Space Engineering Period 2 Summary

1. Two-Dimensional Orbital Mechanics

1.1 Kepler's Laws

Kepler came up with three laws (which are in italic). These laws were equivalent with three rules which we now often use. **Kepler's Laws** are:

1. *Planets move in ellipses with the Sun at one focus.* (Planets/satellites/comets all move along conical sections.)
2. *Planets sweep out equal areas in equal times.* (Law of conservation of angular momentum.)
3. *The square of the orbital period is proportional to the cube of the length of the (semi-)major axis of the ellipse.* ($T^2 \sim a^3$)

1.2 Conservation of Angular Momentum (Second Law)

Define the **gravitational parameter** $\mu = MG$, where M is the mass of the central body (assumed to be much greater than the masses of the objects orbiting it) and G is a universal constant. The gravitational acceleration now is:

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^2}\frac{\mathbf{r}}{r} \tag{1.2.1}$$

From this can be derived that **angular momentum** is conserved (thus there is **conservation of angular momentum**. In formula:

$$\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{r} \times \mathbf{V} = \text{constant} = \mathbf{H} \tag{1.2.2}$$

This implies that the motion is in one plane, and $H = r^2\dot{\phi} = \text{constant}$.

1.3 Conservation of Energy

Define the **energy** E as:

$$E = \frac{V^2}{2} - \frac{\mu}{r} \tag{1.3.1}$$

It can be shown that the energy stays constant. Thus there is **conservation of energy**.

1.4 Conical Sections (First Law)

It can also be shown that $\ddot{r} - r\dot{\phi}^2 = -\frac{\mu}{r^2}$. Combining this with conservation of angular momentum ($r^2\dot{\phi} = H$), and solving the differential equation, gives the **equation for conical sections**:

$$r = \frac{p}{1 + e\cos\theta} \tag{1.4.1}$$

where r is the distance from the origin, θ is the **true anomaly** (the angle with the **pericenter**, being the point of the orbit closest to the origin), p is the **semi-latus rectum** ($p = H^2/\mu$) and e is the **eccentricity**. The eccentricity determines the shape:

- $e = 0 \Rightarrow$ The orbit has the shape of a circle.

- $0 < e < 1 \Rightarrow$ The orbit has the shape of an ellipse.
- $e = 1 \Rightarrow$ The orbit has the shape of a parabola.
- $e > 1 \Rightarrow$ The orbit has the shape of an hyperbola.

1.5 Useful Equations for Elliptical Orbits

Most orbits are **elliptical orbits**. Suppose a is the **semi-major axis** (half of the longest diagonal) and b is the **semi-minor axis** (half of the shortest diagonal) of the ellipse. Then the following equation holds:

$$p = a(1 - e^2) \quad \Rightarrow \quad r = \frac{a(1 - e^2)}{1 + e\cos\theta} \tag{1.5.1}$$

Now if r_p is the minimum distance from the origin (**pericenter**) and r_a is the maximum distance from the origin (**apocenter**), then also:

$$r_p = a(1 - e) \qquad r_a = a(1 + e) \tag{1.5.2}$$

$$a = \frac{r_a + r_p}{2} \qquad e = \frac{r_a - r_p}{r_a + r_p} \tag{1.5.3}$$

1.6 Velocities

From conservation of angular momentum, conservation of energy, and elliptical properties, it follows that the velocities in the pericenter and apocenter can be calculated with:

$$V_p^2 = \frac{\mu}{a}\left(\frac{1 + e}{1 - e}\right) \qquad V_a^2 = \frac{\mu}{a}\left(\frac{1 - e}{1 + e}\right) \tag{1.6.1}$$

With some help of these equations, the **energy equation for elliptical orbits** can be derived:

$$V^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right) \tag{1.6.2}$$

(Note that inserting $r_a = a(1+e)$ and $r_p = a(1-e)$ into equation 1.6.2 transforms it back to the equations in 1.6.1.) For a circle it is clear that $r = a$. So inserting this in equation 1.6.2 gives us the **local circular velocity**:

$$V_c = \sqrt{\frac{\mu}{r}} \tag{1.6.3}$$

1.7 Orbital Period (Third Law)

From Kepler's second law, and by using the equation $H = \sqrt{\mu p}$, the **orbital period** T can be derived, which indicates Kepler's third law was correct:

$$T = 2\pi\sqrt{\frac{a^3}{\mu}} \tag{1.7.1}$$

1.8 The Eccentric Anomaly

The **eccentric anomaly** E is an alternative for the true anomaly θ and often shows up when travel times in elliptical orbits need to be calculated. First of all, it can be derived that:

$$r = a(1 - e\cos E) \tag{1.8.1}$$

From this the relationship between θ and E can be derived:

$$\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2} \tag{1.8.2}$$

where $\theta/2$ and $E/2$ have to be in the same quadrant. After some derivation, **Kepler's Equation** can be found:

$$E - e\sin E = \sqrt{\frac{\mu}{a^3}}(t - t_p) = n(t - t_p) = M \tag{1.8.3}$$

where t_p is the time of the last passage of the pericentre, n is the mean angular velocity ($n = \sqrt{\mu/a^3}$), M is the mean anomaly, E is the eccentric anomaly and θ is the true anomaly. Using the past equations, one can use the following steps to find the time needed, given a certain position: $\theta \rightarrow E \rightarrow M \rightarrow t - t_p$. The other way around is a bit more difficult. Given a certain time span, one can find the change in mean anomaly M. From this value M, one can approximate E recursively, using $E_{n+1} = M + e\sin E_n$ (where $E_0 = 0 \Leftrightarrow E_1 = M$).

