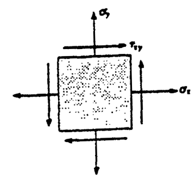
14–1. A material is subjected to a general state of plane stress. Express the strain energy density in terms of the elastic constants E, G, and ν and the stress components σ_x, σ_y, and τ_{xy}.



Strain Energy Due to Normal Stresses : We will consider the application of normal stresses on the element in two successive stages. For the first stage, we apply only σ_x on the element. Since σ_x is a constant, from Eq. 14 – 8, we have

$$(U_i)_1 = \int_V \frac{\sigma_x^2}{2E} dV = \frac{\sigma_x^2 V}{2E}$$

When σ_y is applied in the second stage, the normal strain ε_x will be strained by $\varepsilon_x' = -\nu\varepsilon_y = -\frac{\nu\sigma_y}{E}$. Therefore, the strain energy for the second stage is

$$(U_i)_2 = \int_V \left(\frac{\sigma_y^2}{2E} + \sigma_x \varepsilon_x'\right) dV$$
$$= \int_V \left[\frac{\sigma_y^2}{2E} + \sigma_x\left(-\frac{\nu\sigma_y}{E}\right)\right] dV$$

Since σ_x and σ_y are constants,

$$(U_i)_2 = \frac{V}{2E}\left(\sigma_y^2 - 2\nu\sigma_x\sigma_y\right)$$

Strain Energy Due to Shear Stress : The application of τ_{xy} does not strain the element in normal direction. Thus, from Eq. 14 – 11, we have

$$(U_i)_3 = \int_V \frac{\tau_{xy}^2}{2G} dV = \frac{\tau_{xy}^2 V}{2G}$$

The total strain energy is

$$U_i = (U_i)_1 + (U_i)_2 + (U_i)_3$$
$$= \frac{\sigma_x^2 V}{2E} + \frac{V}{2E}\left(\sigma_y^2 - 2\nu\sigma_x\sigma_y\right) + \frac{\tau_{xy}^2 V}{2G}$$
$$= \frac{V}{2E}\left(\sigma_x^2 + \sigma_y^2 - 2\nu\sigma_x\sigma_y\right) + \frac{\tau_{xy}^2 V}{2G}$$

and the strain energy density is

$$\frac{U_i}{V} = \frac{1}{2E}\left(\sigma_x^2 + \sigma_y^2 - 2\nu\sigma_x\sigma_y\right) + \frac{\tau_{xy}^2}{2G} \qquad \textbf{Ans}$$

14–2 The strain-energy density must be the same whether the state of stress is represented by σ_x, σ_y, and τ_{xy}, or by the principal stresses σ_1 and σ_2. This being the case, equate the strain-energy expressions for each of these two cases and show that $G = E/[2(1 + \nu)]$.

$$U = \int_V [\frac{1}{2E}(\sigma_x^2 + \sigma_y^2) - \frac{\nu}{E}\sigma_x\sigma_y + \frac{1}{2G}\tau_{xy}^2]\, dV$$

$$U = \int_V [\frac{1}{2E}(\sigma_1^2 + \sigma_2^2) - \frac{\nu}{E}\sigma_1\sigma_2]\, dV$$

Equating the above two equations yields.

$$\frac{1}{2E}(\sigma_x^2 + \sigma_y^2) - \frac{\nu}{E}\sigma_x\sigma_y + \frac{1}{2G}\tau_{xy}^2 = \frac{1}{2E}(\sigma_1^2 + \sigma_2^2) - \frac{\nu}{E}\sigma_1\sigma_2 \qquad (1)$$

However, $\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{(\frac{\sigma_x - \sigma_y}{2})^2 + \tau_{xy}^2}$

Thus, $(\sigma_1^2 + \sigma_2^2) = \sigma_x^2 + \sigma_y^2 + 2\tau_{xy}^2$

$$\sigma_1\sigma_2 = \sigma_x\sigma_y - \tau_{xy}^2$$

Substitute into Eq.(1)

$$\frac{1}{2E}(\sigma_x^2 + \sigma_y^2) - \frac{\nu}{E}\sigma_x\sigma_y + \frac{1}{2G}\tau_{xy}^2 = \frac{1}{2E}(\sigma_x^2 + \sigma_y^2 + 2\tau_{xy}^2) - \frac{\nu}{E}\sigma_x\sigma_y + \frac{\nu}{E}\tau_{xy}^2$$

