Introduction to fuzzy sets

In this summary, we will examine various knowledge-based control systems. One type of such systems is based on fuzzy logic. We'll examine the basics of fuzzy logic in this chapter. We'll go more into depth on it in subsequent chapters.

1 Basic properties and representations of fuzzy sets

1.1 Fuzzy sets

Let's examine ordinary set theory. We have a domain X. Now examine a set A with objects $x_i \in X$. The **membership function** $\mu_A(x)$ is defined as

$$\mu_A(x) = \begin{cases} 1 & \text{iff } x \in A, \\ 0 & \text{iff } x \notin A. \end{cases} \tag{1.1}$$

So, an object x is either fully part of A or not at all part of A. We call such a set A a **crisp set**.

However, in **fuzzy logic**, things are different. Now an object x can also be partially in A. In other words, $\mu_A(x)$ can take values between 0 and 1 as well. We call such a set A a **fuzzy set**. Also, the value of $\mu_A(x)$ is called the **membership degree** or **membership grade**.

1.2 Properties of fuzzy sets

We can define various properties for fuzzy sets. The height of a fuzzy set $\text{hgt}(A)$ is the supremum (maximum) of the membership grades of A. So,

$$\text{hgt}(A) = \sup_{x \in X} \mu_A(x). \tag{1.2}$$

A fuzzy set A is **normal** if $\text{hgt}(A) = 1$. In other words, there is an x for which $\mu_A(x) = 1$. Any set that is not normal is called **subnormal**. Such a set A can be normalized using the normalization function $\text{norm}(A)$. It is defined such that, for all $x \in X$, we have

$$B = \text{norm}(A) \qquad \Rightarrow \qquad \mu_B(x) = \frac{\mu_A(x)}{\text{hgt}(A)}. \tag{1.3}$$

The **support** of a set A is the crisp subset of A with nonzero membership grades. Similarly, the **core** of a set A is the crisp subset of A with membership grade equal to one. So,

$$\text{supp}(A) = \{x | \mu_A(x) > 0\} \qquad \text{and} \qquad \text{core}(A) = \{x | \mu_A(x) = 1\}. \tag{1.4}$$

The **α-cut** A_α of a set A is the crisp subset of A with membership grades of at least α. So,

$$A_\alpha = \alpha\text{-cut}(A) = \{x | \mu_A(x) \geq \alpha\}. \tag{1.5}$$

Note that $\text{core}(A) = 1\text{-cut}(A)$. However, $\text{supp}(A) = 0\text{-cut}(A)$ is not always true.

Let's examine a set A. Its membership function $\mu_A(x)$ is called **unimodal** if it only has one global/local maximum. The corresponding set A is then called **convex**. If however $\mu_A(x)$ is **multimodel** (has several local maxima), then A is **non-convex**. Finally, the **cardinality** $\text{card}(A) = |A|$ of a finite discrete set A is the sum of the membership grades. Thus,

$$\text{card}(A) = |A| = \sum_{i=1}^{n} \mu_A(x_i). \tag{1.6}$$

1.3 Representations of fuzzy sets

There are several ways to represent fuzzy sets. We will examine a few.

- **Similarity-based representation** – We use a (dis)similarity measure $d(x, v)$ between two elements x and v. An example of a membership function is now given by

$$\mu(x) = \frac{1}{1 + d(x, v)}. \tag{1.7}$$

- **Trapezoidal membership function** – We choose parameters a, b, c and d ($a < b$, $c > d$) such that

$$\mu(x) = \max\left(0, \min\left(\frac{x-a}{b-a}, 1, \frac{d-x}{d-c}\right)\right). \tag{1.8}$$

  If $b = c$, we obtain the **triangular membership function**.

- **Piece-wise exponential membership function** – We choose the position parameters c_l and c_r ($c_l < c_r$) and the width parameters w_l and w_r ($w_l, w_r > 0$) such that

$$\mu(x) = \begin{cases} \exp\left(-\left(\frac{x-c_l}{2w_l}\right)^2\right) & \text{if } x < c_l, \\ \exp\left(-\left(\frac{x-c_r}{2w_r}\right)^2\right) & \text{if } x > c_r, \\ 1 & \text{otherwise.} \end{cases} \tag{1.9}$$

- **Singleton set** – This is a special fuzzy set. For some chosen element x_0, we have

$$\mu(x) = \begin{cases} 1 & \text{if } x = x_0, \\ 0 & \text{otherwise.} \end{cases} \tag{1.10}$$

- **Universal set** – This is another special fuzzy set. We simply have $\mu(x) = 1$ for all $x \in X$.

- **Point-wise representation** – For every individual element x, we specify the value of $\mu(x)$. Two different methods of notation are

$$A = \{\mu_A(x_1)/x_1, \mu_A(x_2)/x_2, \ldots, \mu_A(x_n)/x_n\} = \mu_A(x_1)/x_1 + \mu_A(x_2)/x_2 + \ldots + \mu_A(x_n)/x_n. \tag{1.11}$$

2 Modifying fuzzy sets

2.1 Basic operations on fuzzy sets

Let's examine a fuzzy set A. In ordinary set theory, we can do several things with sets. (Think of complements, unions, intersections and such.) We can extend these ideas to fuzzy sets. First, let's examine the **complement** $\bar{A}$ of A. The common definition for $\bar{A}$ is that, for all $x \in X$, we have

$$\mu_{\bar{A}}(x) = 1 - \mu_A(x). \tag{2.1}$$

To define the **intersection** $C = A \cap B$ between two sets A and B, we need a ***t*-norm** $T(a, b)$ such that $\mu_C(x) = T(\mu_A(x), \mu_B(x))$ for all $x \in X$. Such a t-norm must satisfy the following conditions.

