Longitudinal stability derivatives

We have seen a lot of stability derivatives in previous chapters. However, it would be nice to know their values. We're therefore going to derive some relations for them. In this chapter, we will look at longitudinal stability derivatives. In the next chapter, we'll examine lateral stability derivatives.

1 Nominal stability derivatives

1.1 Methods of finding the stability derivatives

There are three methods to find the stability derivatives. The first one is of course by performing **flight tests** or **wind tunnel tests**. These tests are, however, quite expensive. An alternative is using **computational fluid dynamics** (CDF). This is usually less expensive, but it still requires a lot of work.

Finally, simple **analytic expressions** can be used. They are usually not very accurate. (Especially not for strange aircraft configurations.) However, they don't require a lot of work. In this chapter, we're going to examine these analytic expressions.

1.2 Equations of motion

Before we will find stability derivatives, we first need to re-examine the equations of motion. The symmetric equations of motion, for an airplane in a steady flight, are

$$X = -D\cos\alpha + L\sin\alpha + T_p\cos(\alpha_0 + i_p) = W\sin\gamma_0, \tag{1.1}$$

$$Z = -L\cos\alpha - D\sin\alpha - T_p\sin(\alpha_0 + i_p) = -W\cos\gamma_0. \tag{1.2}$$

Here, α_0 is the initial angle of attack. (It is now not the zero-lift angle of attack.) Also, α now denotes the deviation from this flight condition. We assume $\alpha_0 + i_p$ is small, so we can use the small angle approximation. If we also non-dimensionalize the above relations, we find that

$$C_X = -C_D\cos\alpha + C_L\sin\alpha + T_c' = \frac{W}{\frac{1}{2}\rho V^2 S}\sin\gamma_0, \tag{1.3}$$

$$C_Z = -C_L\cos\alpha - C_D\sin\alpha - T_c'(\alpha_0 + i_p) = -\frac{W}{\frac{1}{2}\rho V^2 S}\cos\gamma_0, \tag{1.4}$$

where we have defined

$$T_c' = \frac{T_p}{\frac{1}{2}\rho V^2 S}. \tag{1.5}$$

1.3 Nominal flight conditions

Let's examine an aircraft flying a steady horizontal flight. We will now try to find the nominal stability derivatives C_{X_0}, C_{Z_0} and C_{m_0}. Since the aircraft is flying horizontally, we have $\alpha = \gamma_0 = 0$. (Remember that α is the deviation from the steady flight.) The relations of the previous paragraph now turn into

$$C_{X_0} = -C_D + T_c' = 0 \qquad \text{and} \qquad C_{Z_0} = -C_L - T_c'(\alpha_0 + i_p) = \frac{W}{\frac{1}{2}\rho V^2 S}. \tag{1.6}$$

Finally, from moment equilibrium follows that $C_{m_0} = 0$.

2 Velocity stability derivatives

2.1 The basic relations

Now let's find the stability derivatives with respect to the velocity. They are C_{X_u}, C_{Z_u} and C_{m_u}. They are very hard to determine experimentally. This is because wind tunnels and flying aircraft can't change their velocity in a very accurate way. Luckily, we can find expressions for them.

Let's start to examine C_{X_u}. We can recall that

$$C_{X_u} = \frac{1}{\frac{1}{2}\rho V S}\frac{\partial X}{\partial V} \qquad \text{and} \qquad X = C_X \frac{1}{2}\rho V^2 S. \tag{2.1}$$

(We have used the fact that $\partial V/\partial u \approx 1$.) Taking the derivative of the second relation, with respect to V, gives

$$\frac{\partial X}{\partial V} = C_X \rho V S + \frac{\partial C_X}{\partial V}\frac{1}{2}\rho V^2 S. \tag{2.2}$$

Inserting this into the first relation will give

$$C_{X_u} = 2C_X + \frac{\partial C_X}{\partial V}V. \tag{2.3}$$

In a similar way, we can find the expressions for C_{Z_u} and C_{m_u}. They are

$$C_{Z_u} = 2C_Z + \frac{\partial C_Z}{\partial V}V \qquad \text{and} \qquad C_{m_u} = 2C_m + \frac{\partial C_m}{\partial V}V. \tag{2.4}$$

2.2 Rewriting the relations

There are still some terms we don't know in the relations of the previous paragraph. They are C_X, C_Z, C_m, $\partial C_X/\partial V$, $\partial C_Z/\partial V$ and $\partial C_m/\partial V$. How can we rewrite them?