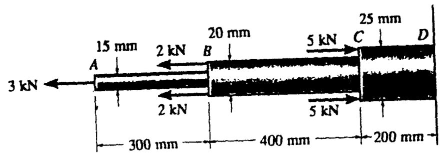
$$\frac{1}{2G}\tau_{xy}^2 = \frac{\tau_{xy}^2}{E} + \frac{\nu}{E}\tau_{xy}^2$$

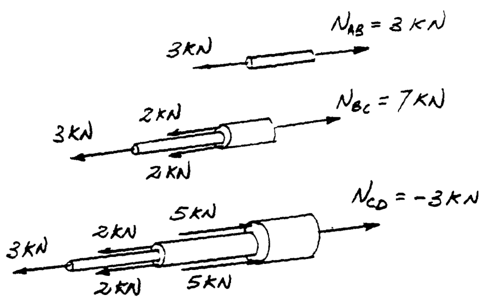
$$\frac{1}{2G} = \frac{1}{E} + \frac{\nu}{E}$$

$$\frac{1}{2G} = \frac{1}{E}(1 + \nu)$$

$$G = \frac{E}{2(1 + \nu)} \qquad \textbf{QED}$$

14–3 Determine the strain energy in the rod assembly. Portion AB is steel, BC is brass, and CD is aluminum. $E_{st} = 200$ GPa, $E_{br} = 101$ GPa, $E_{al} = 73.1$ GPa.



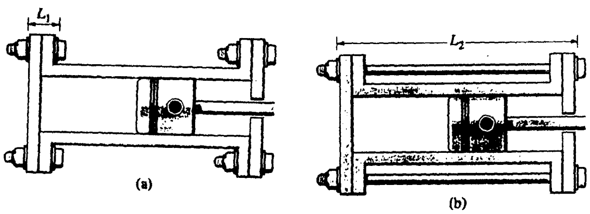


$$U_i = \Sigma \frac{N^2 L}{2AE}$$

$$= \frac{[3(10^3)]^2(0.3)}{2(\frac{\pi}{4})(0.015^2)(200)(10^9)} + \frac{[7(10^3)]^2(0.4)}{2(\frac{\pi}{4})(0.02^2)(101)(10^9)} + \frac{[-3(10^3)]^2(0.2)}{2(\frac{\pi}{4})(0.025^2)(73.1)(10^9)}$$

$= 0.372$ N.m $= 0.372$ J **Ans**

•14–4 Using bolts of the same material and cross-sectional area, two possible attachments for a cylinder head are shown. Compare the strain energy developed in each case, and then explain which design is better for resisting an axial shock or impact load.



Case (a)

$$U_A = \frac{N^2 L_1}{2AE}$$

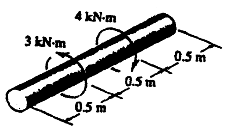
Case (b)

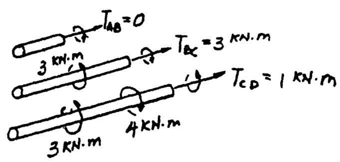
$$U_B = \frac{N^2 L_2}{2AE}$$

Since $U_B > U_A$,i.e., $L_2 > L_1$ the design for case (*b*) is better able to absorb energy.

Case (b) **Ans**

14-5. Determine the torsional strain energy in the A-36 steel shaft. The shaft has a radius of 30 mm.



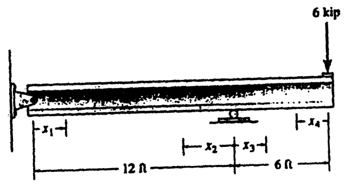


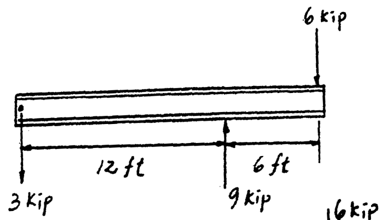
$$U_i = \Sigma \frac{T^2 L}{2JG} = \frac{1}{2JG}[0^2(0.5) + ((3)(10^3))^2(0.5) + ((1)(10^3))^2(0.5)]$$

$$= \frac{2.5(10^6)}{JG}$$

$$= \frac{2.5(10^6)}{75(10^9)(\frac{\pi}{2})(0.03)^4} = 26.2 \text{ N}\cdot\text{m} = 26.2 \text{ J} \qquad \textbf{Ans}$$

