Coordinate Systems

1 System Definitions

In the previous chapters we have derived equations of motion in a rather simple way: By drawing a picture and examining forces. This gets increasingly complicated if the flight is more three-dimensional. That's why it is useful to define various coordinate systems.

We start with the **geodetic system**, which has subscript g. The x_g-axis is defined as pointing north, while the z_g-axis points downward, to the center of the earth. The y_g-axis can now be found by using the right-hand rule. The origin of the coordinate system is at any point on the surface of the earth.

Now let's shift the origin of the geodetic system to the center of gravity (COG) of the aircraft. Now we have found the **moving earth system**, having subscript e.

Now we the road splits up in two paths. We could rotate the coordinate system to the **body system** of the aircraft (subscript b). The x_b-axis then coincides with the longitudinal axis of the aircraft (forward direction being positive). The y_b axis coincides with the latitudinal axis of the aircraft. Its positive direction is such that the z_b-axis (found with the right-hand rule) points downward.

The body system isn't very useful in performance. That's why we'll mostly use the **air path system** (subscript a). Now we rotate the coordinate system such that the x_a-axis points in the direction of the velocity. Just like in the body system, the z_a-axis points downward.

2 Equations of Motion for Symmetric Flight

To demonstrate the use of these coordinate systems, we will derive the equations of motion for a symmetric flight. Let's call E the matrix $\begin{bmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\end{bmatrix}^T$. The position of any point in the air path system can be expressed as

$$\mathbf{r} = \begin{bmatrix}x & y & z\end{bmatrix} E_a, \tag{2.1}$$

where $E_a = \begin{bmatrix}\mathbf{i_a} & \mathbf{j_a} & \mathbf{k_a}\end{bmatrix}^T$. The velocity can then be expressed as

$$\mathbf{V} = \dot{\mathbf{r}} = \begin{bmatrix}V & 0 & 0\end{bmatrix} E_a. \tag{2.2}$$

Now we see why it is convenient to use the air path system. Since the x_a-axis points per definition in the direction of the velocity vector, we have an easy relation here. We can once more take a time derivative to find the acceleration. This is

$$\mathbf{a} = \dot{\mathbf{V}} = \begin{bmatrix}\dot{V} & 0 & 0\end{bmatrix} E_a + \begin{bmatrix}V & 0 & 0\end{bmatrix} \dot{E}_a. \tag{2.3}$$

Note that we had to use the product rule. But what is $\dot{E}_a$? We know that the components of E_a can not change in magnitude, as they are unit vectors. They can only change in direction, when something rotates. We have assumed we're in a symmetric flight, so the only rotation is when γ changes. Expressing $\dot{E}_a$ in $\dot{\gamma}$ gives

$$E_a = \begin{bmatrix}-\mathbf{k}\dot{\gamma} & 0 & \mathbf{i}\dot{\gamma}\end{bmatrix}. \tag{2.4}$$

The **rotation operator** R is defined such that

$$\dot{E}_a = \begin{bmatrix}0 & 0 & -\dot{\gamma} \\ 0 & 0 & 0 \\ \dot{\gamma} & 0 & 0\end{bmatrix} E_a = R\,E_a. \tag{2.5}$$

If we insert this in the relation for **a**, we will find

$$\mathbf{a} = \begin{bmatrix}\dot{V} & 0 & 0\end{bmatrix} E_a + \begin{bmatrix}V & 0 & 0\end{bmatrix} R E_a = \begin{bmatrix}\dot{V} & 0 & 0\end{bmatrix} E_a + \begin{bmatrix}0 & 0 & -V\dot{\gamma}\end{bmatrix} E_a = \begin{bmatrix}\dot{V} & 0 & -V\dot{\gamma}\end{bmatrix} E_a. \quad (2.6)$$

We can also express the forces in the air path system. We once more assume the thrust is in the direction of the velocity. Working everything out gives

$$\mathbf{F} = \begin{bmatrix}T - D - W\sin\gamma & 0 & L - W\cos\gamma\end{bmatrix} E_a. \quad (2.7)$$

We can now use $\mathbf{F} = m\mathbf{a}$. This relation must hold for all three components of these vectors. Looking at the y-component will give $0 = 0$, which isn't altogether interesting. Looking at the x and z-components however, will give two interesting relations, being

$$m\dot{V} = T - D - W\sin\gamma, \qquad \text{and} \qquad mV\dot{\gamma} = L - W\cos\gamma. \quad (2.8)$$

These are exactly the equations of motion we already know. So this method actually works. It may seem like a lot of work now, but for difficult situations this approach is actually a lot easier than what we did in the previous chapters.

3 Switching Between Systems

Suppose we have certain coordinates in the moving earth system and want to transform them to the air path system. How do we do this? The key to answering this question, is the **transformation matrix**.

