

# Thermodynamics

## 1 An Ideal Gas

For a perfect gas, the **gas law** holds:

$$p = \rho RT \quad \Leftrightarrow \quad pv = RT \quad (1)$$

where  $p$  [Pa] is the **pressure** of the air,  $\rho$  [kg/m<sup>3</sup>] is the **density**,  $R$  [J/(kg K)] is the **specific gas constant**,  $T$  [K] is the **temperature** and  $v$  [m<sup>3</sup>/kg] is the **specific volume**. Per definition  $\rho v = 1$ . For an ideal gas, the **internal energy** only depends on its temperature:  $u = c_v T$ , where  $u$  [J/kg] is the **specific internal energy** and  $c_v$  is a constant (depending on the type of gas) about which more will be explained in paragraph 3.

## 2 First Law of Thermodynamics

Let us consider 1 kg of gas. The change in energy, represented by the **heat**  $\delta q$  [J/kg] added, is equal to the increase in internal energy  $du$  [J/kg] plus the **external work** done  $\delta w$  [J/kg]:

$$\delta q = du + \delta w \quad (2)$$

This is the **first law of thermodynamics**. The work can be expressed as  $\delta w = p dv$  such that the first law of thermodynamics becomes  $\delta q = du + p dv$ .

## 3 Specific Heat

If the 1 kg of gas is heated, and if the volume is kept constant (such that the pressure increases), then the **specific heat at constant volume**,  $c_v$  [J/(kg K)], is defined such that:

$$\delta q = c_v dT \quad (3)$$

where  $dT$  [K] is the rise in temperature. In the case of constant volume, no work can be done, so  $\delta q = du$  and thus also  $du = c_v dT$ . Integrating gives the equation for the internal energy:

$$u = c_v T \quad (4)$$

However, if the 1 kg is heated such that the pressure stays constant (such that the volume increases), then the **specific heat at constant pressure**,  $c_p$  [J/(kg K)], is defined such that:

$$\delta q = c_p dT \quad (5)$$

Using the gas law, it can be found that  $\delta q = c_v dT + R dT$ . So we get the important relation:

$$R = c_p - c_v \quad (6)$$

## 4 Enthalpy

Let's not consider a system at which gas is locked and at rest, but let's consider a system at which gas enters with a velocity  $V_1$  [m/s] and exits (with different pressure, temperature, etcetera) with a velocity  $V_2$  [m/s]. The kinetic energy of the gas (when it enters) is  $\frac{1}{2}V_1^2$  (since we consider only 1 kg of the gas). The work done can be shown to be  $w_1 = p_1 v_1$  (note the difference between the specific volume  $v$  and the velocity  $V$ ). If we also keep the internal energy in mind, we can show that the total energy in state 1 is:

$$e_1 = u_1 + p_1 v_1 + \frac{1}{2}V_1^2 \quad (7)$$

Since the combination  $u + pv$  often occurs, it is given a name. The **enthalpy**  $h$  [ $J/kg$ ] is defined as:

$$h = u + pv \quad (8)$$

In case of a perfect gas, the enthalpy can also be written as  $h = c_p T$  (since  $pv = RT$  and  $c_v + R = c_p$ ). So for a perfect gas, the enthalpy is only a function of the temperature. So the energy entering the system can be found by measuring the temperature and the velocity. To simplify this, the **total enthalpy** is introduced and defined as:

$$h_t = h + \frac{1}{2}V^2 = u + pv + \frac{1}{2}V^2 \quad (9)$$

In the same way the **total temperature** is defined:

$$T_t = T + \frac{V^2}{2c_p} \Rightarrow h_t = c_p T_t \quad (10)$$

This makes the **balance of energy** the following:

$$q_{in} + w_{in} = h_{t_2} - h_{t_1} = c_p(T_{t_2} - T_{t_1}) = c_p(T_2 - T_1) + \frac{1}{2}(V_2^2 - V_1^2) \quad (11)$$

where  $q_{in}$  is the added heat and  $w_{in}$  is the added work. The balance of energy is always valid, whether there is friction or not.

## 5 Entropy

The **entropy**  $s$  [ $J/(kg K)$ ] can be calculated using the following equations:

$$s = c_v \ln\left(\frac{T}{T_{ref}}\right) + R \ln\left(\frac{v}{v_{ref}}\right) = c_p \ln\left(\frac{T}{T_{ref}}\right) - R \ln\left(\frac{p}{p_{ref}}\right) = c_v \ln\left(\frac{p}{p_{ref}}\right) - c_p \ln\left(\frac{\rho}{\rho_{ref}}\right) \quad (12)$$

where the first part is the definition of the entropy. All parts in equation 12 are equivalent due to the gas law. Using the entropy, the first law of thermodynamics can be written as:

$$\delta q = T ds \quad (13)$$

For diabatic processes (heat is supplied from the environment) or for processes with friction (heat is supplied by friction)  $\delta q > 0$  and so  $ds > 0$ . Only for an **adiabatic** and **frictionless** process (no heat is supplied to the gas)  $ds = 0$ . Such a process is called an **isentropic** process (or sometimes an adiabatic reversible process). It means entropy is reserved, which gets expressed in the following important equation:

$$s_2 - s_1 = 0 \quad (14)$$

## 6 Mollier Diagram

The enthalpy and the entropy are state variables. If  $(h, s)$  is known, the entire state of the gas is known. Using the definitions of  $h$  and  $s$  all variables concerning the gas can be determined. It is also possible to make a diagram with on the horizontal axis the entropy  $s$  and on the vertical axis the enthalpy  $h$ . Such a diagram is called a **Mollier diagram**. The state of a gas can be represented by a point in a Mollier diagram.

The state of a gas usually changes along certain lines in a Mollier diagram. For an isentropic process, the line in a Mollier Diagram is vertical (as the entropy - the horizontal axis - stays constant). Sometimes lines are drawn in Mollier diagrams for a gas at a constant pressure. Such a line is called a **isobar**. In a jet engine certain processes occur at constant pressure. The state of a gas then changes along one of the isobars of the Mollier diagram. When calculating gas states for piston engines, it is often handy to use **isochors** - lines at which the gas has constant density.

## 7 Equation of Poisson

Let's define  $\gamma$  as  $\gamma = c_p/c_v$ . This variable can come in handy in many equations. For isentropic processes ( $s_2 - s_1 = 0$ ), the equations of Poisson can be derived. These equations state that the following quantities remain constant during isentropic processes:

$$\frac{p}{\rho^\gamma} = \text{constant} \quad \frac{T}{\rho^{\gamma-1}} = \text{constant} \quad \frac{T}{p^{\left(\frac{\gamma}{\gamma-1}\right)}} = \text{constant} \quad (15)$$

## 8 Speed of Sound

The crossing of a sound wave may assumed to be isentropic. From that assumption can be derived that the **speed of sound**  $a$  [m/s] is:

$$a = \sqrt{\gamma RT} = \sqrt{\gamma p v} = \sqrt{\gamma \frac{p}{\rho}} \quad (16)$$

The **Mach number**  $M$  is now defined as:

$$M = \frac{V}{a} \quad (17)$$

Using this Mach number, it can be derived that for an isentropic process the total pressure can be calculated using:

$$p_t = p \left( 1 + \frac{\gamma - 1}{2} M^2 \right)^{\frac{\gamma}{\gamma-1}} \quad (18)$$