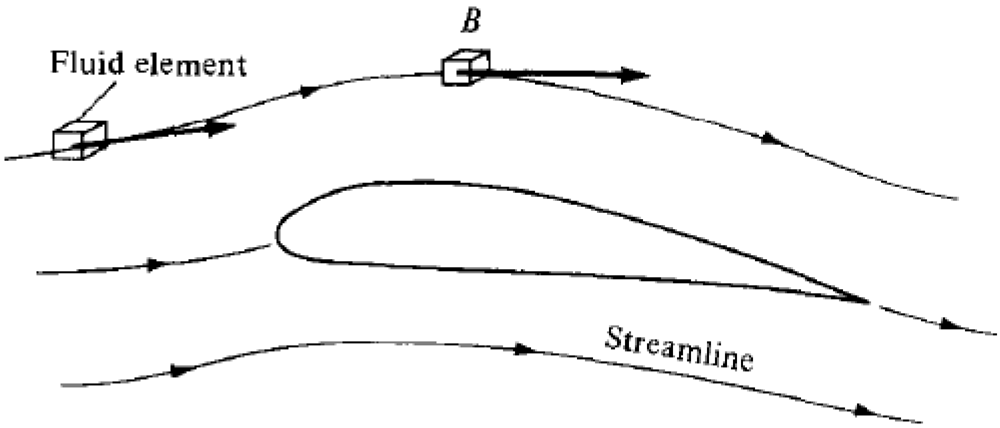
Atmospheric Models

1.0 Aerodynamic Variables

Four of the most frequently used words in aerodynamics:

- Pressure (p)
- Density (ρ)
- Temperature (T)
- Flow Velocity (**V**)

1.1 Pressure



The pressure is defined at a point in the fluid or a point on a solid surface. The pressure can vary from one point to another.

Definition:
"Pressure is the normal force per unit area exerted on a surface due to the time rate of change of momentum of the gas molecules impacting on (or crossing) that surface" (J. Anderson, Fundamentals of Aerodynamics, 2001, pg. 13)

$$(1) \Rightarrow \boxed{p = \lim_{dA \to 0} \left(\frac{dF}{dA}\right)} \; N/m^2$$

dA = Elemental Area at B (m^2)
dF = Force on 1 side of dA due to pressure (N)

1.2 Density

The density of a material is a measure of the amount of material contained in a given volume.

$$(2) \Rightarrow \boxed{\rho = \lim_{dv \to 0} \left(\frac{dm}{dv}\right)} \; kg/m^3$$

dv = Elemental volume around B (m^3)
dm = Mass of the fluid inside dv (kg)

<<<<< Note: 1.3 & 1.4 (up to viscosity) were missing from original handout! >>>>>

1.3 Temperature

"The temperature *T* of a gas is directly proportional to the average kinetic energy of the molecules of the fluid. In fact, if KE is the mean molecular kinetic energy, then temperature is given by":

$$KE = \frac{3}{2}kT$$

Where *k* is the Boltzmann constant (k = 1.38 x 10^{-23} J/°K)

Temperature is "a point property, which can vary from point to point in the gas" & has "an important role in high-speed aerodynamics"

Source: J. Anderson, Fundamentals of Aerodynamics, 2001, pg. 14

1.4 Flow Velocity

A velocity is a vector value & as such must contain both scalar value & direction. In a flowing fluid at each region (or point) in the fluid there is not necessarily the same velocity. Hence this is also a point property which can vary from point to point in the flow.