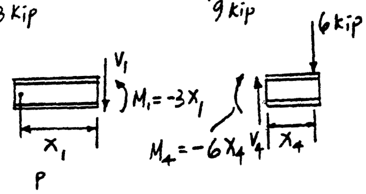
14-6. Determine the bending strain energy in the A-36 structural steel $W10 \times 12$ beam. Obtain the answer using the coordinates (a) x_1 and x_4, and (b) x_2 and x_3.





a) $$U_i = \int_0^L \frac{M^2\,dx}{2EI}$$

$$= \int_0^{144} \frac{(-3x_1)^2\,dx_1}{2EI} + \int_0^{72} \frac{(-6x_4)^2\,dx_4}{2EI}$$

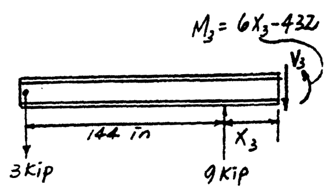


$$= \frac{9}{2EI}\frac{144^3}{3} + \frac{36}{2EI}\frac{72^3}{3}$$

$$= \frac{6718464}{29\ \ (53.8)} = 4306\ \text{lb}\cdot\text{in.}$$ **Ans**

b) $M_3 = 6x_3 - 432$

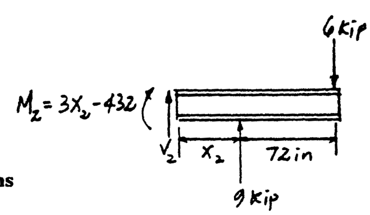
$M_2 = 9x_2 - 6(x_2 + 72) = 3x_2 - 432$



$$U_i = \int_0^{144} \frac{(3x_2 - 432)^2\,dx_2}{2EI} + \int_0^{72} \frac{(6x_3 - 432)^2\,dx_3}{2EI}$$

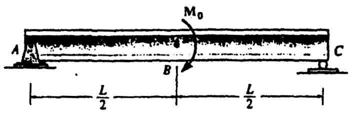
$$= \int_0^{144} \frac{(9x_2^2 - 2592x_2 + 186624)\,dx_2}{2EI} + \int_0^{72} \frac{(36x_3^2 - 5184x_3 + 186624)\,dx_3}{2EI}$$

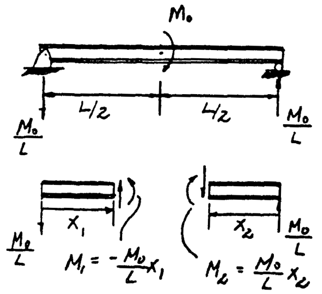
$$= \frac{1}{2EI}[3(144)^3 - \frac{2592}{2}(144)^2 + 186624(144)$$
$$+ 12(72)^3 - \frac{5184}{2}(72)^2 + 186624(72)]$$



$$= \frac{6718464}{29\ \ (53.8)} = 4306\ \text{lb}\cdot\text{in.}$$ **Ans**

14-7. Determine the bending strain energy in the beam due to the loading shown. EI is constant.





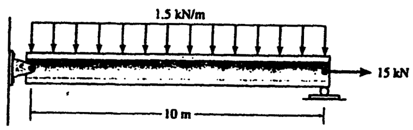
$$U_i = \int_0^L \frac{M^2 dx}{2EI}$$

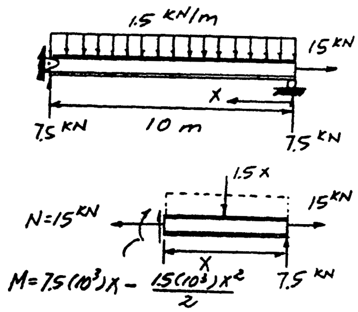
$$= \frac{1}{2EI}\left[\int_0^{L/2} \left(\frac{-M_0}{L}x_1\right)^2 dx_1 + \int_0^{L/2} \left(\frac{M_0}{L}x_2\right)^2 dx_2\right]$$

$$= \frac{M_0^2 L}{24EI} \qquad \textbf{Ans}$$

Note : Strain energy is always positive regardless of the sign of the moment function.