1.9 Parabolic and Hyperbolic Orbits

Parabolic orbits are the orbits at which $V \rightarrow 0$ as $r \rightarrow \infty$. Using this data and the energy equation, one finds the **local escape velocity**:

$$V_o = \sqrt{\frac{2\mu}{r}} = \sqrt{2}V_c \tag{1.9.1}$$

where V_c still is the local circular velocity. While objects in a parabolic orbit lose all their speed as $r \rightarrow \infty$, objects in **hyperbolic orbits** still have a velocity when $r \rightarrow \infty$. Using the energy equation once more, one finds for the actual local satellite velocity:

$$V^2 = V_o^2 + V_\infty^2 \tag{1.9.2}$$

where V_∞ is the velocity the satellite would have as $r \rightarrow \infty$.

2. Three-Dimensional Orbits

2.1 Orbit Descriptions

The projection of the satellite on the Earth's surface is called the **sub-satellite point**. The trace of the sub-satellite point is called the **ground track**. The ground track is a way to visualize the shape of the orbit. But to describe an orbit, we want to know the satellite's position and velocity at a given point, which can not be derived from a ground track.

2.2 Cartesian Elements

There are multiple ways to fully describe an orbit. One such description (which is not often used) is in **Cartesian Elements**: $X, Y, Z, \dot{X}, \dot{Y}, \dot{Z}$. This Cartesian coordinate system has as origin the center of the earth. The X-axis points in the direction of the **vernal equinox**, which is the position of the sun as it cross the equator around March 21$^{\text{st}}$. The Z-axis points to the North Pole and the Y-axis is perpendicular to the past two (its direction can be found with the right-hand-rule). These 6 orbital elements fully determine the orbit, orbital plane and satellite position. Note that X, Y and Z describe the position, while $\dot{X}$ and $\dot{Y}$ and $\dot{Z}$ describe the velocity.

2.3 Keplerian Elements

Another description is in **Keplerian Elements** (see figure 2.1): $a, e, i, \Omega, \omega, \theta$. a is the **semi-major axis** of the ellipse the satellite is traveling on. e is the **eccentricity** of the ellipse. i is the **inclination** of the orbital plane, being the angle between the orbital plane and the equatorial plane. Ω is the **right ascension of the ascending node**, being the angle between the X-axis (pointing to the vernal equinox) and the ascending node. (The **ascending node** is the point at which the sub-satellite point of the satellite crosses the equator in a northward direction.) ω is the **argument of perigee**, being the angle between the perigee and the ascending node. And finally, θ is the true anomaly, which was discussed in the last chapter. Note here that i and Ω describe the orbital plane, a and e describe the shape of the ellipse, ω describes the orientation of the ellipse and θ describes the position of the satellite on the ellipse.

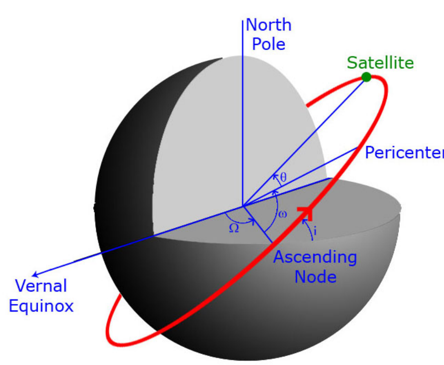

Figure 2.1: Indication of the Keplerian Elements.

2.4 Orbit Perturbations

Orbits are in reality not exact ellipses in a 2D plane. The real (3D) orbits are influenced by a lot aspects. For example, irregularities on earth (non-spherical shape, non-homogeneous division of mass), gravity from moon/sun, atmospheric drag, solar radiation pressure, etcetera.

However, these **orbit perturbations** do not have to be bad, as they can be used. Due to the non-spherical shape of the earth, there is extra mass around the equator, effecting the satellite orbit. This causes **precession** (rotation of the orbital plane). If the angle of inclination is $\phi = 97^\circ$, the precession is about 1° per day, which is sufficient to keep the sun always on the same side of the satellite. More about this will be discussed in chapter 5.10.

3. Rocket Motion

3.1 Tsoilkowski's Formula

Suppose we have a rocket with mass M flying at velocity V in some direction, expelling fuel with mass dM with a velocity w (with respect to the rocket) in the opposite direction. The change in momentum now is:

$$dI = I_{final} - I_{start} = ((M + dM)(V + dV) - (V - w)dM) - (M\,V) = M\;dV + w\;dM \tag{3.1.1}$$

where (in the case of rockets, where fuel is exhausted, and thus the mass decreases) dM will be negative. If we assume no external forces are present (thus $dI = 0$) and if we assume w is constant, then integrating the last equation with respect to time results in **Tsoilkowski's formula**:

$$\Delta V = w\;\ln\frac{M_0}{M_e} = I_{spec}\;g_0\;\ln\Lambda \tag{3.1.2}$$

where M_0 is the **initial mass** of the rocket, M_e is the **final mass**, $\Lambda = M_0/M_e$ is the **mass ratio**, g_0 is the gravitational acceleration at ground level and I_{spec} is the **specific impulse** (per definition equal to $I_{spec} = w/g_0 = T/mg$).