We are considering deviations from the steady horizontal flight. So we can replace C_X by $C_{X_0} = -C_D + T_c'$. Similarly, C_Z is replaced by $C_{Z_0} = -C_L - T_c'(\alpha_0 + i_p)$ and C_m by $C_{m_0} = 0$. That simplifies the equations quite a bit.

The derivatives are a bit harder to rewrite. At a steady horizontal flight, we have $C_X = -C_D + T_c'$ and $C_Z = -C_L - T_c'(\alpha_0 + i_p)$. Differentiating this gives

$$\frac{\partial C_X}{\partial V} = -\frac{\partial C_D}{\partial V} + \frac{\partial T_c'}{\partial V} \qquad \text{and} \qquad \frac{\partial C_Z}{\partial V} = -\frac{\partial C_L}{\partial V} - \frac{\partial T_c'}{\partial V}(\alpha_0 + i_p). \tag{2.5}$$

That leaves us with some more derivatives. First, let's examine $\partial C_D/\partial V$, $\partial C_L/\partial V$ and $\partial C_m/\partial V$. To find them, we have to know why the coefficients C_D, C_L and C_m vary with airspeed. These variations are partly caused by changes in Mach number M, changes in thrust T_c' and changes in Reynolds number Re. Although changes in the Reynolds number may be neglected, we do have to consider M and T_c'. This implies that

$$\frac{\partial C_D}{\partial V} = \frac{\partial C_D}{\partial M}\frac{\partial M}{\partial V} + \frac{\partial C_D}{\partial T_c'}\frac{\partial T_c'}{\partial V}, \tag{2.6}$$

$$\frac{\partial C_L}{\partial V} = \frac{\partial C_L}{\partial M}\frac{\partial M}{\partial V} + \frac{\partial C_L}{\partial T_c'}\frac{\partial T_c'}{\partial V}, \tag{2.7}$$

$$\frac{\partial C_m}{\partial V} = \frac{\partial C_m}{\partial M}\frac{\partial M}{\partial V} + \frac{\partial C_m}{\partial T_c'}\frac{\partial T_c'}{\partial V}. \tag{2.8}$$

If we use this, in combination with earlier equations, we will find that

$$C_{X_u} = -2C_D + 2T_c' + \left(1 - \frac{\partial C_D}{\partial T_c'}\right)\frac{dT_c'}{dV}V - \frac{\partial C_D}{\partial M}M, \tag{2.9}$$

$$C_{Z_u} = -2C_L - 2T_c'(\alpha_0 + i_p) - \left((\alpha_0 + i_p) + \frac{\partial C_L}{\partial T_c'}\right)\frac{dT_c'}{dV}V - \frac{\partial C_L}{\partial M}M, \tag{2.10}$$

$$C_{m_u} = \frac{\partial C_m}{\partial T_c'}\frac{dT_c'}{dV}V + \frac{\partial C_m}{\partial M}M. \tag{2.11}$$

2.3 The thrust derivative

The equations of the previous paragraph still have a lot of derivatives. We won't go into detail on derivatives with respect T_c' or M. However, we will consider dT_c'/dV. It turns out that we can write this derivative as

$$\frac{dT_c'}{dV} = -k\frac{T_c'}{V}, \tag{2.12}$$

where the constant k depends on the flight type. If we have a **gliding flight**, then $T_c' = 0$. Thus also $dT_c'/dV = 0$ and therefore $k = 0$. If we have a **jet-powered aircraft**, then $T_p = T_c'\frac{1}{2}\rho V^2 S = \text{constant}$. From this follows that $k = 2$. Finally, for **propeller-powered aircraft**, we have $T_p V = T_c'\frac{1}{2}\rho V^3 S = \text{constant}$. This implies that $k = 3$.