$$T(a,1) = a, \tag{2.2}$$
$$b \leq c \Rightarrow T(a,b) \leq T(a,c), \tag{2.3}$$
$$T(a,b) = T(b,a), \tag{2.4}$$
$$T(a,T(b,c)) = T(T(a,b),c). \tag{2.5}$$

The most commonly used t-norms are the **standard intersection** (also known as the **minimum**) and the **algebraic product**, which are respectively defined as

$$T(a,b) = \min(a,b) \qquad \text{and} \qquad T(a,b) = ab. \tag{2.6}$$

The minimum is the largest possible t-norm.

To define the **union** $C = A \cup B$ between two sets A and B, we need a **t-conorm** $S(a,b)$ such that $\mu_C(x) = S(\mu_A(x), \mu_B(x))$ for all $x \in X$. Such a t-conorm must satisfy the following conditions.

$$S(a,0) = a, \tag{2.7}$$

$$b \leq c \Rightarrow S(a,b) \leq S(a,c), \tag{2.8}$$

$$S(a,b) = S(b,a), \tag{2.9}$$

$$S(a, S(b,c)) = S(S(a,b),c). \tag{2.10}$$

The most commonly used t-conorms are the **standard union** (also known as the **maximum**) and the **algebraic sum**, which are respectively defined as

$$S(a,b) = \max(a,b) \qquad \text{and} \qquad S(a,b) = 1-(1-a)(1-b) = a+b-ab. \tag{2.11}$$

The maximum is the smallest possible t-norm.

We can also change fuzzy sets by using **hedges**. Let's suppose that the fuzzy set A indicates expensive cars. If some element x (say, x = 10,000 euros) has a low membership degree, it is not expensive. But if its membership degree is high, it is expensive. How can we find the set B that indicates very expensive cars or the set C that indicates mildly expensive cars? There are two methods. We can use **shifted hedges**: we shift the membership function along the domain. So, $\mu_B(x) = \mu_A(x - 5{,}000)$ and $\mu_C(x) = \mu_A(x + 3000)$. We can also use **powered hedges**: $\mu_B(x) = \mu_A(x)^2$ and $\mu_C(x) = \sqrt{\mu_A(x)}$.

2.2 Modifications of fuzzy sets

Let's examine some domain X and another domain Y. We can define a fuzzy set A in X or in Y, but we can also define it in $X \times Y$. We then have to define $\mu_A(x,y)$ for every combination $x \in X, y \in Y$. The same can be done for higher-dimensional spaces. Such spaces are known as **Cartesian product spaces**.

Let's examine some Cartesian product spaces U, U_1 and U_2 with $U_1 \subseteq U \subseteq U_1 \times U_2$. (In other words, U_1 and U_2 together encompass U, which in turn encompasses U_1.) We also have some set A defined in U. We can now find the **projection** of A onto U_1 using

$$\text{proj}_{U_1}(A) = \left\{ \sup_{U_2} \mu_A(u)/u_1 | u_1 \in U_1 \right\}. \tag{2.12}$$

In other words, for each set of parameters of U_1, we browse through all combinations of the parameters of U_2 and look for the one with the highest value of $\mu_A(u)/u_1$.

Again, examine Cartesian product spaces U, U_1 and U_2 with $U_1 \subseteq U \subseteq U_1 \times U_2$. But now, we have a set A defined on U. We can find the **cylindrical extension** to U_1 using

$$\text{ext}_U(A) = \{\mu_A(u_1)/u | u \in U\}. \tag{2.13}$$

It is important to note that, with a projection, you go to a lower-dimensional space. This generally results in a loss of data. However, with a cylindrical extension, you go to a higher-dimensional space. You now do not lose data.

Let's examine two fuzzy sets A_1 and A_2, defined on domains X_1 and X_2, respectively. We would like to take the intersection between the two sets. But, because the two sets are defined on different domains, we can't do this using the normal definition. As a solution, we use cylindrical extensions. So,

$$A_1 \times A_2 = \text{ext}_{X_2}(A_1) \cap \text{ext}_{X_1}(A_2) \qquad \Rightarrow \qquad \mu_{A_1 \times A_2}(x_1, x_2) = T(\mu_{A_1}(x_1), \mu_{A_2}(x_2)). \tag{2.14}$$

2.3 Fuzzy relations

A **fuzzy relation** R is a fuzzy set in the Cartesian product space $X_1 \times X_2 \times \ldots \times X_n$. This fuzzy set has a membership function $\mu_R(x_1, x_2, \ldots, x_n)$ which gives a value between 0 and 1 (inclusive) for all combinations of parameters $x_1, x_2, \ldots, x_n$.

Now let's examine a fuzzy relation R in $X \times Y$ and a fuzzy set A in X. We can find a fuzzy set B in Y through the **composition** of A and R:

$$B = A \circ R = \text{proj}_Y \left(R \cap \text{ext}_{X \times Y}(A) \right). \tag{2.15}$$

We thus extend A to $X \times Y$, intersect it with R, and then project the result on Y. It can be shown that the membership function of B now satisfies

$$\mu_B(y) = \max_x \min(\mu_A(x), \mu_R(x, y)). \tag{2.16}$$