Determinants

1 Definitions and Terms

1.1 Determinants

For any square matrix, let A_{ij} denote the submatrix formed by deleting the ith row and the jth column of A. For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1j} \text{det } A_{1j}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of A. In symbols:

$$\text{det } A = a_{11}\text{det } A_{11} - a_{12}\text{det } A_{12} + \ldots + (-1)^{n+1} a_{1n}\text{det } A_{1n} = \sum_{j=1}^{n} (-1)^{1+j} a_{1j}\text{det } A_{1j}$$

Next to writing det A to indicate a determinant, it is also often used to write $|A|$.

1.2 Cofactors

Given $A = [a_{ij}]$, the **(i, j)-cofactor** of A is the number C_{ij} given by $C_{ij} = (-1)^{i+j}\text{det } A_{ij}$. The determinant of A can be determined using a cofactor expansion. The formula $\text{det } A = a_{11}C_{11} + a_{12}C_{12} + \ldots a_{1n}C_{1n}$ is called a **cofactor expansion across the first row** of A.

A matrix $B = [b_{ij}]$ of cofactors of A, where $b_{ij} = C_{ij}$, is called the **adjugate** (or **classical adjoint**) of A. This is denoted by $\text{adj}A$.

2 Theorems

1. The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion. The expansion across the ith row is: $\text{det } A = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}$. The expansion down the jth column is: $\text{det } A = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}$.
2. If A is a triangular matrix, then det A is the product of the entries on the main diagonal of A.
3. **Row Operations:** Let A be a square matrix.
    - If a multiple of one row of A is added to another row to produce a matrix B, then $\text{det } A = \text{det } B$.
    - If two rows of A are interchanged to produce B, then $\text{detA} = -\text{detB}$.
    - If one row of A is multiplied by k to produce B, then $\text{det } B = k \cdot \text{det } A$.
4. A square matrix A is invertible if, and only if $\text{det } A \neq 0$.
5. If A is an $n \times n$ matrix, then $\text{det } A^T = \text{det } A$.
6. If A and B are $n \times n$ matrices, then $\text{det } AB = (\text{det } A)(\text{det } B)$.
7. **Cramer's Rule:** Let A be an invertible $n \times n$ matrix. For any $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ has entries given by $x_i = \frac{\text{det } A_i(\mathbf{b})}{\text{det } A}$, where $i = 1, 2, \ldots, n$ and $A_i\mathbf{b}$ is the matrix obtained from A by replacing column i for the vector $\mathbf{b}$.

8. Let A be an invertible $n \times n$ matrix. Then: $A^{-1} = \frac{1}{\det\ A}\mathrm{adj}A$.

9. If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is $|\det\ A|$. If A is a 3×3 matrix, the volume of the parallelepiped determined by the columns of A is $|\det\ A|$.

10. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation determined by a 2×2 matrix A. If S is any region in $\mathbb{R}^2$ with finite area, then $\{\text{area of } T(S)\} = |\det\ \mathrm{A}| \cdot \{\text{area of } S\}$. Also, if T is determined by a 3×3 matrix A, and if S is any region in $\mathbb{R}^3$ with finite volume, then $\{\text{volume of } T(S)\} = |\det\ \mathrm{A}| \cdot \{\text{volume of } S\}$.