Bending Deflection

1 The Differential Equations

In the last chapter we saw the effect of a torque: A twist angle. What is the effect of bending a beam? There are two things that are of interest: The beam **rotation** θ and the **deflection** $u = \delta$.

To be able to analyze this problem without having awfully difficult equations, we have to assume that the beam rotations and deflections are small. If we make that assumption, we can derive five equations. These are

$$q(x) = EIu'''' = EI\frac{d^4u}{dx^4} \quad (1.1)$$

$$V(x) = EIu''' = EI\frac{d^3u}{dx^3} \quad (1.2)$$

$$M(x) = EIu'' = EI\frac{d^2u}{dx^2} = EI\kappa \quad (1.3)$$

$$\theta(x) = u' = \frac{du}{dx} \quad (1.4)$$

$$\delta(x) = u \quad (1.5)$$

Let's discuss the sign convention. The distributed load q is assumed to act downward, and so is the vertical force V. The moment M is assumed to be clockwise. The rotation θ is assumed to be clockwise as well. Finally, the displacement is positive when the beam is deflected downward.

So, if we have a distributed load q acting on a beam, all we have to do is integrate four times and we have the deflection. It sounds easy, but actually, it isn't easy at all. Not only is it annoying to integrate four times. But during every integration, a new constant pops up. That'll be a lot of constants! You need to solve for these constants using boundary conditions.

So, it's a lot of work. And it's very easy to make an error. I can imagine you've got better things to do. So let's look for an easier method to find the rotation/deflection of a beam.

2 Vergeet-Me-Nietjes/Forget-Me-Nots

The second (and mostly used) method to determine deflections is by considering standard load types. Sounds vague so far? Well, the idea is as follows. We have a clamped beam. On this beam is a certain load type working. The load types we will be considering are shown in figure 1.

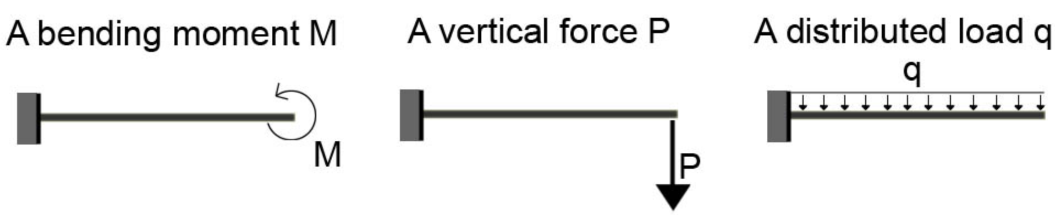

Figure 1: The three load types we will be considering.

These are the three standard load types. The rotation and displacement of the tip has been calculated for these examples. The results are shown in table 1. These equations are called the **vergeet-me-nietjes** (in Dutch) or the **forget-me-nots** (in English), named after the myosotis flower. They are therefore sometimes also called the **myosotis equations**.

Load type:	θ:	δ:
A distributed load q, pointing downward	$\frac{qL^3}{6EI}$	$\frac{qL^4}{8EI}$
A single force P, pointing downward	$\frac{PL^2}{2EI}$	$\frac{PL^3}{3EI}$
A single moment M, applied CCW	$\frac{ML}{EI}$	$\frac{ML^2}{2EI}$

Table 1: The vergeet-me-nietjes/forget-me-nots.

3 Applying the Forget-Me-Nots

So, how do we apply the forget-me-nots? Just follow the following steps:

- Split the beam up in segments. Every segment should be one of the standard load types.
- For every segment, do the following:
  - Let's call the left side of the segment point A and the right side point B. First find the forces and moments acting on B. (Don't forget the internal forces in the beam!)
  - Now express the rotation and displacement in B as a function of the rotation and displacement in A. For that, use the equations

$$\theta_B = \theta_A + \frac{q_{AB}L_{AB}^3}{6EI} + \frac{P_B L_{AB}^2}{2EI} + \frac{M_B L_{AB}}{EI}, \tag{3.1}$$

$$\delta_B = \delta_A + \frac{q_{AB}L_{AB}^4}{8EI} + \frac{P_B L_{AB}^3}{3EI} + \frac{M_B L_{AB}^2}{2EI} + L_{AB}\theta_A. \tag{3.2}$$

(3.3)

An important part is the term $L\theta_A$ at the end of the last equation. This term is present due to the so-called **wagging tail effect**. It is often forgotten, so pay special attention to it.