3.2 Thrust and Burn Time

The thrust T of a rocket can be calculated as follows:

$$T = m\frac{dV}{dt} = -w\frac{dM}{dt} = w\,m \tag{3.2.1}$$

where $m = -dM/dt$ is the mass flow. Assuming the mass flow is constant, the **burn time** t_b can be calculated as follows:

$$t_b = \frac{M_0 - M_e}{m} = \frac{w}{T}(M_0 - M_e) = \frac{w}{g_0}\frac{M_0 g_0}{T}\left(1 - \frac{M_e}{M_0}\right) = \frac{I_{spec}}{\Psi_0}\left(1 - \frac{1}{\Lambda}\right) \tag{3.2.2}$$

where Ψ_0 is the **thrust-to-weight-ratio** (defined as the thrust per unit of initial weight $\Psi_0 = \frac{T}{M_0 g_0}$).

3.3 Traveled Distance

To find the distance that a rocket has traveled when the rocket has burned out, we once more use $m = -dM/dt$ and find the following:

$$s_e = \int_0^{t_b} V\;dt = \int_0^{t_b} w\;\ln\frac{M_0}{M}dt = -\int_0^{t_b} w\,(\ln M - \ln M_0)\,dt = \int_{M_0}^{M_e}\frac{w}{m}\,(\ln M - \ln M_0)\,dM \tag{3.3.1}$$

Let's also assume that the mass flow is constant. Integrating and working out the result gives the following equation:

$$s_e = w\,t_b\left(1 - \frac{\ln\Lambda}{\Lambda - 1}\right) + V_0\,t_b \tag{3.3.2}$$

3.4 Gravity and Drag

When gravity and drag are included in the calculations, things get more difficult. Using $T = m\,w = -w\frac{dM}{dt}$, it follows that:

$$M\frac{dV}{dt} = T - Mg - D \quad \Rightarrow \quad \Delta V = w \ln \Lambda - \int_0^{t_b} g\,dt - \int_0^{t_b} \frac{D}{M} dt \tag{3.4.1}$$

This is a complicated integral. When it is used, it is often simplified in several ways. First of all, when a rocket is launched from earth, the burn times are often short enough to assume the gravity stays constant (thus $g = g_0$). We also still assume that w is constant. And finally drag is often neglected. This gives the following two equations:

$$\Delta V = w\,\ln \Lambda - g\,t_b \tag{3.4.2}$$

$$s_e = w\,t_b\left(1 - \frac{\ln \Lambda}{\Lambda - 1}\right) - \frac{1}{2}g t_b^2 + V_0\,t_b \tag{3.4.3}$$

Note that usually $V_0 = 0$ and thus $\Delta V = V_e$.

3.5 Rocket Restrictions

When a rocket is lifting off from earth, there are several restrictions to its parameters. First of all, there are restrictions on the thrust-to-weight-ratio and to the burn time, if the rocket ever wants to come loose from the ground:

$$T > M_0 g_0 \quad \Rightarrow \quad \Psi_0 > 1 \quad \Rightarrow \quad t_b < I_{spec}\left(1 - \frac{1}{\Lambda}\right) \tag{3.5.1}$$

There is also a restriction for the maximum thrust-to-weight-ratio and the maximum burn time. It can be shown that the acceleration at burn-out (at which it is at a maximum, for constant thrust, since the mass is at a minimum) is $a_e = g_0(\Psi_0\Lambda - 1)$. Thus the following restrictions are in order:

$$\Psi_0 < \frac{1}{\Lambda}\left(\frac{(a_e)_{max}}{g_0} + 1\right) \quad \Rightarrow \quad t_b > \frac{I_{spec}(\Lambda - 1)}{\frac{(a_e)_{max}}{g_0} + 1} \tag{3.5.2}$$

where the second part was derived using equation 3.2.2.