***14-8.** Determine the total axial and bending strain energy in the A-36 steel beam. $A = 2300\ \text{mm}^2$, $I = 9.5(10^6)\ \text{mm}^4$,





Axial load :

$$(U_a)_i = \int_0^L \frac{N^2 dx}{2EA} = \frac{N^2 L}{2EA}$$

$$(U_a)_i = \frac{((15)(10^3))^2(10)}{2(200)(10^9)(2.3)(10^{-3})} = 2.4456\ \text{J}$$

Bending :

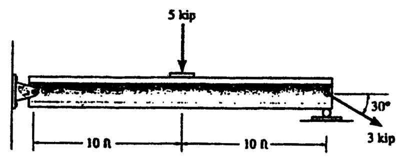
$$(U_b)_i = \int_0^L \frac{M^2 dx}{2EI} = \frac{1}{2EI}\int_0^{10} [(7.5)(10^3)x - 0.75(10^3)x^2]^2 dx$$

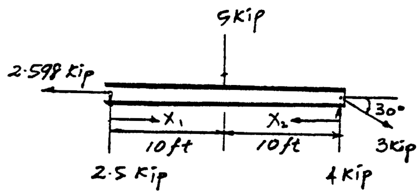
$$= \frac{1}{2EI}\int_0^{10} [56.25(10^6)x^2 + 562.5(10^3)x^4 - 11.25(10^6)x^3]dx$$

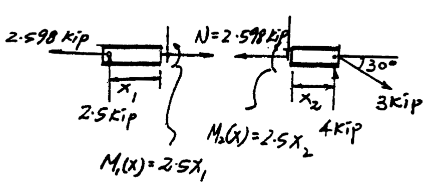
$$(U_b)_i = \frac{0.9375(10^9)}{200(10^9)(9.5)(10^{-6})} = 493.4210\ \text{J}$$

$$U_i = (U_a)_i + (U_b)_i = 2.4456 + 493.4210 = 496\ \text{J} \qquad \textbf{Ans}$$

14-9. Determine the total axial and bending strain energy in the A-36 structural steel $W8 \times 58$ beam.







Axial load :

$$(U_a)_i = \int_0^L \frac{N^2\,dx}{2AE} = \frac{N^2 L}{2AE}$$

$$= \frac{[2.598]^2(20)(12)}{2(17.1)(29)(10^3)} = 1.6334\,(10^{-3})\text{ in}\cdot\text{kip}$$

$$= 0.1361\,(10^{-3})\text{ ft}\cdot\text{kip}$$

Bending :

$$(U_b)_i = \int_0^L \frac{M^2\,dx}{2EI} = \frac{2}{2EI}\int_0^{120\text{ in.}}(2.5\,x)^2\,dx$$

$$= \frac{3.6\,(10^6)}{EI} = \frac{3.6\,(10^6)}{29\,(10^3)(228)}$$

$$= 0.5446\text{ in.}\cdot\text{kip} = 0.04537\text{ ft}\cdot\text{kip}$$

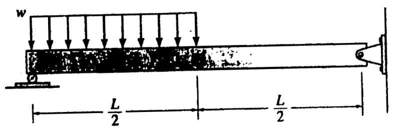
Total strain energy :

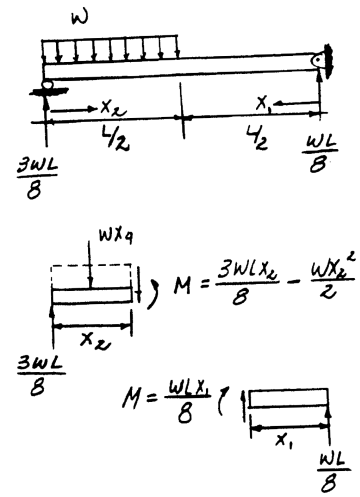
$$U_i = (U_a)_i + (U_b)_i$$

$$= 0.1361\,(10^{-3}) + 0.04537$$

$$= 0.0455\text{ ft}\cdot\text{kip} = 45.5\text{ ft}\cdot\text{lb} \qquad \textbf{Ans}$$

14–10 The simply supported beam is subjected to the loading shown. Determine the bending strain energy in the beam.



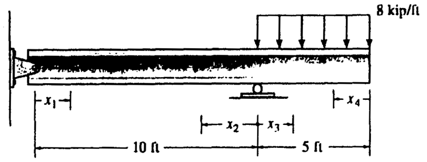


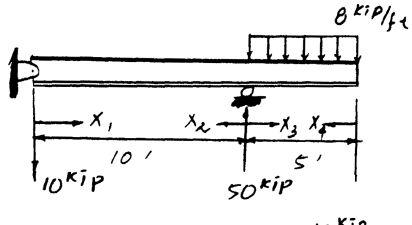
$$U_i = \int_0^L \frac{M^2 dx}{2EI} = \frac{1}{2EI}\left[\int_0^{L/2}\left(\frac{3wLx_2}{8} - \frac{wx_2^2}{2}\right)^2 dx_2 + \int_0^{L/2}\left(\frac{wLx_1}{8}\right)^2 dx_1\right.$$