Let's use the above relation to simplify our equations. The equations for C_{X_u}, C_{Z_u} and C_{m_u} now become

$$C_{X_u} = -2C_D + \left(2 - k\left(1 - \frac{\partial C_D}{\partial T_c'}\right)\right)T_c' - \frac{\partial C_D}{\partial M}M, \tag{2.13}$$

$$C_{Z_u} = -2C_L + \left((k-2)(\alpha_0 + i_p) + k\frac{\partial C_L}{\partial T_c'}\right)T_c' - \frac{\partial C_L}{\partial M}M, \tag{2.14}$$

$$C_{m_u} = -k\frac{\partial C_m}{\partial T_c'}T_c' + \frac{\partial C_m}{\partial M}M. \tag{2.15}$$

When specific data about the type of flight is known, the above equations can be simplified even further. For example, when the flight is at low subsonic velocities, then Mach effects may be neglected. Thus $\partial C_D/\partial M = \partial C_L/\partial M = \partial C_m/\partial M = 0$. In other cases, there are often other simplifications that can be performed.

3 Angle of attack stability derivatives

3.1 The basic relations for C_{X_α} and C_{Z_α}

We will now try to find relations for C_{X_α}, C_{Z_α} and C_{m_α}. First we examine C_{X_α} and C_{Z_α}. They are defined as

$$C_{X_\alpha} = \frac{1}{\frac{1}{2}\rho V S}\frac{\partial X}{\partial w} = \frac{\partial C_X}{\partial \alpha} \qquad \text{and} \qquad C_{Z_\alpha} = \frac{1}{\frac{1}{2}\rho V S}\frac{\partial Z}{\partial w} = \frac{\partial C_Z}{\partial \alpha}. \tag{3.1}$$

If we take the derivative of equations (1.3) and (1.4), we find that

$$C_{X_\alpha} = -C_{D_\alpha}\cos\alpha + C_D\sin\alpha + C_{L_\alpha}\sin\alpha + C_L\cos\alpha, \tag{3.2}$$

$$C_{Z_\alpha} = -C_{L_\alpha}\cos\alpha + C_L\sin\alpha - C_{D_\alpha}\sin\alpha - C_D\cos\alpha. \tag{3.3}$$

We are examining an aircraft performing a steady horizontal flight. Thus $\alpha = 0$. This simplifies the above equations to

$$C_{X_\alpha} = C_L - C_{D_\alpha} \qquad \text{and} \qquad C_{Z_\alpha} = -C_{L_\alpha} - C_D \approx -C_{L_\alpha} \approx -C_{N_\alpha}. \tag{3.4}$$

In the last part of the above equation, we have used the fact that C_D is much smaller than C_{L_α}.

3.2 Rewriting the relation for C_{X_α}

We can try to rewrite the relation for C_{X_α}. To do this, we examine C_{D_α}. Let's assume that the aircraft has a parabolic drag curve. This implies that

$$C_D = C_{D_0} + \frac{C_L^2}{\pi A e}, \qquad \text{which, in turn, implies that} \qquad C_{D_\alpha} = 2\frac{C_{L_\alpha}}{\pi A e} C_L. \tag{3.5}$$

If we combine this with the former expression for C_{X_α}, we wind up with

$$C_{X_\alpha} = C_L \left(1 - 2\frac{C_{L_\alpha}}{\pi A e}\right). \tag{3.6}$$

3.3 The relation for C_{m_α}

In a previous chapter, we have already considered C_{m_α}. After neglecting the effects of many parts of the aircraft, we wound up with

$$C_{m_\alpha} = C_{N_{w_\alpha}} \frac{x_{cg} - x_w}{\bar{c}} - C_{N_{h_\alpha}} \left(1 - \frac{d\varepsilon}{d\alpha}\right) \left(\frac{V_h}{V}\right)^2 \frac{S_h l_h}{S\bar{c}}. \tag{3.7}$$

