Eigenvalues and Eigenvectors

1 Definitions and Terms

1.1 Introduction to Eigenvectors and Eigenvalues

An **eigenvector** of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$. Such an $\mathbf{x}$ is called an **eigenvector corresponding to** λ. The set of all eigenvectors corresponding to λ is a subspace of $\mathbb{R}^n$ and is called the **eigenspace of** A **corresponding to** λ.

1.2 The Characteristic Equation

The scalar equation $\det(A - \lambda I) = 0$ is called the **characteristic equation** of A. If A is an $n \times n$ matrix, then $\det(A - \lambda I)$ is a polynomial of degree n called the **characteristic polynomial** of A. A specific eigenvalue λ_s is said to have **multiplicity** r if $(\lambda - \lambda_s)$ occurs r times as a factor of the characteristic polynomial.

If A and B are $n \times n$ matrices, then A and B are **similar** if there is an invertible matrix P such that $P^{-1}AP = B$. Changing A into $P^{-1}AP$ is called a **similarity transformation**.

1.3 Diagonalization

A square matrix A is said to be **diagonalizable** if A is similar to a diagonal matrix, that is, if $A = PDP^{-1}$ for some invertible matrix P and some diagonal matrix D. The basis formed by all the eigenvectors of a matrix A is called the **eigenvector basis**.

1.4 Eigenvectors and Linear Transformations

Let V be an n-dimensional vector space, W an m-dimensional vector space, T any linear transformation from V to W, β a basis for V and γ a basis for W. Now the image of any vector $[\mathbf{x}]_\beta$ (the vector $\mathbf{x}$ relative to the base β) to $[T(\mathbf{x})]_\gamma$ is given by $[T(\mathbf{x})]_\gamma = M[\mathbf{x}]_\beta$, where $M = [[T(\mathbf{b_1})]_\gamma \ [T(\mathbf{b_2})]_\gamma \ \dots \ [T(\mathbf{b_n})]_\gamma]$. The $m \times n$ matrix M is a matrix representation of T, called the **matrix for T relative to the bases** β **and** γ.

In the common case when W is the same as V, and the basis γ is the same as β, the matrix M is called the **matrix for** T **relative to** β or simply the β**-matrix for** T and is denoted by $[T]_\beta$. Now $[T(\mathbf{x})]_\beta = [T]_\beta[\mathbf{x}]_\beta$.

1.5 Complex Eigenvalues

A complex scalar λ satisfies $\det(A - \lambda I) = 0$ if, and only if there is a nonzero vector $\mathbf{x}$ in $\mathbb{C}^n$ such that $A\mathbf{x} = \lambda\mathbf{x}$. We call λ a **(complex) eigenvalue** and $\mathbf{x}$ a **(complex) eigenvector** corresponding to λ.

1.6 Dynamical Systems

Many dynamical systems can be described or approximated by a series of vectors $\mathbf{x_k}$ where $\mathbf{x_{k+1}} = A\mathbf{x_k}$. The variable k often indicates a certain time variable. If A is a diagonal matrix, having n eigenvalues forming a basis for $\mathbb{R}^n$, any vector $\mathbf{x_k}$ can be described by $\mathbf{x_k} = c_1(\lambda_1)^k\mathbf{v_1} + \ldots + c_n(\lambda_n)^k\mathbf{v_n}$. This is called the **eigenvector decomposition** of $\mathbf{x_k}$.

The graph $\mathbf{x_0}, \mathbf{x_1}, \ldots$ is called a **trajectory** of the dynamical system. If, for every $\mathbf{x_0}$, the trajectory goes to the origin $\mathbf{0}$ as k increases, the origin is called an **attractor** (or sometimes **sink**). If, for every $\mathbf{x_0}$, the trajectory goes away from the origin, it is called a **repellor** (or sometimes **source**). If $\mathbf{0}$ attracts for certain $\mathbf{x_0}$ and repels for other $\mathbf{x_0}$, then it is called a **saddle point**. For matrices having complex eigenvalues/eigenvectors, it often occurs that the trajectory spirals inward to the origin (attractor) or outward (repellor) from the origin (the origin is then called a **spiral point**).

