Symmetric Matrices and Quadratic Forms

1 Definitions and Terms

1.1 Diagonalization of Symmetric Matrices

A **symmetric matrix** is a square matrix such that $A^T = A$. A matrix A is said to be **orthogonally diagonalizable** if there are an orthogonal matrix P (so $P^{-1} = P^T$) and a diagonal matrix D such that $A = PDP^T = PDP^{-1}$. An orthogonally diagonalizable matrix A with orthonormal eigenvectors $\mathbf{u_1} \ldots \mathbf{u_n}$ can be written as $A = \lambda_1 \mathbf{u_1}\mathbf{u_1}^T + \ldots + \lambda_n \mathbf{u_n}\mathbf{u_n}^T$. This representation of A is called a **spectral decomposition** of A. Furthermore, each matrix $\mathbf{u_j}\mathbf{u_j}^T$ is a **projection matrix**.

1.2 Quadratic Forms

A **quadratic form** on $\mathbb{R}^n$ is a function Q defined on $\mathbb{R}^n$ whose value at a vector $\mathbf{x}$ in $\mathbb{R}^n$ can be computed by an expression of the form $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$, where A is an $n \times n$ symmetric matrix. The matrix A is called the **matrix of the quadratic form**.

If $\mathbf{x}$ represents a variable vector in $\mathbb{R}^n$, then a **change of variable** is an equation of the form $\mathbf{x} = P\mathbf{y}$, where P is an invertible matrix and $\mathbf{y}$ is a new variable vector in $\mathbb{R}^n$. Now $\mathbf{x}^T A\mathbf{x} = \mathbf{y}^T(P^T AP)\mathbf{y}$. If P diagonalizes A, then $P^T AP = P^{-1}AP = D$, in which D is a diagonal matrix. The columns op P are called the **principal axes** of the quadratic form $\mathbf{x}^T A\mathbf{x}$.

A quadratic form Q is per definition:

- **positive definite** if $Q(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$.
- **negative definite** if $Q(\mathbf{x}) < 0$ for all $\mathbf{x} \neq \mathbf{0}$.
- **positive semidefinite** if $Q(\mathbf{x}) \geq 0$ for all $\mathbf{x} \neq \mathbf{0}$.
- **negative semidefinite** if $Q(\mathbf{x}) \leq 0$ for all $\mathbf{x} \neq \mathbf{0}$.
- **indefinite** if $Q(\mathbf{x})$ assumes both positive and negative values.

The classification of a quadratic form is often carried over to the matrix of the form. Thus a **positive definite matrix** A is a symmetric matrix for which the quadratic form $\mathbf{x}^T A\mathbf{x}$ is positive definite.

1.3 Geometric View of Principal Axes

When $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$, where A is an invertible 2×2 symmetric matrix, and c is a constant, then the set of all $\mathbf{x}$ such that $\mathbf{x}^T A\mathbf{x} = c$ corresponds to an ellipse (or circle) or a hyperbola. An ellipse is described by the following equation in standard form: $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$ $(a > b > 0)$, where a is the semi-mayor axes and b is the semi-minor axes. A hyperbola is described by the following equation in standard form: $\frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} = 1$ $(a > b > 0)$, where the asymptotes are given by the equations $x_2 = \pm\frac{b}{a}x_1$.

2 Theorems

1. If A is symmetric, then any two eigenvectors from different eigenspaces are orthogonal.
2. An $n \times n$ matrix A is orthogonally diagonalizable if, and only if A is a symmetric matrix.

3. **The Spectral Theorem for Symmetric Matrices:** An $n \times n$ symmetric matrix A has the following properties:

   (a) A has n real eigenvalues, counting multiplicities.

   (b) The dimension of the eigenspace for each eigenvalue λ equals the multiplicity of λ as a root of the characteristic equation.

   (c) The eigenspaces are mutually orthogonal, in the sense that eigenvectors corresponding to different eigenvalues are orthogonal.

   (d) A is orthogonally diagonalizable.

4. **The Principal Axes Theorem:** Let A be an $n \times n$ symmetric matrix. Then there is an orthogonal change of variable $\mathbf{x} = P\mathbf{y}$, that transforms the quadratic form $\mathbf{x}^T A\mathbf{x}$ into a quadratic form $\mathbf{y}^T D\mathbf{y}$ with no cross-product term.

5. **Quadratic Forms and Eigenvalues:** Let A be an $n \times n$ symmetric matrix. Then a quadratic form $\mathbf{x}^T A\mathbf{x}$ is:

   (a) positive definite if, and only if the eigenvalues of A are all positive.

   (b) negative definite if, and only if the eigenvalues of A are all negative.

   (c) positive semidefinite if, and only if one eigenvalue of A is 0, and the others are positive.

   (d) negative semidefinite if, and only if one eigenvalue of A is 0, and the others are negative.

   (e) indefinite if, and only if A has both positive and negative eigenvalues.