4 Pitch rate stability derivatives

4.1 The reasons behind the changing coefficients

We will now try to find C_{X_q}, C_{Z_q} and C_{m_q}. Luckily, C_X doesn't get influenced a lot by q. So it is usually assumed that $C_{X_q} = 0$. That saves us some work. We now only need to find C_{Z_q} and C_{m_q}. They are defined as

$$C_{Z_q} = \frac{1}{\frac{1}{2}\rho V S \bar{c}} \frac{\partial Z}{\partial q} = \frac{\partial C_Z}{\partial \frac{q\bar{c}}{V}} \qquad \text{and} \qquad C_{m_q} = \frac{1}{\frac{1}{2}\rho V S \bar{c}^2} \frac{\partial M}{\partial q} = \frac{\partial C_m}{\partial \frac{q\bar{c}}{V}}. \tag{4.1}$$

To find C_{Z_q} and C_{m_q}, we first have to understand some theory behind rotations. Why do the coefficients change, when the aircraft rotates? This is because the effective angle of attack changes. Imagine an aircraft with its nose pitching upward. The tailplane of the aircraft is thus pitching downward. Now imagine you're sitting on the tailplane. As seen from the tailplane, it looks like the flow of air is coming upward. This means that the tailplane experiences a bigger angle of attack.

To find the exact value of the change in angle of attack $\Delta\alpha$, we examine the **center of rotation**. This is the point about which the aircraft appears to be rotating. The center of rotation lies on the Z_s-axis. The apparent rotation itself is performed with a radius R, which is given by

$$R = \frac{V}{q}. \tag{4.2}$$

The change in angle of attack $\Delta\alpha$, at any point x on the airplane, is then given by

$$\Delta\alpha = \frac{x - x_{cg}}{R} = \frac{x - x_{cg}}{\bar{c}} \frac{q\bar{c}}{V}. \tag{4.3}$$

4.2 The changing coefficients

We know that the apparent angle of attack changes across the aircraft. This is especially important for the horizontal tailplane. In fact, the change in angle of attack of the tailplane is given by

$$\Delta\alpha_h = \frac{x_h - x_{cg}}{\bar{c}} \frac{q\bar{c}}{V} \approx \frac{l_h}{\bar{c}} \frac{q\bar{c}}{V}. \tag{4.4}$$

This change in angle of attack causes the normal force of the tailplane to change. In fact, it changes by an amount

$$\Delta C_{N_h} = C_{N_{h_\alpha}} \left(\frac{V_h}{V}\right)^2 \frac{S_h}{S} \Delta\alpha_h = C_{N_{h_\alpha}} \left(\frac{V_h}{V}\right)^2 \frac{S_h l_h}{S\bar{c}} \frac{q\bar{c}}{V}. \tag{4.5}$$

Similarly, the change of the moment is given by

$$\Delta C_m = -C_{N_{h_\alpha}} \left(\frac{V_h}{V}\right)^2 \frac{S_h l_h}{S\bar{c}} \Delta\alpha_h = -C_{N_{h_\alpha}} \left(\frac{V_h}{V}\right)^2 \frac{S_h l_h^2}{S\bar{c}^2} \frac{q\bar{c}}{V}. \tag{4.6}$$

We know that $C_{Z_q} = \partial C_Z / \partial \frac{q\bar{c}}{V}$ and $C_{m_q} = \partial C_m / \partial \frac{q\bar{c}}{V}$. By using this, we can find the contributions of the horizontal tailplane to C_{z_q} and C_{m_q}. They are

$$\left(C_{Z_q}\right)_h = -C_{N_{h_\alpha}} \left(\frac{V_h}{V}\right)^2 \frac{S_h l_h}{S\bar{c}} \qquad \text{and} \qquad \left(C_{m_q}\right)_h = -C_{N_{h_\alpha}} \left(\frac{V_h}{V}\right)^2 \frac{S_h l_h^2}{S\bar{c}^2}. \tag{4.7}$$