3.6 Coasting

After burn-out, the rocket pursues its vertical flight. During this flight, its velocity is $V = V_e - gt$. This vertical flight continues for a time of $t_c = V_e/g_0$ (where g is approximated as being constant) until it has reached its highest point. The distance traveled after burn-out of the rocket, can be calculated as follows:

$$s_c = \int_0^{t_c} V\,dt = V_e\,t_c - \frac{1}{2}g_0 t_c^2 = \frac{1}{2}\frac{V_e^2}{g_0} \tag{3.6.1}$$

Now the time taken to reach the highest point (the **culmination point**) and the maximum height reached can be calculated, simply by adding things up:

$$t_{tot} = t_b + t_c \qquad h_{tot} = h_b + h_c \tag{3.6.2}$$

3.7 Multi-Stage Rockets

Normal rockets usually do not reach the high velocity necessary for low earth orbit. That's why multi-stage rockets are used. As the name indicates, a multi-stage rocket consists of multiple stages, each with its own mass. Usually a division is made as follows:

$$M_0 = M_p + M_e = M_p + M_c + M_u \tag{3.7.1}$$

where M_0 is the total mass, M_p is the mass of the fuel of the current stage, M_e is the empty rocket, M_c is the structural weight of that stage and M_u is the payload of that stage. So the rocket burns up its fuel with mass M_p to give the rocket with mass M_e a certain velocity, which can be calculated using Tsoilkowski's equation. Then the structure with mass M_c is ejected, and the new stage, consisting of M_u is activated. This stage can now be seen as a new rocket, so if we call $M_{0_2} = M_{u_1}$, we can do all the calculations again for the new stage and calculate the velocity of the rocket at the end of the second stage. The primary advantage lies in the fact that structural weight is dumped as soon as it's unnecessary, and thus does not need to accelerate further.

4. Satellite Tracking Systems

4.1 Measuring Ways

There are multiple tracking systems, and they sometimes measure different aspects of the satellite trajectory. Some tracking systems measure the **range**, which is the distance between the ground station and the satellite. Others measure the **range-rate**, which is the time derivative of the distance between the ground station and the satellite.

4.2 Visibility

When a satellite is passing over a ground station, it can be seen. But when it's on the other side of the earth it can not. When an observer can see a satellite (indicated by the visibility circle radius s), depends on the satellite altitude h and the cut-off elevation E. Figure 4.1 shows more details. From this picture can be derived that:

$$s = R\alpha = \frac{h}{\tan(\alpha + E)} \tag{4.2.1}$$

The cut-off elevation depends on buildings and trees around the observatory, but it can minimally be $E = 0°$. Then the line PS in figure 4.1 will be tangent to the earth's surface.

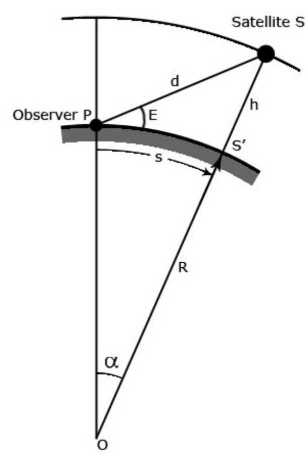

Figure 4.1: Definition of the variables concerning visibility circles.

4.3 Contact Time

If there's a ground station observing a satellite in a circular orbit, how long would it be able to see it if the satellite passes directly over it? This is an important aspect, since it's an indication of how long data can be transmitted/recieved. This time depends on the height h, the orbit time T and the cut-off angle E. For simplicity we will assume that the ground station's view doesn't get hindered, so $E = 0$. The angle α can now be calculated using:

$$\cos\alpha = \frac{R}{R+h} \quad \Rightarrow \quad \alpha = \arccos\frac{R}{R+h} \tag{4.3.1}$$

The part of the orbit which the observer can see spans an angle of 2α. So the satellite is visible for a part $(2\alpha)/(2\pi) = \alpha/\pi$. The total contact time now is:

$$T_{contact} = \frac{T}{\pi} \arccos\left(\frac{R}{R+h}\right) \tag{4.3.2}$$

4.4 Number of Ground Stations

To calculate how many ground stations are necessary to continuously keep track of a satellite, use is made of a simplification. Simply divide the area of the earth by the area of one visibility circle:

$$\text{Number of stations} = \frac{4\pi R^2}{\pi s^2} = \left(\frac{2R}{s}\right)^2 \tag{4.4.1}$$

5. Traveling to Planets

5.1 Minimum Energy Travel

There are many ways to travel to other planets. However, the method that uses minimum energy is the so-called **Hohmann transfer orbit**. This orbit is half of an ellipse, with one planet at its apocenter and the other at its pericenter. Various aspects of this Hohmann orbit can be calculated, but they are based on a few assumptions. It is assumed that the orbit trajectories of planets are circles, and it is also assumed that the orbital planes of all planets are in the **ecliptic** (the orbital plane of the earth). When calculating these aspects, use is made of the **patched conic approach**. The entire orbit is split up into pieces, which all have the shape of a conic section, and those pieces are then 'patched' together.

5.2 Example Travel to Mars

Suppose we want to travel to Mars. We start in a parking orbit around earth (at height $h_e = 300\,km$, and want to wind up in a parking orbit around Mars (at height $h_m = 200\,km$). What would our velocity be at the important parts of our journey? We know the trajectory for the satellite in the sun-centered part already, so it's easiest to start calculating there.

