Flight Mechanics

1 Two-Dimensional Airfoils

The **Reynolds number** is defined as:

$$Re = \frac{\rho V c}{\mu} \tag{1}$$

where c $[m]$ is the chord length and μ $[Pa\,s]$ is the viscosity of the air. The **lift** L $[N]$ and **drag** D $[N]$ can be calculated using:

$$L = c_l \frac{1}{2}\rho V^2 S \qquad\qquad D = c_d \frac{1}{2}\rho V^2 S \tag{2}$$

where the **lift coefficient** c_l and the **drag coefficient** c_d depend on the **angle of attack** α [rad] (being the angle between the longitudinal axis of the aircraft and the direction of flight), the shape of the wing, the Mach number and the Reynolds number. Also S $[m^2]$ is the wing surface. It is possible to plot the lift and drag coefficients with respect to the angle of attack. However, it is also possible to plot the lift coefficient with respect to the drag coefficient. The diagram that results is called a **lift-drag polar**.

2 Three-Dimensional Airfoils

In reality wings aren't two-dimensional but three-dimensional. And in three dimensions also wing vortices occur, causing induced drag. The drag coefficient now consists of two parts. The part being present when there is zero lift C_{D_0}, which is thus called the **zero lift drag coefficient**, and the part belonging to the induced drag C_{D_i}. (Note that since we're talking about three-dimensional airfoils, we use capital letters to denote the coefficients.) The induced drag coefficient is:

$$C_{D_i} = \frac{C_L^2}{\pi A e} \tag{3}$$

where A is the aspect ratio of the wing, defined as b^2/S (with $b\,[m]$ the wing span), and e is Oswald's factor, depending on the lift distribution of the wing. The drag coefficient now is:

$$C_D = C_{D_0} + C_{D_i} = C_{D_0} + \frac{C_L^2}{\pi A e} \tag{4}$$

3 Flight Types

There are multiple ways of flying. Some of them have gotten a specific name. These are their definitions:

- **Gliding Flight** - Flight in which the thrust is 0: $T = 0$.
- **Steady Flight** - Flight in which the forces and moments do not vary in time, neither in magnitude nor direction.
- **Straight Flight** - Flight in which the center of gravity of the aircraft travels along a straight line.
- **Symmetric Flight** - Flight in which both the angle of sideslip (angle between the direction of motion and the longitudinal axis of the airplane) is zero, and the plane of symmetry of the airplane is perpendicular to the normal plane of the earth.

Let's look at a **steady horizontal flight**. The lift is equal to the weight: $L = W$. It follows that:

$$V = \sqrt{\frac{W}{S}\frac{2}{\rho}\frac{1}{C_L}} \tag{5}$$

with V_{min} as $C_{L_{max}}$. The factor W/S is called the **wing loading**. So the minimum velocity is low when there is a low wing loading, or when the lift coefficient is high. The latter is often achieved by using slats.

4 Straight Symmetric Flight

Now let's look at a **straight symmetric flight** in which the aircraft is climbing. The angle between the direction of flight and the ground plane is the **flight path angle** γ [rad]. The angle between the longitudinal axis of the aircraft and the direction of flight is the angle of attack. The angle between the longitudinal axis of the aircraft and the ground plane is the **pitch angle** θ [rad]. Note that $\theta = \gamma + \alpha$. Figure 1 visualizes the definitions of these angles.

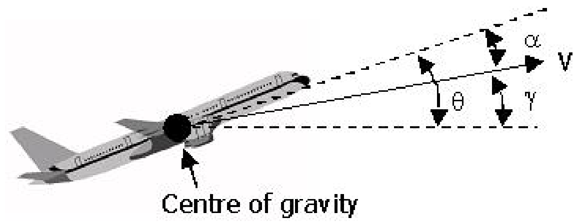

Figure 1: Visualization of angles in symmetric flight.

