Lateral stability derivatives

In the previous chapter, we found relations for the longitudinal stability derivatives. Now we'll examine the lateral stability derivatives.

1 Sideslip angle stability derivatives

1.1 Horizontal forces

We start by examining derivatives with respect to the **sideslip angle** β. This angle is defined as

$$\beta = \arcsin\left(\frac{v}{V}\right) \approx \frac{v}{V}. \tag{1.1}$$

We will now examine C_{Y_β}. Let's examine an aircraft with a sideslip angle β. This sideslip angle causes a horizontal force Y on the aircraft. The most important contributors to this horizontal force are the fuselage and the vertical tailplane.

First let's examine the vertical tailplane. Luckily, this tailplane has a lot of analogies with the horizontal tailplane, so we can use some short cuts. For example, the force acting on the vertical tailplane is given by

$$\left(C_{Y_\beta}\right)_v = C_{Y_{v_\alpha}} \frac{d\alpha_v}{d\beta} \left(\frac{V_v}{V}\right)^2 \frac{S_v}{S}, \tag{1.2}$$

where S_v is the vertical tailplane surface area and V_v is the average velocity over it. Also, $C_{Y_{v_\alpha}} = \frac{\partial C_{Y_v}}{\partial \alpha_v}$. Let's take a closer look at the **effective angle of attack of the tailplane** α_v. It's not equal to β. This is because the fuselage also alters the flow by an angle σ. (σ is similar to the downwash ε for the horizontal tailplane.) The vertical tailplane thus has an angle of attack of $\alpha_v = -(\beta - \sigma)$. (The minus is present due to sign convention.) Inserting this relation into the above equation gives

$$\left(C_{Y_\beta}\right)_v = -C_{Y_{v_\alpha}} \left(1 - \frac{d\sigma}{d\beta}\right) \left(\frac{V_h}{V}\right)^2 \frac{S_v}{S}. \tag{1.3}$$

Usually, most terms in the above equation are known. Only $d\sigma/d\beta$ is still a bit of a mystery. It is very hard to determine. However, it usually is negative. (So $d\sigma/d\beta < 0$.)

Next to the tailplane contribution, there is usually also a contribution by the fuselage. However, we don't go into depth on that here.

1.2 Rolling momemts

Now let's examine the so-called **effective dihedral** C_{l_β}. The coefficient C_l was defined as

$$C_l = \frac{L}{\frac{1}{2}\rho V^2 b}. \tag{1.4}$$

It is important to note that L is not the lift. It is the moment about the X axis. C_l is thus not the lift coefficient either.

The effective dihedral C_{l_β} mostly depends on the wing set-up. Both the **wing-dihedral** Γ and the **sweep angle** Λ strongly effect C_l. (The wing-dihedral Γ is the angle with which the wings have been tilted upward, when seen from the fuselage.)

First let's examine an aircraft with a wing-dihedral Γ. We suppose that the aircraft is sideslipping to the right. From the aircraft, it now appears as if part of the flow is coming from the right. This flow 'crouches' under the right wing, pushing it more upward. However, it flows over the left wing, pushing that one downward a bit. This thus causes the aircraft to roll to the left.

To find more info about the moment caused by the wing-dihedral, we need to examine the new angle of attacks of the wings α_{w_l} and α_{w_r}. By using small angle approximations, we can find that

$$\alpha_{w_l} \approx \alpha - \beta\Gamma \qquad \text{and} \qquad \alpha_{w_r} \approx \alpha + \beta\Gamma. \tag{1.5}$$

The changes in the angles of attack are thus $\Delta\alpha_{w_l} = -\beta\Gamma$ and $\Delta\alpha_{w_r} = \beta\Gamma$. So the moment caused by the wing-dihedral is approximately linearly dependend on both β and Γ. (We thus have $C_{l_\beta} \sim \Gamma$.)