"The velocity of a flowing gas at any fixed point B in space is the velocity of an infinitesimally small fluid element as it sweeps through B." (J. Anderson, Fundamentals of Aerodynamics, 2001, pg. 14)

Viscosity of oil > Viscosity of air

Sutherland's Law:

$$(3) \Rightarrow \boxed{\frac{\mu}{\mu_0} = \left(\frac{T}{T_0}\right)^{3/2} \frac{T_0+110}{T+110}}$$

Where: $_0$ is reference data
T_0 = 288.16 °K
μ_0 = 1.7894 x 10^{-5} kg/ms

Dynamic Viscosity:

$$(4) \Rightarrow \boxed{\nu = \frac{\mu}{\rho}}$$

Newtonian Fluids

For many simple fluids, such as air and water, μ is a thermodynamic property which depends only on Temperature & Pressure, but not on the shear rate $\left(\tau = \mu \frac{\partial u}{\partial y}\right)$

Note: $\boxed{\text{Newtonian} \neq \text{Non Newtonian Fluids}}$

2.0 Equation of State

This may also be referred to as the *Ideal Gas Law* or the *Perfect Gas Law.*

$$(5) \Rightarrow \boxed{p = \rho RT}$$

Using SI units

R = Specific Gas Constant = 287.05287 J/kg.°K

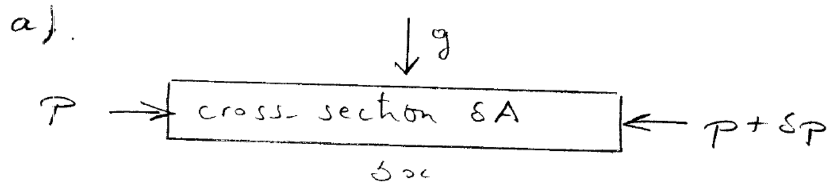
$$\boxed{R = c_p - c_v} \quad \& \quad \boxed{\gamma = \frac{c_p}{c_v}}$$

Where c_p & c_v are the specific heats

Atmospheric Models

3.0 The Hydrostatic Equation

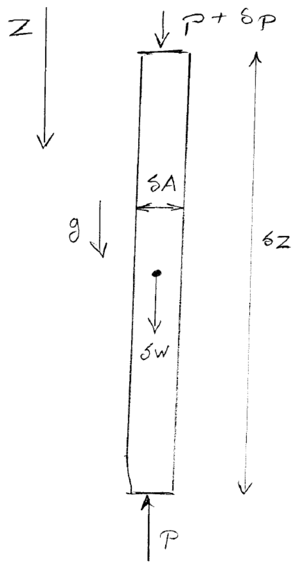
For a fluid at rest, the pressure is constant over any horizontal surface but decreases with altitude.



If the pressure is assumed to change from 'p' at one end of the cylinder to 'p + δp' at the other end, then the net force acting on the cylinder is:

$$\boxed{p.\delta A - (p + \delta p)\delta A = -\delta p.\delta A}$$

This net force <u>must be</u> 0 unless the fluid cylinder is accelerating to the right (δp < 0) or to the left (δp > 0). Hence for a fluid at rest, δp = 0 & the pressure is constant along any horizontal line (or surface).



The net downwards force acting on the fluid cylinder is:

$$-p.\delta A + \delta W + (p + \delta p)\delta A = 0$$

Where: $\delta W = \rho(\delta z.\delta A)g$

$$\therefore -p.\delta A + \rho(\delta z.\delta A)g + p.\delta A + \delta p.\delta A = 0$$
$$\therefore \rho.\delta z.\delta A.g + \delta p.\delta A = 0$$

$$(6) \Rightarrow \boxed{\delta p = -\rho.\delta z.g}$$

OR

$$(7) \Rightarrow \boxed{\frac{dp}{dz} = -\rho.g}$$

4.0 An Energy Balance Equation

4.1 Some Definitions

- <u>Adiabatic Process</u>
  One in which no heat is added to or taken away from the system

- <u>Isothermal Process</u>
  One in which the temperature remains constant

- <u>Reversible Process</u>
  One in which no dissipative phenomena occur, that is, where the effects of viscosity, thermal conductivity & mass diffusion are absent

- <u>Isentropic Process</u>
  One that is both Adiabatic & Reversible
  OR
  If there is no heat transfer or friction in the process