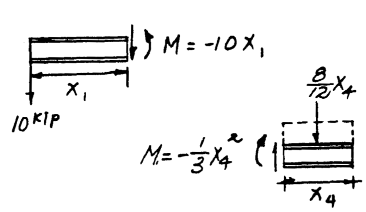
$$= \frac{1}{2EI}\left[\int_0^{L/2}\left(\frac{9w^2L^2x_2^2}{64} + \frac{w^2x_2^4}{4} - \frac{3w^2Lx_2^3}{8}\right)dx_2 + \int_0^{L/2}\frac{w^2L^2x_1^2}{64}dx_1\right]$$

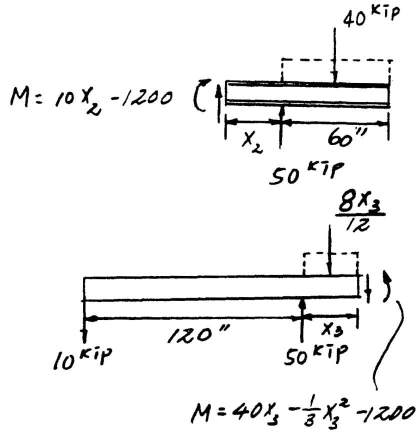
$$= \frac{0.00111w^2L^5}{EI}$$ **Ans**

14–11 Determine the bending strain energy in the A-36 steel beam due to the loading shown. Obtain the answer using the coordinates (*a*) x_1 and x_4, and (*b*) x_2 and x_3. $I = 53.8\ \text{in}^4$.









a)

$$U_i = \int_0^L \frac{M^2 dx}{2EI} = \frac{1}{2EI}\left[\int_0^{120\text{ in.}} (-10x_1)^2 dx_1 + \int_0^{60\text{ in.}} \left(-\frac{1}{3}x_4^2\right)^2 dx_4\right]$$

$$= \frac{37.44(10^6)}{EI} = \frac{37.44(10^6)}{29(10^3)(53.8)} = 24.00\ \text{in.}\cdot\text{kip} = 2.00\ \text{ft}\cdot\text{kip} \qquad \textbf{Ans}$$

b)

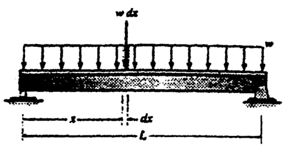
$$U_i = \int_0^L \frac{M^2 dx}{2EI} = \frac{1}{2EI}\left[\int_0^{60\text{ in.}} \left(40x_3 - \frac{1}{3}x_3^2 - 1200\right)^2 dx_3 + \int_0^{120\text{ in.}} (10x_2 - 1200)^2 dx_2\right]$$

$$= \frac{1}{2EI}\left[\int_0^{60\text{in.}} \left(\frac{1}{9}x_3^4 - \frac{80}{3}x_3^3 + 2400x_3^2 - 96000x_3 + 1440000\right)dx_3 + \right.$$

$$\left.\int_0^{120\text{ in.}} (100x_2^2 - 24000x_2 + 1440000)dx_2\right]$$

$$= \frac{37.44(10^6)}{EI} = \frac{37.44(10^6)}{29(10^3)(53.8)} = 24.00\ \text{in.}\cdot\text{kip} = 2.00\ \text{ft}\cdot\text{kip} \qquad \textbf{Ans}$$

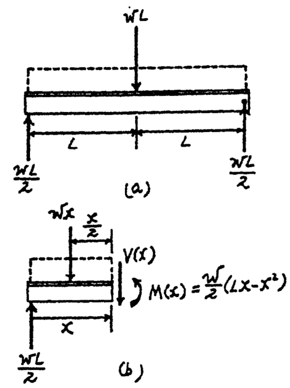
***14-12.** Determine the bending strain energy in the simply supported beam due to a uniform load w. Solve the problem two ways. (a) Apply Eq. 14–17. (b) The load $w\,dx$ acting on the segment dx of the beam is displaced a distance y, where $y = w(-x^4 + 2Lx^3 - L^3x)/(24EI)$, the equation of the elastic curve. Hence the internal strain energy in the differential segment dx of the beam is equal to the external work, i.e., $dU_i = \frac{1}{2}(w\,dx)(-y)$. Integrate this equation to obtain the total strain energy in the beam. EI is constant.



