Normal Forces and Bending Moments

1 Introduction to Bending

Let's consider a beam. We can bend this beam with only a bending moment M. This form of bending (bending without any normal forces) is called **pure bending**. But what will happen to the beam under pure bending? To find that out, we have to look at a small part dx of the beam, as is done in figure 1.

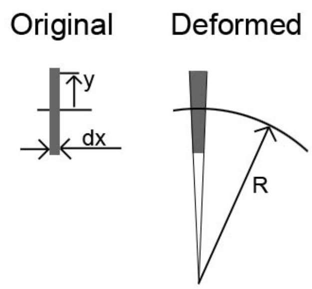

Figure 1: Deformation of a small part of beam under bending.

We see that one part of the beam (in this case the top part) elongates, while the other part is being contracted. So part of the beam has tensile stress, while the other has compressive stress. Also, there is some part in the beam without any stresses. This part is called the **neutral axis** (also called **neutral line**).

Let's see if we can find an expression for the stress. For that, we first ought to consider the strain. This strain depends on the **bend radius** R and the position in the beam y. Here y is the distance from the neutral axis. We will find that the strain due to a bending moment is

$$\varepsilon_M(y) = \frac{L_{new} - L_{old}}{L_{old}} = \frac{\frac{R+y}{R}dx - dx}{dx} = \frac{y}{R}. \tag{1.1}$$

For some reason engineers don't like to work with a bend radius. Instead, the **curvature** is defined as $\kappa = 1/R$. Let's now assume that no yielding (no permanent deformation) occurs. Then we have as stress

$$\sigma_M = \varepsilon_M E = \frac{Ey}{R} = Ey\kappa. \tag{1.2}$$

So the stress varies linearly with the distance y. That's nice to know! It's often relatively easy to work with linear relations. However, our job of analyzing bending is not finished yet.

2 Stress as a Function of the Bending Moment

One thing we would still like to know, is where the neutral axis will be. To find it, we have two pieces of data: The equations of the previous paragraph and the fact that there is no normal force (since we are dealing with pure bending). So let's see if we can find the normal force. This force is

$$0 = N = \int_A dF = \int_A \sigma_M dA = \int_A Ey\kappa\, dA = E\kappa \int_A y\, dA. \tag{2.1}$$

E and κ are constant for this cross-section. They are also nonzero, so the integral must be zero. It can be shown that this is only the case if y is the distance from the center of gravity of the cross-section. So the neutral line is the center of gravity of the cross-section!

That's very nice to know, but we still can't calculate much. Although we know the stress as a function of the curvature κ, this curvature is usually unknown. Can we express the stress as a function of the bending moment M? In fact, we can. This bending moment can be found using

$$M = -\int_A y\,dF = -E\kappa \int_A y^2\,dA = -E\kappa I, \tag{2.2}$$

where I is the moment of inertia about the horizontal axis of the cross-section. The minus sign in the equation is present due to the sign convention of the bending moment. A couple of engineers decided that to bend the beam as shown in figure 1, a negative bending moment was required.

So now we have expressed the curvature κ as a function of the bending moment. Combining this with the equations of the previous paragraph, we find the equation for the stress under pure bending (the so-called **flexure formula**). It is

$$\sigma_M = -\frac{My}{I}. \tag{2.3}$$

We can derive another important fact from this equation. We can find where maximum stress occurs. M and I are constant for a cross-section. So maximum stress occurs if y is maximal. This is thus at the top/bottom of the cross-section.

3 Adding a Normal Force

So now we know the normal stress in a beam when it is subject to only a bending moment. What happens when we subject a beam to a normal force N? Well, if there is only a normal force, then the normal stress is easy to find. It is

$$\sigma_N = \frac{N}{A}. \tag{3.1}$$

But what do we do if we have a normal force and a bending moment simultaneously? The answer to that question is quite simple. We add the stress up. So the total stress σ_T then becomes

$$\sigma_T = \sigma_N + \sigma_M = \frac{N}{A} - \frac{My}{I}. \tag{3.2}$$

The principle used in this relation is the principle of **superposition**.

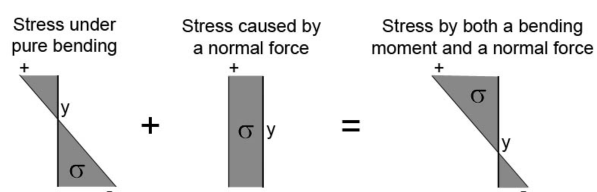

Figure 2: Stress diagram for different loading cases.

Now let's examine the stress in a beam. We do this using the stress diagrams of figure 2. When there was only a bending moment, the neutral axis was at the center of gravity of the cross-section. Applying a normal force shifts the neutral line by a distance d. This distance can be found using

$$\frac{N}{A} - \frac{Md}{I} = 0 \qquad \Rightarrow \qquad d = \frac{NI}{MA}. \tag{3.3}$$

When the normal force gets sufficiently big, the neutral line may move outside of the cross-section. Then there isn't any part in the beam anymore without stress.