4.2 Isothermal Case

If: $T = T_0$

Then the equation of state becomes:

$$p = \rho R T_0$$

$$(8) \Rightarrow \boxed{\rho = \frac{p}{RT_0}}$$

Substitute (8) into (7):

$$(9) \Rightarrow \boxed{\frac{dp}{dz} = -\frac{p}{RT_0} \times g}$$

$$\frac{dp}{p} = -\frac{g}{RT_0} \times dz$$

$$(10) \Rightarrow \boxed{p = p_0, e^{\left(-g/_{RT_0}\right)z}}$$

Where p_0 occurs at z = 0

Substitute (10) into (8):

$$\rho = \frac{p_0}{RT_0}, e^{\left(-g/_{RT_0}\right)z}$$

$$(11) \Rightarrow \boxed{\rho = \rho_0, e^{\left(-g/_{RT_0}\right)z}}$$

Where ρ_0 occurs at z = 0

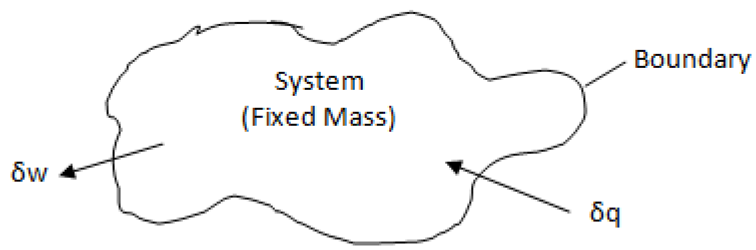
4.3 Adiabatic Process

$$(12) \Rightarrow \boxed{\delta q = 0}$$

For an isentropic process, ds = 0, where s is the entropy of the system.

Note: Entropy & the Second Law of Thermodynamics:

$$(13) \Rightarrow \boxed{ds = \frac{\delta q}{T}}$$



Thermodynamic System

First Law of Thermodynamics:

The change in energy in a system is:

$$(14) \Rightarrow \boxed{de = \delta w + \delta q}$$

For a reversible process: $\delta w = -pdv$

$$(15) \Rightarrow \boxed{de = \delta q - pdv}$$

Where dv is an elemental change in the volume due to a displacement of the boundary of the system

By deriving (13) it can be shown that $\delta q = Tds$

$$\therefore de = Tds - pdv$$

$$(16) \Rightarrow \boxed{Tds = de + p.dv}$$

The enthalpy is defined by:

$$(17) \Rightarrow \boxed{h = e + pv}$$

$$\therefore dh = de + d(pv) = de + p.dv + v.dp$$

$$\therefore de + p.dv = dh - v.dp$$

Substitute this into (16):

$$(18) \Rightarrow \boxed{Tds = dh - v.dp}$$

For a Perfect Gas:

$$de = c_v.dT \quad \& \quad dh = c_p.dT$$

Therefore by substituting into (16) & (18):

$$(19) \Rightarrow \boxed{ds = c_v \frac{dT}{T} + \frac{p.dv}{T}}$$

$$(20) \Rightarrow \boxed{ds = c_p \frac{dT}{T} - \frac{v.dp}{T}}$$

Using the Equation of State:

$$\frac{p}{\rho} = RT \quad OR \quad pv = RT$$

$$\therefore v = \frac{RT}{p} \quad OR \quad p = \frac{RT}{v}$$

Substitute these into (19) & (20):

$$\boxed{ds = c_v \frac{dT}{T} + R \frac{dv}{v}} \Leftarrow (21) \Rightarrow \boxed{ds = c_p \frac{dT}{T} - R \frac{dp}{p}}$$

$$\boxed{s_2 - s_1 = c_v.\ln\left(\frac{T_2}{T_1}\right) + R.\ln\left(\frac{v_2}{v_1}\right)} \Leftarrow (22) \Rightarrow \boxed{s_2 - s_1 = c_p.\ln\left(\frac{T_2}{T_1}\right) + R.\ln\left(\frac{p_2}{p_1}\right)}$$