5.3 Example Travel - Sun-Centered Part

The circular velocity of the earth with respect to the sun is:

$$V_{c_e} = \sqrt{\frac{\mu_s}{r_e}} = \sqrt{\frac{132.7 \cdot 10^9}{149.6 \cdot 10^6}} = 29.784\,km/s \tag{5.3.1}$$

where r_e is the distance between Earth and the sun. The circular velocity of Mars around the sun is now easily calculated:

$$V_{c_m} = V_{c_e}\sqrt{\frac{r_e}{r_m}} = 29.784\sqrt{\frac{1AU}{1.524AU}} = 24.126 \tag{5.3.2}$$

where r_m is the distance between Mars and the sun. Also, since the satellite will be traveling in an ellipse from Earth to Mars, we can calculate the eccentricity of the ellipse:

$$e = \frac{r_m - r_e}{r_m + r_e} = \frac{1.524AU - 1AU}{1.524AU + 1AU} = 0.2076 \tag{5.3.3}$$

Since we'll travel to Mars (which is further away from the sun than the earth), the earth is the perihelion (pericenter, but then for the sun) of the elliptical orbit and Mars is the apohelion (note that this is opposite if we traveled to Venus and Mercury!). Using these three numbers, the velocities at perihelion and the apohelion (so the velocities at earth and at Mars) can be found:

$$V_p = V_{c_e}\sqrt{1+e} = 29.784\sqrt{1+0.2076} = 32.729\,km/s \tag{5.3.4}$$

$$V_a = V_{c_m}\sqrt{1-e} = 24.126\sqrt{1-0.2076} = 21.477\,km/s \tag{5.3.5}$$

These velocities come in very handy during the upcoming calculations. So now we will lift-off at earth, and see which velocities we will need.

5.4 Example Travel - Earth-Centered Part

We're in a parking orbit around earth at height $h_e = 300\,km$. So our velocity is the circular velocity around earth at $r = R_e + h_e$ (where $R_e = 6378\,km$ is the radius of the earth), which is:

$$V_{e_c} = \sqrt{\frac{\mu_e}{r}} = \sqrt{\frac{\mu_e}{R_e + h_e}} = \sqrt{\frac{398.6 \cdot 10^3}{6378 + 300}} = 7.726\,km/s \tag{5.4.1}$$

But we want to get away from the earth, so we need to escape. For this, the escape velocity of $\sqrt{2}V_{e_c}$ isn't sufficient. We want to escape, but also want to have sufficient velocity to travel to Mars. We just found out that we needed a velocity of $V_p = 32.729\,km/s$ to travel to Mars. But this velocity is with respect to the sun. We want to know the velocity with respect to the earth. So we simply subtract the velocity of the earth with respect to the sun. So the final velocity V_{n_e} (after escape), with respect to the earth, would be:

$$V_{\infty_e} = V_p - V_{c_e} = 32.729 - 29.784 = 2.945\,km/s \tag{5.4.2}$$

(Note that if we wanted to travel to Venus, the velocity at the earth would be the velocity at the apohelion, so we would have to use $V_{\infty_e} = V_{c_e} - V_a$.) To reach this velocity, we get a velocity increase. Suppose we have a velocity of V_{n_e} after the increase. This velocity can be calculated (using the energy equation):

$$\frac{V_{n_e}^2}{2} - \frac{\mu_e}{R_e + h_e} = \frac{V_{\infty_e}^2}{2} - \frac{\mu_e}{\infty} = \frac{V_{\infty_e}^2}{2} \tag{5.4.3}$$

$$V_{n_e} = \sqrt{V_{\infty_e}^2 + 2\frac{\mu_e}{R_e + h_e}} = \sqrt{V_{\infty_e}^2 + 2V_{e_c}^2} = \sqrt{2.945^2 + 2 \cdot 7.726^2} = 11.316\,km/s \tag{5.4.4}$$

So the necessary velocity increase at earth is:

$$\Delta V_e = V_{n_e} - V_{e_c} = 11.316 - 7.726 = 3.590\,km/s \tag{5.4.5}$$

5.5 Example Travel - Mars-Centered Part

And finally we arrive at Mars with a velocity $V_a = 21.477\,km/s$ with respect to the sun. This velocity with respect to Mars is:

$$V_{\infty_m} = V_{c_m} - V_a = 24.126 - 21.477 = 2.649\,km/s \tag{5.5.1}$$

We want to get in a parking orbit at $h_m = 200\,km$. So first we need to descend, but while doing that, our velocity increases. And if we do not do anything, then we will just leave Mars again. So we need a velocity decrease. To be able to stay in orbit at $h_m = 200\,km$, we need the following circular velocity:

$$V_{m_c} = \sqrt{\frac{\mu_m}{r}} = \sqrt{\frac{\mu_m}{R_m + h_m}} = \sqrt{\frac{43.01 \cdot 10^3}{3397 + 200}} = 3.458\,km/s \tag{5.5.2}$$

Suppose we have a velocity of V_{n_m} when we are $h_m = 200\,km$, before the velocity decrease. Then V_{n_m} can be calculated:

$$\frac{V_{n_m}^2}{2} - \frac{\mu_m}{R_m + h_m} = \frac{V_{\infty_m}^2}{2} - \frac{\mu_m}{\infty} = \frac{V_{\infty_m}^2}{2} \tag{5.5.3}$$

$$V_{n_m} = \sqrt{V_{\infty_m}^2 + 2\frac{\mu_m}{R_e + h_m}} = \sqrt{V_{\infty_m}^2 + 2V_{m_c}^2} = \sqrt{2.649^2 + 2 \cdot 3.458^2} = 5.562\,km/s \tag{5.5.4}$$