1.7 Differential Equations

Linear algebra comes in handy when differential equations take the form $\mathbf{x}' = A\mathbf{x}$. The **solution** is then a vector-valued function that satisfies $\mathbf{x}' = A\mathbf{x}$ for all t in some interval. There is always a **fundamental set of solutions**, being a basis for the set of all solutions. If a vector $\mathbf{x_0}$ is specified, then the **initial value problem** is to construct the function such that $\mathbf{x}' = A\mathbf{x}$ and $\mathbf{x}(0) = \mathbf{x_0}$.

2 Theorems

1. The solution set of $A\mathbf{x} = \lambda\mathbf{x}$ is the null space of $A - \lambda I$. This is the eigenspace corresponding to λ.
2. The eigenvalues of a triangular matrix are the entries on its main diagonal.
3. 0 is an eigenvalue of A if, and only if A is not invertible.
4. If $\mathbf{v_1}, \ldots, \mathbf{v_r}$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v_1}, \ldots, \mathbf{v_r}\}$ is linearly independent.
5. A scalar λ is an eigenvalue of an $n \times n$ matrix A if, and only if λ satisfies the characteristic equation $\det(A - \lambda I) = 0$.
6. If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities).
7. **The Diagonalization Theorem:** An $n \times n$ matrix A is diagonalizable if, and only if A has n linearly independent eigenvectors. In fact, $A = PDP^{-1}$, with D a diagonal matrix, if, and only if the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.
8. If a $n \times n$ matrix has n distinct eigenvalues, then it is diagonalizable. (Note that the opposite is not always true.)
9. Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \ldots, \lambda_p$.
   (a) For $1 \leq k \leq p$, the dimension of the eigenspace for λ_k is less than or equal to the multiplicity of the eigenvalue λ_k.
   (b) The matrix A is diagonalizable if, and only if the sum of the dimensions of the distinct eigenspaces equals n, and this happens if, and only if the dimension of the eigenspace for each λ_k equals the multiplicity of λ_k.

(c) If A is diagonalizable and β_k is a basis for the eigenspace corresponding to λ_k for each k, then the total collection of vectors in the sets $\beta_1, \ldots, \beta_p$ forms an eigenvector basis for $\mathbb{R}^n$.

10. **Diagonal Matrix Representation:** Suppose $A = PDP^{-1}$, where D is a diagonal $n \times n$ matrix. If $\beta = \{\mathbf{b_1}, \ldots, \mathbf{b_n}\}$ is the basis for $\mathbb{R}^n$ formed from the columns of P, then D is the β-matrix for the transformation $\mathbf{x} \mapsto A\mathbf{x}$.

11. If P is the matrix whose columns come from the vectors in β (that is, $P = [\mathbf{b_1} \ \ldots \ \mathbf{b_n}]$), then $[T]_\beta = P^{-1}AP$.

12. If λ is an eigenvalue of A and $\mathbf{x}$ is a corresponding eigenvector in $\mathbb{C}^n$, then the complex conjugate $\bar{\lambda}$ is also an eigenvalue of A, with $\bar{\mathbf{x}}$ as the corresponding eigenvector.

13. Let A be a real 2×2 matrix with a complex eigenvalue $\lambda = a - bi$ ($b \neq 0$) and an associated eigenvector $\mathbf{v}$ in $\mathbb{C}^2$. Then $A = PCP^{-1}$, where $P = [\operatorname{Re}\mathbf{v} \ \ \operatorname{Im}\mathbf{v}]$ and $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

14. For discrete dynamical systems, multiple possibilities are present:

    (a) $|\lambda_i| < 1$ for every i. In that case the system is an attractor.

    (b) $|\lambda_i| > 1$ for every i. In that case the system is a repellor.

    (c) $|\lambda_i| > 1$ for some i and $|\lambda_i| < 1$ for all other i. In that case the system is a saddle point.

    (d) $|\lambda_i| = 1$ for some i. In that case the trajectory can converge to any vector in the eigenspace corresponding to the eigenvalue 1. However, it can also diverge.

15. For linear differential equations, each eigenvalue-eigenvector pair provides a solution of $\mathbf{x}' = A\mathbf{x}$. This solution is $\mathbf{x}(t) = \mathbf{v}e^{\lambda t}$.

16. For linear differential equations, any linear combination of solutions is also a solution for the differential equation. So if $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are solutions, then $c\mathbf{u}(t) + d\mathbf{v}(t)$ is also a solution for any scalars c and d.