We go from the moving earth system to the air path system in several steps. First we rotate the moving earth system about its z-axis, such that the x-axis points in the direction of the velocity vector as much as possible. (The y-component of $\mathbf{V}$ should then be 0.) We now have rotated over the so-called **azimuth angle** χ. We call this intermediate system E_1. We can now show that

$$\begin{bmatrix}x & y & z\end{bmatrix} E_1 = \begin{bmatrix}x\cos\chi - y\sin\chi & x\sin\chi + y\cos\chi & 1\end{bmatrix} E_e = \begin{bmatrix}x & y & z\end{bmatrix} \begin{bmatrix}\cos\chi & \sin\chi & 0 \\ -\sin\chi & \cos\chi & 0 \\ 0 & 0 & 1\end{bmatrix} E_e. \quad (3.1)$$

The matrix on the right-hand side (the one with the sines and cosines) is the transformation matrix, denoted by $[\chi]$. It transforms E_e to E_1. Note that we could have removed the matrix with the x, y and z in the above equations. The equation would then simply be $E_1 = [\chi]\, E_e$.

But we haven't reached the air path system yet. To continue, we rotate system 1 around the y-axis over the **flight path angle** γ, such that the x-axis actually coincides with the velocity vector $\mathbf{V}$. We give this intermediate system number 2. We now have

$$E_2 = [\gamma]\, E_1 = [\gamma]\, [\chi]\, E_e, \quad (3.2)$$

where the transformation matrix $[\gamma]$ is derived similar to $[\chi]$. We're almost at the air path system. There's only one remaining thing that we have neglected in all previous chapters. Previously we have only considered symmetric flights. But now we consider any type of flight. So we still need to rotate around the x-axis by the **aerodynamic roll angle** μ. The result will be

$$E_a = [\mu]\, E_2 = [\mu]\, [\gamma]\, [\chi]\, E_e. \quad (3.3)$$

We have now expressed the air path system as a function of the moving earth system. It is very important to note the order of the transformation matrices. Mixing up the order of the matrices will give wrong results.

But what if we want to reverse the transformation - go from the air path system to the moving earth system? That is relatively simple. All the transformation matrices are orthonormal matrices (meaning that their columns are linearly independent and all have unit length). Any orthonormal matrix A has the property $A^{-1} = A^T$. Using this fact, we can write

$$E_e = ([\mu]\,[\gamma]\,[\chi])^{-1}\,E_a = [\chi]^{-1}\,[\gamma]^{-1}\,[\mu]^{-1}\,E_a = [\chi]^T\,[\gamma]^T\,[\mu]^T\,E_a. \tag{3.4}$$

Note that now also the order of the matrices has reversed.

4 General Equations of Aircraft Motion

We have derived the equations of motion for symmetric flights previously in this chapter. We can now derive the general equations of motion for any type of flight. First we need to consider the acceleration of the aircraft. This is still given by equation 2.3. However, the value of $\dot{E}_a$ has been drastically changed. By examining the aircraft motion, we can find that

$$\dot{E}_a = \begin{bmatrix} 0 & r & -q \\ -r & 0 & p \\ q & -p & 0 \end{bmatrix} E_a = R\,E_a, \tag{4.1}$$

where p is the rotational velocity about the x-axis, q is the rotational velocity about the y-axis and r is the rotational velocity about the z-axis.

Now we know the acceleration of the aircraft, we need to look at the forces acting on the aircraft. These are

$$\mathbf{F} = \mathbf{T} + \mathbf{D} + \mathbf{L} + \mathbf{W} = \begin{bmatrix} T-D & 0 & -L \end{bmatrix} E_a + \begin{bmatrix} 0 & 0 & W \end{bmatrix} E_e. \tag{4.2}$$

Now we see why we need the transformation matrices. To be able to work with these equations, we need to express them in the same coordinate system. It's wisest to use E_a as the coordinate system. Luckily we know how to convert something from E_e to E_a. If we work everything out and use $\mathbf{F} = m\mathbf{a}$ we will get

$$\begin{bmatrix} T-D & 0 & -L \end{bmatrix} E_a + \begin{bmatrix} 0 & 0 & W \end{bmatrix} [\chi]^T\,[\gamma]^T\,[\mu]^T\,E_a = \begin{bmatrix} m\dot{V} & 0 & 0 \end{bmatrix} E_a + \begin{bmatrix} mV & 0 & 0 \end{bmatrix} R\,E_a. \tag{4.3}$$

Now we can derive the general equations of motion. By examining all three components, we find three equation. These are

$$T - D - W\sin\gamma = m\dot{V}, \tag{4.4}$$

$$W\cos\gamma\sin\mu = mVr, \tag{4.5}$$

$$L - W\cos\gamma\cos\mu = mVq. \tag{4.6}$$

We can use transformation matrices to rewrite these equations to a more useful form, being

$$m\dot{V} = T - D - W\sin\gamma, \tag{4.7}$$

$$mV\dot{\chi}\cos\gamma = L\sin\mu, \tag{4.8}$$

$$mV\dot{\gamma} = L\cos\mu - W\cos\gamma. \tag{4.9}$$

These equations are useful when describing many flight types. However, they do have their constraints. They don't take into account forces/moments caused by the elevators, the ailerons or the rudder. So keep that in mind when applying them.