Matrix Algebra

1 Definitions and Terms

1.1 Matrix Entries

If A is an $m \times n$ matrix, then the scalar in the ith row and the jth column is denoted by a_{ij}. The **diagonal entries** in a matrix are the numbers a_{ij} where $i = j$. They form the **main diagonal** of A. A **diagonal matrix** is a square matrix whose nondiagonal entries are 0. An example is I_n. A matrix whose entries are all zero is called a **zero matrix**, and denoted as 0. To matrices are **equal** if they have the same size, and all their corresponding entries are equal.

1.2 Matrix Operations

If A and B are both $m \times n$ matrices, and $A + B = C$ then C is also an $m \times n$ matrix whose entries are the sum of the corresponding entries of A and B. If r is a scalar, then the **scalar multiple** $C = rA$ is the matrix whose entries are r times the corresponding entries of A.

Two matrices can be multiplied, by multiplying one matrix by the columns of the other matrix. If A is an $m \times n$ matrix and B is an $n \times p$ matrix with columns $\mathbf{b_1}$, $\mathbf{b_2}$, ..., $\mathbf{b_p}$, then the product AB is the $m \times p$ matrix $AB = A\,[\,\mathbf{b_1}\ \ \mathbf{b_2}\ \ \ldots\ \ \mathbf{b_p}] = [\,A\mathbf{b_1}\ \ A\mathbf{b_2}\ \ \ldots\ \ A\mathbf{b_p}]$. Note that usually $AB \neq BA$. If $AB = BA$, then we say that A and B **commute** with one another.

Since it is possible to multiply matrices, it is also possible to take their power. If A is a square matrix, then $A^k = A \ldots A$, where there should be k A's. Also A^0 is defined as I_n. Given an $m \times n$ matrix, the **transpose** of A is the $n \times m$ matrix, denoted by A^T, whose columns are formed from the corresponding rows of A. So $\text{row}_i(A) = \text{col}_i(A^T)$. The transpose should not be confused by a matrix to the power T.

1.3 Inverses

An $n \times n$ (square) matrix A is said to be invertible if there is an $n \times n$ matrix C such that $CA = I_n = AC$. In this case C is the **inverse** of A, denoted as A^{-1}. A matrix that is not invertible is called a **singular matrix**. For a 2-dimensional matrix, the quantity $a_{11}a_{22} - a_{12}a_{21}$ is called the determinant, noted as $\det A = ad - bc$. An **elementary matrix** is a matrix that is obtained by performing a single elementary row operation on an identity matrix.

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be **invertible** if there exists a function $S : \mathbb{R}^n \to \mathbb{R}^n$ such that $S(T(\mathbf{x})) = \mathbf{x}$ and $T(S(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$. We call S the **inverse** of T and write it as $S = T^{-1}$ or $S(\mathbf{x}) = T^{-1}(\mathbf{x})$. If $T(\mathbf{x}) = A\mathbf{x}$, then A is called the **standard matrix** of the linear transformation T.

1.4 Subspaces

A **subspace** of $\mathbb{R}^n$ is any set H in $\mathbb{R}^n$ for which three properties apply. The zero vector $\mathbf{0}$ is in H, for each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u}+\mathbf{v}$ is in H, and for each $\mathbf{u}$ in H, the vector $c\mathbf{u}$ is in H (for every scalar c). Subspaces are always a point (0-dimensional) on the origin, a line (1-dimensional) through the origin, a plane (2-dimensional) through the origin, or any other multidimensional plane through the origin.

The **column space** of a matrix A is the set Col A of all linear combinations of the columns of A. The column space of an $m \times n$ matrix is a subspace of $\mathbb{R}^m$. The **row space** of a matrix A is the set Row A of all linear combinations of the rows of A. The **null space** of a matrix A is the set Nul A of all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$. The null space of an $m \times n$ matrix is a subspace of $\mathbb{R}^n$. A **basis** for a subspace H or $\mathbb{R}^n$ is a linearly independent set in H that spans H.