So the necessary velocity decrease will be:

$$\Delta V_m = V_{n_m} - V_{m_c} = 5.562 - 3.458 = 2.104\,km/s \tag{5.5.5}$$

5.6 Example Travel - Summary

We were in orbit around the earth at $h_e = 300\,km$ with a velocity of $V_{e_c} = 7.726\,km/s$. Then we got a velocity increase of $\Delta V_e = 3.590\,km/s$, and thus got a velocity of $V_{n_e} = 11.316\,km/s$. We ascended, and when we finally left earth, we only had a velocity left of $V_{\infty_e} = 2.945\,km/s$ with respect to the earth, which was a velocity of $V_p = 32.729\,km/s$ with respect to the sun. We traveled all the way to Mars, and when we finally got there, our remaining velocity was $V_a = 21.477\,km/s$ with respect to the sun, which was $V_{\infty_m} = 2.649\,km/s$ with respect to Mars. We descended, and picked up velocity, until at $h_m = 200\,km$ we had a velocity of $V_{n_m} = 5.562\,km/s$. Then we got a velocity decrease of $\Delta V_m = 2.104\,km/s$, such tat we remained in orbit around Mars with a velocity of $V_{m_c} = 3.458\,km/s$.

The total velocity change we have had during our trip was $\Delta V_t = \Delta V_e + \Delta V_m = 3.590 + 2.104 = 5.694\,km/s$, which is about the minimum possible for a travel from Earth to Mars.

5.7 Travel Time

When traveling from one planet to another, a certain amount of time is taken. This time depends on the orbit type. But for a Hohmann transfer orbit, the travel time can be rather easily calculated. If we travel from planet 1 to planet 2, then suppose those two planets have a distance r_1 and r_2, respectively, to the star they orbit. The ellipse with its pericenter at one of the planets and its apocenter at the other has a semi-major axis of:

$$a = \frac{r_1 + r_2}{2} \tag{5.7.1}$$

We only travel half an ellipse, so the travel time is only half of the orbit time, which is:

$$T_{travel} = \pi\sqrt{\frac{a^3}{\mu}} = \pi\sqrt{\frac{(r_1 + r_2)^3}{8\mu}} \tag{5.7.2}$$

And since the orbit times of the two planets, T_1 and T_2 respectively (which are for the whole orbits), can also be calculated using $T = 2\pi\sqrt{r^3/\mu}$, it can be determined that:

$$T_{travel} = \frac{1}{2}\sqrt{\frac{a^3}{r_1^3}}T_1 = \frac{1}{2}\sqrt{\frac{a^3}{r_2^3}}T_2 = \frac{\sqrt{2}}{8}\sqrt{\left(T_1^{2/3} + T_2^{2/3}\right)^3} \tag{5.7.3}$$

5.8 Synodic Period

Interplanetary missions can not be started at any time. It takes a time until the planet configuration is right. If you miss an opportunity to launch, you have to wait until the opportunity occurs again. This time between two consecutive opportunities (thus the time after which the planets have the same configuration again) is called the **synodic period**. With the same configuration is meant that the angle between the two planets and the sun is equal.

Let's define an arbitrary axis somewhere, and let's suppose we have two planets, planet 1 and planet 2, just like in figure 5.1. The two planets have angles ϕ_1 and ϕ_2, respectively, with the axis. Their relative angle is defined as $\phi_{rel} = \phi_2 - \phi_1$. Suppose the relative angle ϕ_{nes} is necessary to start a mission. Also suppose the last time the relative angle was ϕ_{nes} was on $t = t_0$. We have to wait until ϕ_{rel} once more becomes ϕ_{nes}.

To calculate the time taken, we first express ϕ_1 and ϕ_2 as a function of the time t:

$$\phi_1(t) = \phi_{1_0} + \omega_1(t - t_0) = \phi_{1_0} + \frac{2\pi}{T_1}(t - t_0) \quad \text{and} \quad \phi_2(t) = \phi_{2_0} + \omega_2(t - t_0) = \phi_{2_0} + \frac{2\pi}{T_2}(t - t_0) \tag{5.8.1}$$

And now we can express ϕ_{rel} as a function of t:

$$\phi_{rel}(t) = \phi_2(t) - \phi_1(t) = (\phi_{2_0} - \phi_{1_0}) + \left(\frac{2\pi}{T_2} - \frac{2\pi}{T_1}\right)(t - t_0) = \phi_{rel_0} + 2\pi\left(\frac{T_1 - T_2}{T_1 T_2}\right)(t - t_0) \quad (5.8.2)$$

We assumed that at $t = t_0$ also $\phi_{rel} = \phi_{nes}$ so $\phi_{rel_0} = \phi_{nes}$. To be able to launch again, we know that $\phi_{rel}(t)$ should be ϕ_{nes} plus or minus $k \cdot 2\pi$ (since if one planet has turned exactly k more rounds than the other, the configuration would be the same again, but ϕ_{rel} would have increased by $k \cdot 2\pi$). So we need to solve the following equation:

$$\phi_{rel}(t) = \phi_{nes} + 2\pi\left(\frac{T_1 - T_2}{T_1 T_2}\right)(t - t_0) = \phi_{nes} + k \cdot 2\pi \quad \Rightarrow \quad (t - t_0) = k\frac{T_1 T_2}{T_1 - T_2} \quad (5.8.3)$$

