Vector Spaces and Subspaces

1 Definitions and Terms

1.1 Vector Spaces

A **vector space** is a nonempty set V of objects, called **vectors**, on which are defined two operations, called addition and multiplication by scalars, subject to the ten axioms listed in paragraph 3. As was already mentioned in the chapter Matrix Algebra, a **subspace** of a vector space V is a subset H of V that has three properties:

1. The zero vector of V is in H.
2. H is closed under vector addition. That is, for each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u} + \mathbf{v}$ is in H.
3. H is closed under multiplication by scalars. That is, for each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

If $\mathbf{v_1}, \dots, \mathbf{v_p}$ are in a vector space V, then $\text{Span}\{\mathbf{v_1}, \dots, \mathbf{v_p}\}$ is called **the subspace spanned** by $\mathbf{v_1}, \dots, \mathbf{v_p}$. Given any subspace H of V, a **spanning set** for H is a set $\mathbf{v_1}, \dots, \mathbf{v_p}$ in H such that $H = \text{Span}\{\mathbf{v_1}, \dots, \mathbf{v_p}\}$.

1.2 Bases

Let H be a subspace of a vector space V. An indexed set of vectors $\beta = \{\mathbf{b_1}, \dots, \mathbf{b_p}\}$ in V is a **basis** for H if β is a linearly independent set, and the subspace spanned by β coincides with H, that is, $H = \text{Span}\{\mathbf{b_1}, \dots, \mathbf{b_p}\}$. The set $\{\mathbf{e_1}, \dots, \mathbf{e_n}\}$ is a **standard basis** for $\mathbb{R}^n$. The set $\{1, t, \dots, t^n\}$ is a **standard basis** for $\mathbb{P}^n$.

1.3 Coordinate Systems

Suppose $\beta = \{\mathbf{b_1}, \dots, \mathbf{b_n}\}$ is a basis for V and $\mathbf{x}$ is in V. The **coordinates of x relative to the basis** β (or the **β-coordinates of x**) are the weights $c_1, \dots, c_n$ such that $\mathbf{x} = c_1\mathbf{b_1} + \dots + c_n\mathbf{b_n}$. If $c_1, \dots, c_n$ are the β-coordinates of $\mathbf{x}$, then the vector $[\mathbf{x}]_\beta$ in $\mathbb{R}^n$ (consisting of $c_1, \dots, c_n$) is the **coordinate vector of x (relative to β)**, or the **β-coordinate vector of x**. The mapping $\mathbf{x} \mapsto [\mathbf{x}]_\beta$ is the **coordinate mapping (determined by β)**.

If $P_\beta = [\,\mathbf{b_1}\ \dots\ \mathbf{b_n}\,]$, then the vector equation $\mathbf{x} = c_1\mathbf{b_1} + \dots + c_n\mathbf{b_n}$ is equivalent to $\mathbf{x} = P_\beta[\mathbf{x}]_\beta$. We call P_β the **change-of-coordinates matrix** from β to the standard basis $\mathbb{R}^n$. Since P_β is invertible (invertible matrix theorem), also $[\mathbf{x}]_\beta = P_\beta^{-1}\mathbf{x}$.

1.4 Vector Space Dimensions

If V is spanned by a finite set, then V is said to be **finite-dimensional**, and the **dimension** of V, written as dim V, is the number of vectors in a basis for V. The dimension of the zero vector space $\{\mathbf{0}\}$ is defined to be zero. If V is not spanned by a finite set, then V is said to be **infinite-dimensional**.

2 Theorems

1. If $\mathbf{v_1}, \ldots, \mathbf{v_p}$ are in a vector space V, then Span$\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ is a subspace of V.
2. Let $S = \{\mathbf{e_1}, \ldots, \mathbf{e_n}\}$ be a set in V, and let $H = \text{Span}\{\mathbf{e_1}, \ldots, \mathbf{e_n}\}$. If one of the vectors in S, say, $\mathbf{v_k}$, is a linear combination of the remaining vectors in S, then the set formed from S by removing $\mathbf{v_k}$ still spans H.
3. Let $S = \{\mathbf{e_1}, \ldots, \mathbf{e_n}\}$ be a set in V, and let $H = \text{Span}\{\mathbf{e_1}, \ldots, \mathbf{e_n}\}$. If $H \neq \{\mathbf{0}\}$, some subset of S is a basis for H.
4. Let $\beta = \{\mathbf{b_1}, \ldots, \mathbf{b_n}\}$ be a basis for a vector space V. Ten for each $\mathbf{x}$ in V, there exists a unique set of scalars $c_1, \ldots, c_n$ such that $\mathbf{x} = c_1\mathbf{b_1} + \ldots + c_n\mathbf{b_n}$.
5. Let $\beta = \{\mathbf{b_1}, \ldots, \mathbf{b_n}\}$ be a basis for a vector space V, and let $P_\beta = [\,\mathbf{b_1} \ \ldots \ \mathbf{b_n}\,]$. Then the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_\beta$ is a one-to-one linear transformation from V onto $\mathbb{R}^n$.
6. If a vector space V has a basis $\beta = \{\mathbf{b_1}, \ldots, \mathbf{b_n}\}$, then any set in V containing more than n vectors must be linearly dependent.
7. If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.
8. Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded, if necessary, to a basis for H. Also, H is finite-dimensional and dim $H \leq$ dim V.
9. **The Basis Theorem:** Let V be a p-dimensional vector space, $p \geq 1$. Any linearly independent set of exactly p elements in V is automatically a basis for V. Any set of exactly p elements that spans V is automatically a basis for V.

3 Vector Space Axioms

The following axioms must hold for all the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in the vector space V and all scalars c and d.

1. The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u} + \mathbf{v}$, is in V.
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
4. There is a **zero** vector $\mathbf{0}$ in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = 0$.
6. The scalar multiple of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$, is in V.
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$.
10. $1\mathbf{u} = \mathbf{u}$.