1.5 Dimension and Rank

Suppose the set $\beta = \{\mathbf{b_1}, \ldots, \mathbf{b_p}\}$ is a basis for a subspace H. For each $\mathbf{x}$ in H, the **coordinates of x relative to the basis** β are the weights $c_1, \ldots, c_p$ such that $\mathbf{x} = c_1\mathbf{b_1} + \ldots + c_p\mathbf{b_p}$. The vector $[\mathbf{x}]_\beta$ in $\mathbb{R}^p$ with coordinates $c_1, \ldots, c_p$ is called the **coordinate vector of x (relative to** β**)** or the $beta$**-coordinate vector of x**.

The **dimension** of a subspace H, denoted by dim H, is the number of vectors in any basis for H. The zero subspace has no basis, since the zero vector itself forms a linearly dependent set. The **rank** of a matrix A, denoted by rank A, is the dimension of the column space of A. So per definition rank A = dim Col A.

1.6 Kernel and Range

Let T be a linear transformation. The **kernel** (or **null space**) of T, denoted as ker T, is the set of all $\mathbf{u}$ such that $T(\mathbf{u}) = \mathbf{0}$. The **range** of T, denoted as range T, is the set of all vectors $\mathbf{v}$ for which $T(\mathbf{x}) = \mathbf{v}$ has a solution. If $T(\mathbf{x}) = A\mathbf{x}$, then the kernel of T is the null space of A, and the range of T is the column space of A.

2 Theorems

1. **The Row-Column Rule**. If A is an $m \times n$ matrix, and B is an $n \times p$ matrix, then the entry in the ith row and the jth column of AB is $(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}$.
2. From the Row-Column Rule can be found that $\text{row}_i(AB) = \text{row}_i(A) \cdot B$.
3. If A has size 2×2. If $ad - bc \neq 0$, then A is invertible, and $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
4. If A is an invertible matrix, then for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.
5. If A is invertible, then A^{-1} is invertible, and $(A^{-1})^{-1} = A$.
6. If A and B are $n \times n$ matrices, then so is AB, and the inverse of AB is the product of the inverses of A and B in the reverse order. That is, $(AB)^{-1} = B^{-1}A^{-1}$. This also goes for any number of matrices. That is, if $A_1, \ldots, A_p$ are $n \times n$ matrices, then $(A_1A_2 \ldots A_p)^{-1} = A_p^{-1} \ldots A_2^{-1}A_1^{-1}$.
7. If A is an invertible matrix, then so is A^T, and the inverse of A^T is the transpose of A^{-1}. That is, $(A^T)^{-1} = (A^{-1})^T$.
8. If an elementary row operation is performed on an $m \times n$ matrix A, the resulting matrix can be written as EA, where the $m \times m$ elementary matrix E is created by performing the same row operation on I_m.

9. Each elementary matrix E is invertible. The inverse of E is the elementary matrix of the same type that transforms E back into I.

10. An $n \times n$ matrix A is invertible if, and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}.

11. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation, and let A be the standard matrix for T. That is, $T(\mathbf{x}) = A\mathbf{x}$. Then T is invertible if, and only if A is an invertible matrix. In that case, $T^{-1}(\mathbf{x}) = A^{-1}\mathbf{x}$.

12. If $\mathbf{u_1}, \ldots, \mathbf{u_p}$ are in the subspace H, then every vector in $\text{Span}\{\mathbf{u_1}, \ldots, \mathbf{u_p}\}$ is in H.

13. If A is an $m \times n$ matrix with column space Col A, then Col $A = \text{Span}\{\mathbf{a_1}, \ldots, \mathbf{a_n}\}$. Also Col A is the set of all $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has a solution.

14. The pivot columns of a matrix A form a basis for the column space of A.

15. The dimension of Nul A is equal to the number of free variables in $A\mathbf{x} = \mathbf{0}$.

16. The dimension of Col A (which is rank A) is equal to the number of pivot columns in A.

17. **The Rank Theorem**. If a matrix A has n columns, then dim Col A + dim Nul A = rank A + dim Nul $A = n$.

18. **The Basis Theorem**. Let H be a p-dimensional subspace of $\mathbb{R}^n$. Any linearly independent set of exactly p elements in H is automatically a basis for H. Also any set of p elements of H that spans H is automatically a basis for H.