Support Reactions : As shown on FBD(a).

Internal Moment Function : As shown on FBD(b).

Bending Strain Energy : **a)** Applying Eq. 14 – 17 gives



$$U_i = \int_0^L \frac{M^2\,dx}{2EI}$$

$$= \frac{1}{2EI}\left[\int_0^L \left[\frac{w}{2}\left(Lx - x^2\right)\right]^2 dx\right]$$

$$= \frac{w^2}{8EI}\left[\int_0^L \left(L^2x^2 + x^4 - 2Lx^3\right) dx\right]$$

$$= \frac{w^2L^5}{240EI} \qquad \textbf{Ans}$$

b) Integrating $dU_i = \dfrac{1}{2}(w\,dx)(-y)$

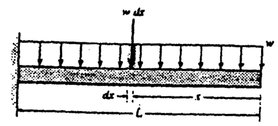
$$dU_i = \frac{1}{2}(w\,dx)\left[-\frac{w}{24EI}\left(-x^4 + 2Lx^3 - L^3x\right)\right]$$

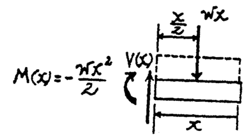
$$dU_i = \frac{w^2}{48EI}\left(x^4 - 2Lx^3 + L^3x\right) dx$$

$$U_i = \frac{w^2}{48EI}\int_0^L \left(x^4 - 2Lx^3 + L^3x\right) dx$$

$$= \frac{w^2L^5}{240EI} \qquad \textbf{Ans}$$

* **14-13.** Determine the bending strain energy in the cantilevered beam due to a uniform load w. Solve the problem two ways. (a) Apply Eq. 14–17. (b) The load $w\,dx$ acting on a segment dx of the beam is displaced a distance y, where $y = w(-x^4 + 4L^3x - 3L^4)/(24EI)$, the equation of the elastic curve. Hence the internal strain energy in the differential segment dx of the beam is equal to the external work, i.e., $dU_i = \frac{1}{2}(w\,dx)(-y)$. Integrate this equation to obtain the total strain energy in the beam. EI is constant.





Internal Moment Function : As shown on FBD.

Bending Strain Energy : **a)** Applying Eq. 14 – 17 gives

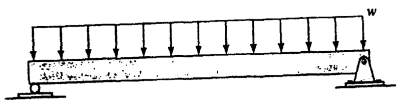
$$U_i = \int_0^L \frac{M^2 dx}{2EI}$$
$$= \frac{1}{2EI}\left[\int_0^L \left[-\frac{w}{2}x^2\right]^2 dx\right]$$
$$= \frac{w^2}{8EI}\left[\int_0^L x^4 dx\right]$$
$$= \frac{w^2L^5}{40EI} \qquad \textbf{Ans}$$

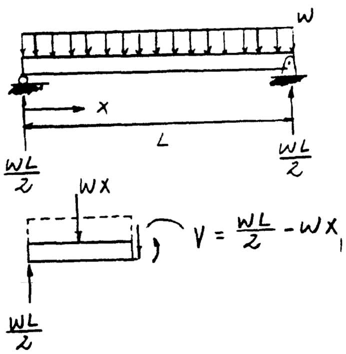
b) Integrating $dU_i = \frac{1}{2}(wdx)(-y)$

$$dU_i = \frac{1}{2}(wdx)\left[-\frac{w}{24EI}(-x^4 + 4L^3x - 3L^4)\right]$$
$$dU_i = \frac{w^2}{48EI}(x^4 - 4L^3x + 3L^4)\,dx$$

$$U_i = \frac{w^2}{48EI}\int_0^L (x^4 - 4L^3x + 3L^4)\,dx$$
$$= \frac{w^2L^5}{40EI} \qquad \textbf{Ans}$$

14–14 Determine the shear strain energy in the beam. The beam has a rectangular cross section of area A, and the shear modulus is G.





$$U_i = \int_0^L \frac{f_s V^2\, dx}{2GA} = \frac{f_s}{2GA}\int_0^L \left(\frac{wL}{2} - wx\right)^2 dx$$

$$= \frac{f_s}{2GA}\int_0^L \left(\frac{w^2L^2}{4} + w^2x^2 - w^2Lx\right)dx$$

$$= \frac{f_s\, w^2 L^3}{24\, GA}$$

For a rectangular section $f_s = \dfrac{6}{5}$

$$U_i = \frac{w^2L^3}{20\,GA} \qquad \textbf{Ans}$$