This equation gives, for different k, the occurrences of all launch possibilities (and for $k = 0$ the launch possibility we just missed). However, we only want to know when the next launch could take place. This is for $k = \pm 1$ (where the sign depends on whether $T_1 - T_2 > 0$). So the synodic period can be calculated using:

$$T_{syn} = (t - t_0) = \left|\frac{T_1 T_2}{T_1 - T_2}\right| \quad (5.8.4)$$

It is interesting to note that if $T_1 = T_2$, the synodic period is infinite.

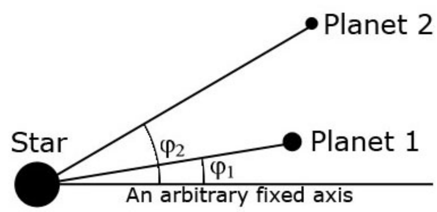

Figure 5.1: Clarification of the synodic period.

5.9 Lay-Over Time

Suppose we have traveled all the way from planet 1 to planet 2, and are in orbit there, but want to get back. How much time must there minimally be between arrival and departure? Or in other words, what is the **lay-over time**? For that, we once more need to look at angles between the two planets and some arbitrary axis. Let's call the lay-over time T_w. The angle which the satellite has at the start and the end of the mission are:

$$\phi_{sat_{begin}} = \phi_0 \quad \text{and} \quad \phi_{sat_{end}} = \phi_0 + \pi + \pi + T_w\omega_2 \quad (5.9.1)$$

where the first π is for the road to planet 2 and the second π is for the road back. Also note that $\omega_2 = \frac{2\pi}{T_2}$. In the meanwhile, the angle which the satellite has at the start and the end of the mission are:

$$\phi_{earth_{begin}} = \phi_0 \quad \text{and} \quad \phi_{earth_{end}} = \phi_0 + T_h\omega_1 + T_h\omega_1 + T_w\omega_1 \quad (5.9.2)$$

where T_h is the time needed for the trip. Also note that $\omega_1 = \frac{2\pi}{T_1}$. At the end of the mission, the condition $\phi_{earth} - \phi_{sat} = k \cdot 2\pi$ must be true, and so:

$$\phi_0 + 2T_h\frac{2\pi}{T_1} + T_w\frac{2\pi}{T_2} - \phi_0 - 2\pi - T_w\frac{2\pi}{T_2} = k \cdot 2\pi \quad (5.9.3)$$

Working this out gives:

$$k + 1 = 2\frac{T_h}{T_1} + \frac{T_w}{T_1} - \frac{T_w}{T_2} = 2\frac{T_h}{T_1} \pm \frac{T_w}{T_{syn}} \quad \Rightarrow \quad T_w = \pm T_{syn}\left(k + 1 - 2\frac{T_h}{T_1}\right) \quad (5.9.4)$$

The $\pm$ is present, because the equation for the synodic period was in absolute value. If $T_2 > T_1$ there should be a plus, and otherwise there should be a minus. To get the minimum lay-over time, we have to find the k for which T_w is minimal, but not smaller than 0. This is usually a matter of trial and error, based on the numbers given.

5.10 Swing-By's

Swing-by's are very handy for long trips, as they give a (free) velocity increase. This velocity increase seems to be inconsistent with conservation of energy, but this is not the case. The satellite approaches the target planet with a certain velocity (with respect to the planet), and it leaves the planet with the same velocity (still with respect to the planet). With respect to the sun, the velocity vector of the satellite has changed direction, and thus, seen from the sun, the velocity of the satellite has increased.

The change in angle of the velocity vector can be calculated, based on the (hyperbolic) orbit around the planet. The deflection angle δ can be calculated using the simple relation $\delta = 2\theta - 180°$, where θ is the angle such that $r \to \infty$. Thus $\cos\theta = -1/e$ and this make δ:

$$\delta = 2\arccos\left(-\frac{1}{e}\right) - 180° = 2\arcsin\left(\frac{1}{e}\right) \tag{5.10.1}$$

The change of velocity of the satellite now depends on the eccentricity, the velocity of the planet and the way of approaching the planet.

6. Special Earth Orbits

6.1 Orbital Properties

The orbital plane (dermined by **r** and **ṙ**), with the angular momentum vector **H** perpendicular to it, always runs through the center of gravity of the earth. The point where the satellite passes the equator in northward direction is called the **ascending node** (AN), while the southward crossing is the termdescending node. The line between the nodes, called the **line of nodes**, also goes through the earth's center of gravity.

The **inclination**, as was already discussed, is the angle between the orbital plane and the equatorial planes, but it's also the angle between the angular momentum **H** and the earth's rotation vector. If $0\circ \leq i < 90°$, then the rotation is **prograde**, meaning in the same direction as the rotation of the earth. If $90° < i \leq 180°$, the rotation is **retrograde** - against the rotation of the earth. The inclination for **geostationary orbits** is always $i = 0°$ and for **polar orbits** it always is $i = 90°$.