19. If the linear transformation $T(\mathbf{x}) = A\mathbf{x}$, then ker T = Nul A and range T = Col A.

20. If $\mathbb{R}^n$ is the domain of T, then dim ker T + dim range $T = n$.

21. If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as for that of B.

22. **The Invertible Matrix Theorem**. The following statements are equivalent for a particular square $n \times n$ matrix A (be careful: these statements are not equivalent for rectangular matrices). That is, if one is true, then all are true, and if one is false, then all are false:

    (a) A is an invertible matrix.

    (b) A is row equivalent to the $n \times n$ identity matrix I_n.

    (c) A has n pivot positions.

    (d) The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

    (e) The columns of A form a linearly independent set.

    (f) The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

    (g) The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$. That is, the mapping $\mathbf{x} \mapsto A\mathbf{x}$ is onto $\mathbb{R}^n$.

    (h) The columns of A span $\mathbb{R}^n$.

    (i) The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

    (j) There is an $n \times n$ matrix C such that $CA = I = AC$.

    (k) The transpose A^T is an invertible matrix.

    (l) The columns of A form a basis of $\mathbb{R}^n$.

    (m) Col $A = \mathbb{R}^n$

(n) dim Col A = rank $A = n$

(o) Nul $A = \mathbf{0}$

(p) dim Nul $A = 0$

(q) det $A \neq 0$ (The definition for determinants will be given in chapter 3.)

(r) The number 0 is not an eigenvalue of A (The definition for eigenvalues will be given in chapter 5.)

3 Calculation Rules

3.1 Algebraic Definitions

If A, B and C are $m \times n$ matrices, then the addition and multiplication is defined as:

$$A+B = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{m1} & \dots & b_{mn} \end{bmatrix} = \begin{bmatrix} (a_{11}+b_{11}) & \dots & (a_{1n}+b_{1n}) \\ \vdots & & \vdots \\ (a_{m1}+b_{m1}) & \dots & (a_{mn}+b_{mn}) \end{bmatrix} \tag{1}$$

$$rA = r\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} = \begin{bmatrix} ra_{11} & \dots & ra_{1n} \\ \vdots & & \vdots \\ ra_{m1} & \dots & ra_{mn} \end{bmatrix} \tag{2}$$

It is also possible to multiply matrices. If A is an $m \times n$ matrix and B is an $n \times p$ matrix with columns $\mathbf{b_1}$, $\mathbf{b_2}$, …, $\mathbf{b_n}$, then the product AB is the $m \times p$ matrix:

$$AB = A\,[\,\mathbf{b_1}\ \ \mathbf{b_2}\ \ \dots\ \ \mathbf{b_p}] = [\,A\mathbf{b_1}\ \ A\mathbf{b_2}\ \ \dots\ \ A\mathbf{b_p}] \tag{3}$$

Note that $AB \neq BA$. Also, their power is:

$$A^k = A \dots A \quad (k \text{ times}) \tag{4}$$

The transpose of a matrix is defined as:

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \qquad \Rightarrow \qquad A^T = \begin{bmatrix} a_{11} & \dots & a_{m1} \\ \vdots & & \vdots \\ a_{1n} & \dots & a_{nm} \end{bmatrix} \tag{5}$$

3.2 Algebraic Rules

The following rules apply for matrix addition.

$$A+B = B+A \tag{6}$$

$$(A+B)+C = A+(B+C) \tag{7}$$

$$A+0 = A \tag{8}$$

$$r(A+B) = rA+rB \tag{9}$$

$$(r + s)A = rA + sA \tag{10}$$

$$r(sA) = (rs)A \tag{11}$$

For matrix multiplication, the following rules apply.

$$A(BC) = (AB)C \tag{12}$$

$$A(B + C) = AB + AC \tag{13}$$

$$(B + C)A = BA + CA \tag{14}$$

$$r(AB) = (rA)B = A(rB) \tag{15}$$

$$I_m A = A = AI_n \tag{16}$$

$$A^0 = I_n \tag{17}$$

$$I\mathbf{u} = \mathbf{u} \tag{18}$$

The following rules apply for matrix transposes.

$$(A^T)^T = A \tag{19}$$

$$(A + B)^T = A^T + B^T \tag{20}$$

$$(rA)^T = r(A^T) \tag{21}$$

$$(AB)^T = B^T A^T \tag{22}$$