(The minus sign in the left part appeared, because C_N is defined upward, while C_Z is defined downward.) There is, however, one small problem. The aircraft doesn't consist of only a horizontal tailplane. It also has various other parts. But it is very difficult to calculate the effects of all these parts. For that reason, we make an estimate. We say that the contribution of the full aircraft C_{Z_q} is twice the contribution of the horizontal tailplane $\left(C_{Z_q}\right)_h$. This implies that

$$C_{Z_q} = 2\left(C_{Z_q}\right)_h = 2C_{N_{h_\alpha}} \left(\frac{V_h}{V}\right)^2 \frac{S_h l_h}{S\bar{c}}. \tag{4.8}$$

For C_{m_q}, we apply the same trick. But instead of a factor 2, a factor between 1.1 to 1.2 should now be used, depending on the aircraft. We thus get

$$C_{m_q} = (1.1 \sim 1.2)\left(C_{m_q}\right)_h = -(1.1 \sim 1.2) C_{N_{h_\alpha}} \left(\frac{V_h}{V}\right)^2 \frac{S_h l_h^2}{S\bar{c}^2}. \tag{4.9}$$

5 Other longitudinal stability derivatives

5.1 Vertical acceleration stability derivatives

We now examine $C_{Z_{\dot\alpha}}$ and $C_{m_{\dot\alpha}}$. (We assume $C_{X_{\dot\alpha}} = 0$.) To do this, we look at the horizontal tailplane. During a steady flight, it has an effective angle of attack

$$\alpha_h = \alpha - \varepsilon + i_h = \alpha - \frac{d\varepsilon}{d\alpha}\alpha + i_h. \tag{5.1}$$

Now let's suppose that the aircraft experiences a change in angle of attack. This causes the downwash angle ε of the wing to change. A time $\Delta t = l_h/V$ later will this change be experienced by the horizontal tailplane. In other words, the downwash $\varepsilon(t)$ at time t depends on the angle of attack $\alpha(t - \Delta t)$ at time $t - \Delta t$. A linear approximation of $\alpha(t - \Delta t)$ is given by

$$\alpha(t - \Delta t) = \alpha(t) - \dot\alpha \Delta t. \tag{5.2}$$

By using this, we find that the downwash is given by

$$\varepsilon(t) = \frac{d\varepsilon}{d\alpha}\alpha(t - \Delta t) = \frac{d\varepsilon}{d\alpha}\alpha(t) - \frac{d\varepsilon}{d\alpha}\dot\alpha \frac{l_h}{V}. \tag{5.3}$$

This implies that the effective angle of attack is given by

$$\alpha_h = \alpha - \frac{d\varepsilon}{d\alpha}\alpha + \frac{d\varepsilon}{d\alpha}\dot\alpha \frac{l_h}{V} + i_h. \tag{5.4}$$

The change in effective angle of attack is

$$\Delta\alpha_h = \frac{d\varepsilon}{d\alpha}\frac{l_h}{\bar{c}}\frac{\dot{\alpha}\bar{c}}{V}. \tag{5.5}$$

We now have enough data to find the coefficients $C_{Z_{\dot{\alpha}}}$ and $C_{m_{\dot{\alpha}}}$. We know that

$$\Delta C_Z = -C_{N_{h_\alpha}}\left(\frac{V_h}{V}\right)^2\frac{S_h}{S}\Delta\alpha_h = -C_{N_{h_\alpha}}\left(\frac{V_h}{V}\right)^2\frac{S_h l_h}{S\bar{c}}\frac{d\varepsilon}{d\alpha}\frac{\dot{\alpha}\bar{c}}{V}, \tag{5.6}$$

$$\Delta C_m = -C_{N_{h_\alpha}}\left(\frac{V_h}{V}\right)^2\frac{S_h l_h}{S\bar{c}}\Delta\alpha_h = -C_{N_{h_\alpha}}\left(\frac{V_h}{V}\right)^2\frac{S_h l_h^2}{S\bar{c}^2}\frac{d\varepsilon}{d\alpha}\frac{\dot{\alpha}\bar{c}}{V}. \tag{5.7}$$

