x)}{2}. \tag{1.3}$$

Note that we have $f(x) = f_o(x) + f_e(x)$. Also, if $f(x)$ is already odd, then $f_o(x) = f(x)$ and $f_e(x) = 0$.

2 Fourier series and its convergence

Now it is time to examine Fourier series. What are they? And when do they actually converge?

2.1 Definition of the Fourier series

The **Fourier series** of a function $f(x)$ is the series satisfying

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}. \tag{2.1}$$

Here, a_0, a_n and b_n are the so-called **Fourier coefficients**. We can find them using the property of orthogonality. In fact, we will find that

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx, \tag{2.2}$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx, \tag{2.3}$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx. \tag{2.4}$$

It is important to note that this series is periodic with period $2L$. So in fact, the Fourier series is only valid for the interval $[-L, L]$.

2.2 Fourier series and odd and even functions

The Fourier series of odd and even functions are quite interesting. It can be shown that, for odd functions $g(x)$, we always have $a_n = 0$. On the other hand, for even functions $h(x)$, we always have $b_n = 0$. We thus find that

$$g(x) \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \qquad \text{and} \qquad h(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}. \tag{2.5}$$

The Fourier series of $g(x)$ is now called a **Fourier cosine series** (since it only consists of cosines). Similarly, the Fourier series of $h(x)$ is called a **Fourier sine series**.

Sometimes we only want the Fourier series of a function $f(x)$ on the interval $[0, L]$. In this case we have a certain advantage — we can choose whether we use a cosine series or a sine series. If we use a cosine series, then we actually find the Fourier series of $f_{even,ext}(x)$. Similarly, if we use a sine series, then we find the Fourier series of $f_{odd,ext}(x)$.

2.3 Notation for convergence of Fourier series

There is an important question mathematicians like to ask. Will the Fourier series of $f(x)$ actually converge to $f(x)$? It turns out that this is not always the case. If this is not the case, then we may not write an equality sign =. Instead, we usually write

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}. \tag{2.6}$$

The $\sim$ sign means 'has the Fourier series'. But it doesn't imply convergence. If the series does converge, we of course can write an = sign.

2.4 Rules for convergence of Fourier series

Of course it would be nice to know when the Fourier series of $f(x)$ actually converges to $f(x)$. There are rules for that. First we examine the rules of normal Fourier series on the interval $[-L, L]$.

- Let's suppose $f(x)$ is piecewise smooth on the interval $[-L, L]$. Now the Fourier series of $f(x)$ converges everywhere on the interval $[-L, L]$, except at jump discontinuities. At these points the series converges to the average of the jump, being

$$\frac{f(x-) + f(x+)}{2}. \tag{2.7}$$

- Let's suppose $f(x)$ is both piecewise smooth and continuous on the interval $[-L, L]$. Also suppose we have $f(-L) = f(L)$. Now the Fourier series of $f(x)$ converges everywhere on the interval $[-L, L]$. (Note that the conditions simply demand that there are no jump discontinuities.)

We can state similar rules for the cosine/sine series. As you know, these series are only valid on the interval $[0, L]$.

- Let's suppose $f(x)$ is both piecewise smooth and continuous on the interval $[0, L]$. In this case, the Fourier cosine series converges everywhere on the interval $[0, L]$.
- Let's suppose $f(x)$ is both piecewise smooth and continuous on the interval $[0, L]$. Also suppose that $f(0) = f(L) = 0$. Only in this case, the Fourier sine series converges everywhere on the interval $[0, L]$.

3 Differentiating and integrating Fourier series

3.1 Differentiating Fourier series term by term

Let's suppose we have a Fourier series of some function $f(x)$. We now want to find the Fourier series of the derivative df/dx. Can we then simply take the derivative of the Fourier series? Well, it turns out that we can only do that under certain conditions. We can only differentiate the Fourier series of $f(x)$ term by term if...

- $f(x)$ is piecewise smooth on the interval $[-L, L]$,
- $f(x)$ is continuous on the interval $[-L, L]$,
- we have $f(-L) = f(L)$.

All of the above conditions must hold. (It can be noted that the above conditions simply mean that there are no jump discontinuities in $f(x)$.)

Now let's ask ourselves, when we can we differentiate a Fourier cosine series term by term? We can simply modify the above rule for that. It can be noted that cosine series always automatically have $f(-L) = f(L)$. So, we may drop that condition. We thus find that we may differentiate cosine series if $f(x)$ is both piecewise smooth and continuous on the interval $[0, L]$.

Now let's ask ourselves, when can we differentiate a Fourier sine series term by term? Sadly, we can not ignore any conditions now. In fact, there is an extra condition. We can only differentiate a Fourier sine series if $f(x)$ is both piecewise smooth and continuous on $[0, L]$ and also $f(0) = f(L) = 0$.

You may wonder, what happens if we differentiate a sine series, but $f(0) \neq f(L) \neq 0$? We then have a special case. Let's suppose we differentiate the Fourier sine series

$$f(x) \sim \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}. \tag{3.1}$$

Our result will then be

$$\frac{df(x)}{dx} \sim \frac{f(L) - f(0)}{L} + \sum_{n=1}^{\infty} \left(\frac{n\pi}{L} B_n + \frac{2}{L} \left((-1)^n f(L) - f(0) \right) \right) \cos \frac{n\pi x}{L}. \tag{3.2}$$

3.2 Integrating Fourier series term by term

Let's examine the Fourier series of $f(x)$, being

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}. \tag{3.3}$$

Now we want to find the integral of $f(x)$, being

$$F(x) = \int_{-L}^{x} f(x) dx. \tag{3.4}$$

Are we allowed to integrate the Fourier series term by term? Well, luckily it turns out that we can. We are always allowed to integrate a Fourier series term by term. And the integral always converges. There are no special conditions attached. We can thus say that

$$F(x) = a_0(x + L) + \sum_{n=1}^{\infty} \frac{a_n}{n\pi/L} \sin \frac{n\pi x}{L} + \sum_{n=1}^{\infty} \frac{b_n}{n\pi/L} \left(\cos n\pi - \cos \frac{n\pi x}{L} \right). \tag{3.5}$$