6.2 Orientation Changes

If a satellite, going in some direction, wants to change its direction by an angle of β, while not changing the magnitude of the velocity, the necessary velocity change is:

$$\Delta V = 2V \sin \frac{\beta}{2} \tag{6.2.1}$$

This is called an **out-of-plane manoeuvre**, and changes the orientation of the plane. For out-of-plane manoeuvres executed in the equatorial plane, only the inclination changes. For out-of-plane manoeuvres in any other point, both the inclination as the line of nodes changes.

When launching from a point on earth with latitude ϕ, the inclination of the orbit is limited to the range $|\phi| \leq i \leq 180° - |\phi|$. So if a launch site not on the equator wants to put a satellite in a geostationary orbit, an additional velocity change is needed when the satellite crosses the equatorial plane. At the end of its transfer orbit from **Low Earth Orbit** (LEO) to **Geostationary Orbit** (GEO), normally a velocity change of $\Delta V = V_{geo} - V_a$ is needed, where V_a is the velocity in the apogee of the transfer orbit and V_{geo} is the velocity for a geostationary orbit. This velocity changes sets the satellite in the right orbit. However, when also a change in angle, due to the inclination of the orbit, is necessary, the velocity change, due toe the cosine rule, is the following:

$$\Delta V = \sqrt{V_{geo}^2 + V_a^2 - 2V_{geo}V_a \cos i} \tag{6.2.2}$$

6.3 Orbit Perturbations

As was already discussed earlier, real orbits aren't perfect **unperturbed** (**Kepler**) **orbits**, but are influenced by **perturbing** forces. These forces don't have to be bad, since they can be used. The largest perturbation is the so-called J_2**-effect**, caused by the flattening of the earth. The **geometric flattening** is $f = (R_e - R_p)/R_e = 1/298.257$, where R_e is the equatorial radius and R_p the polar radius.

The geometric flattening causes the earth's gravitational field to be flattened to. This is expressed by the parameter $J_2 = 1.082627 \cdot 10^{-3}$. J_2 causes periodic orbit perturbations which cancel out over an entire orbit. However, secular effects cause a rotation of the orbital plane around the polar axis, and a slow precessing of the orientation of the ellipse in the orbital plane. The secular effects thus influence ω and Ω (remember the Keplerian elements), but do not influence i, a and e.

6.4 Nodal Regression

The extra mass of the equator causes a torque which changes the angular momentum vector. This causes the line of nodes to rotate against the rotation of the satellite in its orbit. This is called **Nodal Regression**. After one revolution, the line of nodes has rotated by:

$$\Delta\Omega = -3\pi J_2 \left(\frac{R_e}{p}\right)^2 \cos i \tag{6.4.1}$$

where p is still the Semi-latus Rectum of the orbit (for circles it is just the radius r). Note that this is maximum when $i = 0°$ and minimum when $i = 90°$. A special application of nodal regression is the so-called sun-synchronous orbit. Now the orbit is chosen such that $\dot{\Omega} = 360°$ per year. The sun always makes the same angle with the orbit. This is used because the satellite can continuously see the sun, the satellite overflies a certain latitude always at the same local solar time (illumination conditions are the same) and instruments looking outward can cover the entire celestial sphere in half a year, without disruption by the sun.

When calculating the inclination of a circular, sun-synchronous orbit, you first have to calculate $\dot{\Omega} = \frac{\Delta\Omega}{\Delta T}$. We know $\Delta\Omega$ after on revolution. The change in time after one revolution is $\Delta T = 2\pi\sqrt{r^3/\mu}$. Now $\dot{\Omega}$ is:

$$\dot{\Omega} = \frac{\Delta\Omega}{\Delta T} = \left(-3\pi J_2 \left(\frac{R_e}{p}\right)^2 \cos i\right)\left(\frac{1}{2\pi}\sqrt{\frac{\mu}{r^3}}\right) \tag{6.4.2}$$

For a sun-synchronous orbit, $\dot{\Omega} = 360°/\text{year} = (2\pi)/(365.25 \cdot 24 \cdot 60 \cdot 60) = 1.99102 \cdot 10^{-7}$ rad/s. From this condition the necessary inclination i can be found.

6.5 Precession of Perigee

Also the perigee 'suffers' from precession. The **precession of the perigee** per orbital revolution is expressed by:

$$\Delta\omega = \frac{3}{2}\pi J_2 \left(\frac{R_e}{p}\right)^2 (5\cos^2 i - 1) \tag{6.5.1}$$

When $\cos i = 1/\sqrt{5}$, the orientation of the ellipse in the orbital plane is fixed. This is called the **critical inclination** and is for earth: $i = 63.43°$ and $i = (180 - 63.43) = 116.57°$. This principle is used by the Russian Molnya satellite orbits. To have long contact time, the Molnya satellite orbits are eccentric orbits with its apogee (where the velocity is low) above Russia. To prevent the position of the apogee (and thus also the position of the perigee) from changing, these orbits have the critical inclination.