The coefficients $C_{Z_{\dot{\alpha}}}$ and $C_{m_{\dot{\alpha}}}$ are now given by

$$C_{Z_{\dot{\alpha}}} = \frac{1}{\frac{1}{2}\rho S\bar{c}}\frac{\partial Z}{\partial\dot{w}} = \frac{\partial C_Z}{\partial\frac{\dot{\alpha}\bar{c}}{V}} = -C_{N_{h_\alpha}}\left(\frac{V_h}{V}\right)^2\frac{S_h l_h}{S\bar{c}}\frac{d\varepsilon}{d\alpha}, \tag{5.8}$$

$$C_{m_{\dot{\alpha}}} = \frac{1}{\frac{1}{2}\rho S\bar{c}^2}\frac{\partial M}{\partial\dot{w}} = \frac{\partial C_m}{\partial\frac{\dot{\alpha}\bar{c}}{V}} = -C_{N_{h_\alpha}}\left(\frac{V_h}{V}\right)^2\frac{S_h l_h^2}{S\bar{c}^2}\frac{d\varepsilon}{d\alpha}. \tag{5.9}$$

5.2 Elevator angle stability derivatives

The last stability derivatives we will consider in this chapter are $C_{X_{\delta_e}}$, $C_{Z_{\delta_e}}$ and $C_{m_{\delta_e}}$. Usually C_X doesn't vary a lot with δ_e, so we assume that $C_{X_{\delta_e}} = 0$. But what about $C_{Z_{\delta_e}}$? Well, this one is given by

$$C_{Z_{\delta_e}} = -C_{N_{h_{\delta_e}}}\left(\frac{V_h}{V}\right)^2\frac{S_h}{S}. \tag{5.10}$$

Finally there is $C_{m_{\delta_e}}$. We can find that it is

$$C_{m_{\delta_e}} = C_{Z_{\delta_e}}\frac{x_h - x_{cg}}{\bar{c}} \approx -C_{N_{\delta_e}}\left(\frac{V_h}{V}\right)^2\frac{S_h l_h}{S\bar{c}}. \tag{5.11}$$

The coefficient $C_{Z_{\delta_e}}$ usually isn't very important. However, $C_{m_{\delta_e}}$ is very important. This is because the whole goal of an elevator is to apply a moment to the aircraft.

5.3 Effects of moving the center of gravity

We have one topic left to discuss. What happens when the CG moves from position 1 to position 2? In this case, several coefficients change. This goes according to

$$C_{m_{\alpha_2}} = C_{m_{\alpha_1}} - C_{Z_\alpha}\frac{x_{cg_2} - x_{cg_1}}{\bar{c}}, \tag{5.12}$$

$$C_{Z_{q_2}} = C_{Z_{q_1}} - C_{Z_\alpha}\frac{x_{cg_2} - x_{cg_1}}{\bar{c}}, \tag{5.13}$$

$$C_{m_{q_2}} = C_{m_{q_1}} - \left(C_{Z_{q_1}} + C_{m_{\alpha_1}}\right)\frac{x_{cg_2} - x_{cg_1}}{\bar{c}} + C_{Z_\alpha}\left(\frac{x_{cg_2} - x_{cg_1}}{\bar{c}}\right)^2, \tag{5.14}$$

$$C_{m_{\dot{\alpha}_2}} = C_{m_{\dot{\alpha}_1}} - C_{Z_{\dot{\alpha}}}\frac{x_{cg_2} - x_{cg_1}}{\bar{c}}. \tag{5.15}$$