By drawing a free body diagram of the aircraft, the sum of the forces in multiple directions can be calculated. If we assume that the thrust vector is in the direction of flight, we get:

$$T - D - W \sin\gamma = \frac{W}{g}\frac{dV}{dt} \qquad\qquad L - W\cos\gamma = 0 \tag{6}$$

Now let's define the power required P_r $[J/s]$ as follows:

$$P_r = DV = C_D \frac{1}{2}\rho V^3 S = \frac{C_D}{C_L} LV = \frac{C_D}{C_L} WV \tag{7}$$

In the last part of this equation the assumption $\cos\gamma = 1$ was used, which is accurate for normal climb angles. Multiplying the first part of equation 6 by V, and by using the relation $V(dV/dt) = (1/2)\,(d(V^2)/dt)$, it can be rewritten as:

$$\frac{1}{2}\frac{W}{g}\frac{dV^2}{dt} = TV - DV - WV\sin\gamma = P_a - P_r - P_c \tag{8}$$

where P_c $[J/s]$ is the **climb power**. (Note that $P_c = WV\sin\gamma = W\,dh/dt$, with h $[m]$ as the height.) So the quantity $P_a - P_r$ can be seen as the power left to climb and accelerate - to increase the potential/kinetic energy of the aircraft.

5 Steady Gliding

When the engines of an airplane aren't active (or if the airplane doesn't have any engines), the airplane is gliding. So $T = 0$ and $P_a = 0$ and thus:

$$-P_r = W\frac{dh}{dt} + \frac{W}{g}\frac{dV^2}{dt} \tag{9}$$

If the airplane still follows a horizontal path ($dh/dt = 0$), the velocity decreases. In most situations this isn't favorable, so pilots usually try to keep a constant velocity, at the cost of height. Thus the aircraft descends. The **descend angle** $\bar{\gamma}$ [rad] is defined as $\bar{\gamma} = -\gamma$. Now the following applies:

$$C_D\frac{1}{2}\rho V^2 S = W\sin\bar{\gamma} \quad \text{and} \quad C_L\frac{1}{2}\rho V^2 S = W\cos\bar{\gamma} \quad \Rightarrow \quad \tan\bar{\gamma} = \frac{C_D}{C_L} \tag{10}$$

Suppose we want to travel as much distance as possible. To accomplish this, we should minimize $\sin\bar{\gamma}$. It can be shown that $\sin\bar{\gamma} = D/W$, so $\sin\bar{\gamma} \downarrow$ as $D \downarrow$. So we want to chose our V and C_L such that the drag D is minimal. The drag can be calculated with equation 2. However, this equation still has the velocity V in it, which is a function of the lift coefficient C_L . To solve this problem, we use equation 5. From this follows that the drag is $D = \frac{C_D}{C_L} W$. Since the weight W is constant, the drag is minimal if C_D/C_L is minimal. This is the case if:

$$\frac{d\left(\frac{C_D}{C_L}\right)}{dC_L} = 0 \quad \Rightarrow \quad \frac{C_L\left(\frac{dC_D}{dC_L}\right) - C_D}{C_L^2} = 0 \quad \Rightarrow \quad \frac{dC_D}{dC_L} = \frac{C_D}{C_L} \tag{11}$$

This can be solved using equation 4:

$$\frac{C_D}{C_L} = \frac{dC_D}{dC_L} = \frac{d\left(C_{D_0} + \frac{C_L^2}{\pi Ae}\right)}{dC_L} = 0 + \frac{2C_L}{\pi Ae} \quad \Rightarrow \quad \frac{C_{D_0}}{C_L} + \frac{C_L}{\pi Ae} = 2\frac{C_L}{\pi Ae} \quad \Rightarrow \quad C_L = \sqrt{C_{D_0}\pi Ae} \tag{12}$$

Using equation 5 the corresponding velocity can be found. So if we know our zero lift drag coefficient, we know how to fly to get as far as possible.