To demonstrate this method, we consider an example. Let's take a look at figure 2. Can we determine the displacement at point C?

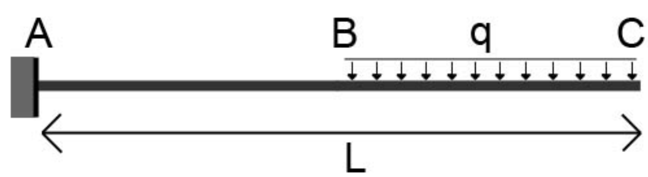

Figure 2: An example problem for the forget-me-nots.

It's quite clear that we need to split the beam up in two parts. First let's consider part BC. We have

$$\delta_C = \delta_B + \frac{q_{BC}L_{BC}^4}{8EI} + L_{BC}\theta_B = \delta_B + \frac{q\,(L/2)^4}{8EI} + (L/2)\theta_B = \delta_B + \frac{qL^4}{128EI} + (L/2)\theta_B. \tag{3.4}$$

Now let's look at part AB. Since the beam is clamped, we have $\theta_A = 0$ and $\delta_A = 0$, which is nice. We do need to take into account the internal forces in point B though. In B is a shear force $P_B = \frac{1}{2}qL$, pointing downward, and a bending moment $M_B = \frac{1}{8}qL^2$, directed clockwise. So we have

$$\theta_B = \frac{P_B L_{AB}^2}{2EI} + \frac{M_B L_{AB}}{EI} = \frac{qL^3}{16EI} + \frac{qL^3}{8EI}, \tag{3.5}$$

$$\delta_B = \frac{P_B L^3}{3EI} + \frac{M_B L^2}{2EI} = \frac{qL^4}{48EI} + \frac{qL^4}{64EI}. \tag{3.6}$$

Fill this in into equation (3.4) and you have your answer! That's a lot easier than difficult integrations.

4 Moment Area Method

Another way to find the rotation/deflection is by using the **moment area method**. For that, we first need to plot the moment diagram of the beam. Now let's consider a certain segment AB. Let A_M be the area under the bending moment diagram between points A and B. Now we have

$$\theta_B = \theta_A + \frac{A_M}{EI}. \tag{4.1}$$

Let's examine the moment diagram a bit closer. The diagram has a center of gravity. For example, if the diagram is a triangle (with $M = 0$ at A and $M =$ something at B), then the center of gravity of the moment diagram will be at $2/3^{rd}$ of length AB. Let's call x_{cog} the distance between the center of gravity of the moment diagram and point B. We then have

$$\delta_B = \delta_A + \frac{A_M}{EI} x_{cog} + L\theta_A. \tag{4.2}$$

Note that the wagging tail effect once more occurs in the moment area method. Although the moment area method is usually a bit more difficult to apply than the forget-me-nots, it can sometimes save time. You do ought to be familiar with the method though.

5 Statically Indeterminate Beams

Let's call R the amount of reaction forces acting on a beam. (A clamp has 3 reaction forces, a hinge has 2, etcetera.) Also, let's call J the amount of hinges in the system. If $R - 2J = 3$, the structure is statically determinate. You can use the above methods to solve reaction forces.

If, however, $R - 2J > 3$, the structure is statically indeterminate. We need compatibility equations to solve the problem. How do we approach such a problem? Simply follow the following steps.

- Remove constraints and replace them by reaction forces/moments, until the structure is statically determinant. Carefully choose the constraints to replace, since it might simplify your calculations.
- Express all the remaining reaction forces/moments in the reaction forces/moments you just added.
- Express the displacement of the removed constraints in the new reaction forces/moments.
- Assume that the displacement of the removed constraints are 0, and solve the equations.

6 Symmetry

Sometimes you are dealing with structures that are symmetric. You can make use of that fact. There are actually two kinds of symmetry. Let's consider both of them.

A structure is **symmetric**, if you can mirror it, and get back the original structure. If this is the case, then the shear force and the rotation at the center are zero. In an equation this becomes

$$V_{center} = 0 \qquad \text{and} \qquad \theta_{center} = 0. \tag{6.1}$$

A structure is **skew symmetric** if you can rotate it 180° about a certain point and get back the original structure. If this is the case, then

$$M_{center} = 0 \qquad \text{and} \qquad \delta_{center} = 0. \tag{6.2}$$