For an isentropic process, δs = 0, this can be substituted into (22)

$$\boxed{0 = c_v.\ln\left(\frac{T_2}{T_1}\right) + R.\ln\left(\frac{v_2}{v_1}\right)} \quad \& \quad \boxed{0 = c_p.\ln\left(\frac{T_2}{T_1}\right) + R.\ln\left(\frac{p_2}{p_1}\right)}$$

Therefore:

$$\boxed{\ln\left(\frac{v_2}{v_1}\right) = \frac{-c_v}{R}.\ln\left(\frac{T_2}{T_1}\right)} \quad \& \quad \boxed{\ln\left(\frac{p_2}{p_1}\right) = \frac{c_p}{R}.\ln\left(\frac{T_2}{T_1}\right)}$$

Therefore:

$$\boxed{\frac{v_2}{v_1} = \left(\frac{T_2}{T_1}\right)^{\frac{-c_v}{R}}} \Leftarrow (23) \Rightarrow \boxed{\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{\frac{c_p}{R}}}$$

Where:

$$\boxed{R = c_p - c_v} \quad \& \quad \boxed{\gamma = \frac{c_p}{c_v}}$$

Therefore:

$$\boxed{c_v = \frac{R}{\gamma - 1}} \Leftarrow (24) \Rightarrow \boxed{c_p = \frac{\gamma R}{\gamma - 1}}$$

Substitute (24) into (23):

$$\boxed{\frac{v_2}{v_1} = \left(\frac{T_2}{T_1}\right)^{\frac{-1}{(\gamma-1)}}} \Leftarrow (25) \Rightarrow \boxed{\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{(\gamma-1)}}}$$

Where: $v = \frac{1}{\rho}$ $\therefore \frac{v_2}{v_1} = \frac{\rho_1}{\rho_2}$

Substitute this into (25):

$$(26) \Rightarrow \boxed{\frac{\rho_1}{\rho_2} = \left(\frac{T_2}{T_1}\right)^{\frac{-1}{(\gamma-1)}}}$$

Therefore:

$$(27) \Rightarrow \boxed{\frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma} = \left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{(\gamma-1)}}}$$

An adiabatic process is characterised by:

$$(28) \Rightarrow \boxed{\frac{p}{\rho^{\gamma}} = cst}$$

4.4 Constant Lapse Rate

In the troposphere, the temperature decreases with altitude z, according to the relation:

$$(29) \Rightarrow \boxed{T = T_o - \gamma z}$$

Where T_0 = 288.16 °K at z = 0

& γ is the lapse rate

$$\therefore dT = -\gamma . dz$$

$$(30) \Rightarrow \boxed{\gamma = -\frac{dT}{dz}}$$

<u>Note:</u> In isothermal conditions T = T_0 = Constant. Therefore γ = 0

Using the hydrostatic equation derived from (9):

$$\frac{dp}{p} = \frac{-g}{RT} dz$$

By substituting equation (29) into it we find:

$$(31) \Rightarrow \boxed{\frac{dp}{p} = \frac{-g}{R(T_0 - \gamma z)} dz}$$

$$\therefore \frac{dp}{p} = \frac{\gamma g}{-\gamma R} \frac{dz}{(T_0 - \gamma z)}$$

$$\therefore \int \frac{dp}{p} = \int \frac{g}{\gamma R} \left(\frac{-\gamma . dz}{(T_0 - \gamma z)}\right)$$

$$(32) \Rightarrow \boxed{\frac{p}{p_0} = \left(\frac{T}{T_0}\right)^{g/\gamma R}}$$

Using the Equation of State:

$$(33) \Rightarrow \boxed{\frac{\rho}{\rho_0} = \left(\frac{T}{T_0}\right)^{\left(g/\gamma R\right).1}}$$

Unsure of character on notes!