But now suppose we do not want to go as far as possible, but just want to stay in the air as long as possible. For that, we first introduce the **rate of descent** RD $[m/s]$, which is defined as:

$$RD = -\frac{dh}{dt} = -V\sin\gamma = V\sin\bar{\gamma} \tag{13}$$

The aircraft stays as long as possible in the air if RD is minimal. We know that:

$$RD = -\frac{dh}{dt} = V\sin\bar{\gamma} = V\frac{C_D}{C_L}\cos\bar{\gamma} = \sqrt{\frac{W}{S}\frac{2}{\rho}\frac{C_D^2}{C_L^3}\cos^3\bar{\gamma}} \tag{14}$$

Since W, S and ρ are constants, the rate of descend is minimal if C_D^2/C_L^3 is minimal. Using a method analogue to what we just did, we find that:

$$0 = \frac{d\left(\frac{C_D^2}{C_L^3}\right)}{dC_L} = \frac{C_L^3 \cdot 2C_D \cdot 2\frac{C_L}{\pi Ae} - 3C_L^2 \cdot C_D^2}{C_L^6} \quad \Rightarrow \quad 4\frac{C_L^2}{\pi Ae} = 3C_D \quad \Rightarrow \quad C_L = \sqrt{3C_{D_0}\pi Ae} \tag{15}$$

The time until we reach the ground (the **endurance**) t $[s]$ and the traveled distance (the **range**) s $[m]$ can then be calculated using:

$$t = \frac{h}{RD} \qquad\qquad s = \frac{h}{\tan\bar{\gamma}} \tag{16}$$

6 Propeller Aircraft Range and Endurance

The jet engine and the propeller are very different. So their thrust develops differently. In a propeller aircraft the thrust T decreases as the velocity increases. This happens in such a way that the power available P_a is (approximately) constant. In a jet aircraft the thrust T simply stays constant for any velocity.

Suppose we have a propeller aircraft that wants to fly as far as possible with the fuel it has. Let's assume that the velocity and height stay constant, and thus $P_a = P_r$. The distance per amount of fuel should be maximized. If W_f is the weight of the fuel that is left, the following quantity should be maximized:

$$\frac{s}{W_f} = \frac{ds}{dW_f} = \frac{ds/dt}{dW_f/dt} = \frac{V}{F} \tag{17}$$

So the quantity V/F should be maximized. If we define the **power coefficient** C_P $[N/J]$ (which can assumed to be constant for propeller aircrafts) such that $F = C_P P_{br}$, we can find that:

$$\frac{V}{F} = \frac{V}{C_P P_{br}} = \frac{V\eta_j}{C_P P_a} = \frac{V\eta_j}{C_P P_r} = \frac{V\eta_j}{C_P D V} = \frac{\eta_j}{C_P}\frac{1}{D} \tag{18}$$

Since η_j and C_P are constants for the aircraft, the quantity V/F is at a maximum if the drag is at a minimum. In the last paragraph we already found out when this was the case. So the aircraft has a maximum range if $C_L = \sqrt{C_{D_0}\pi A e}$.

But what if we want to stay in the air as long as possible with the amount of fuel we have? Then we ought to minimize the fuel flow F. We can find that:

$$F = VD\frac{C_P}{\eta_j} = \frac{C_P}{\eta_j}W\sqrt{\frac{W}{S}\frac{2}{\rho}\frac{C_D^2}{C_L^3}} \tag{19}$$

Since C_P, η_j, W, S and ρ are all constants, this is minimal if C_D^2/C_L^3 is minimal. So the aircraft has maximum endurance if $C_L = \sqrt{3C_{D_0}\pi A e}$.

7 Jet Aircraft Range and Endurance

Suppose we have a jet aircraft that wants to fly as far as possible with the fuel it has. Once more we assume $P_a = P_r$ and thus $T = D$. We should once more maximize V/F. If we define the **thrust coefficient** C_T (which can assumed to be constant for jet aircrafts) such that $F = C_T T$, we can find that:

$$\frac{V}{F} = \frac{V}{C_T T} = \frac{V}{C_T D} = \frac{1}{C_T W}\sqrt{\frac{W}{S}\frac{2}{\rho}\frac{C_L}{C_D^2}} \tag{20}$$

So the aircraft has maximum range if C_L/C_D^2 is minimal. It can be derived that this is the case if $C_D = 4C_{D_i}$ and thus $C_L = \sqrt{\frac{1}{3}C_{D_0}\pi A e}$.

If the jet aircraft wants to stay in the air as long as possible, the quantity F should be minimized. This is the case if C_D/C_L is minimal. Now it's easy to see that the aircraft has maximum endurance if $C_L = \sqrt{C_{D_0}\pi A e}$.