Second, we look at an aircraft with a wing sweep angle Λ. The lift of a wing strongly depends on the flow velocity perpendicular to the leading edge. Again, we suppose that part of the flow is coming in from the right. This causes the flow to be more perpendicular w.r.t. to the right wing leading edge, thus increasing the lift. However, the flow is more parallel w.r.t. the leading edge of the left wing. The left wing thus has reduced lift. It can be shown that the change in lift for the aircraft, due to a sweep angle Λ, is

$$\Delta L = C_L \frac{1}{2}\rho V^2 \frac{S}{2}\left(\cos^2(\Lambda - \beta) - \cos^2(\Lambda + \beta)\right) \approx C_L \frac{1}{2}\rho V^2 S \sin(2\Lambda\beta). \tag{1.6}$$

The rightmost part of the equation is an approximation. It only works for small values of β. The above equation shows that the lift more or less linearly depends on Λ and β. It can be shown that the same holds for the moment C_l. The effective dihedral C_{l_β} is thus proportional to Λ.

Next to the wing, also the horizontal tailplane and the fuselage effect C_{l_β}. However, we won't examine these effects.

1.3 Yawing moments

The stability derivative C_{n_β} is called the **static directional stability**. (It's also known as the **Weathercock stability**.) It is about just as important as C_{m_α}. It can be shown that, if C_{n_β} is positive, then the aircraft is stable for yawing motions. However, if C_{n_β} is negative, then the aircraft is unstable for yawing motions.

Naturally, we want to have $C_{n_\beta} > 0$. Luckily, the wings and the horizontal tailplane have a slightly positive effect on C_{n_β}. However, the fuselage causes C_{n_β} to decrease. To compensate for this, a vertical tailplane is used, strongly increasing C_{n_β}.

Let's examine the effects of this tailplane. You may remember that the normal force on it was

$$\left(C_{Y_\beta}\right)_v = -C_{Y_{v_\alpha}}\left(1 - \frac{d\sigma}{d\beta}\right)\left(\frac{V_v}{V}\right)^2 \frac{S_v}{S}. \tag{1.7}$$

This normal force causes a moment

$$\left(C_{n_\beta}\right)_v = -\left(C_{Y_\beta}\right)_v\left(\frac{z_v - z_{cg}}{b}\sin\alpha_0 + \frac{x_v - x_{cg}}{b}\cos\alpha_0\right). \tag{1.8}$$

We can usually assume α_0 to be small. (Thus $\cos\alpha_0 \approx 1$.) Also, $\frac{z_v - z_{cg}}{b}\sin\alpha_0$ is usually quite small, compared to the other term, so we neglect it. If we also use the **tail length** of the vertical tailplane $l_v = x_v - x_{cg}$, we can rewrite the above equation to

$$\left(C_{n_\beta}\right)_v = C_{Y_{v_\alpha}}\left(1 - \frac{d\sigma}{d\beta}\right)\left(\frac{V_v}{V}\right)^2 \frac{S_v l_v}{Sb}. \tag{1.9}$$

From this, the correspondence to C_{m_α} again becomes clear. To emphasize this, we once more show the equation for the horizontal tailplane contribution to C_{m_α}. Rather similar to $\left(C_{n_\beta}\right)_v$, it was given by

$$\left(C_{m_\alpha}\right)_h = -C_{N_{h_\alpha}}\left(1 - \frac{d\varepsilon}{d\alpha}\right)\left(\frac{V_h}{V}\right)^2 \frac{S_h l_h}{S\bar{c}}. \tag{1.10}$$

2 Roll rate stability derivatives

2.1 Horizontal forces

It is time to investigate the effects of roll. In other words, we will try to find the stability derivatives C_{Y_p}, C_{l_p} and C_{n_p}. (Of these three, C_{l_p} is the most important.) First we examine C_{Y_p}. It is defined such that

$$Y_p = C_{Y_p} \frac{pb}{2V} \frac{1}{2} \rho V^2 S. \tag{2.1}$$

The only part having a more or less significant contribution to C_{Y_p} is the vertical tailplane. Let's examine a rolling aircraft. Due to this rolling, the vertical tailplane is moving horizontally. It will therefore get an effective angle of attack. This causes a horizontal force. A positive roll rate gives a negative horizontal force. C_{Y_p} is thus negative.

However, C_{Y_p} is usually rather small. For this reason it is often neglected. So we say that $C_{Y_p} \approx 0$.

2.2 Rolling moments

Now we will try to find C_{l_p}. Again, we examine a rolling aircraft. One wing of the aircraft goes up, while the other one goes down. This motion changes the effective angle of attack and thus also the lift of the wings. The upward going wing will get a lower lift, while the downward moving wing will experience a bigger amount of lift. The wing forces thus cause a moment opposite to the rolling motion. This means that C_{l_p} is highly negative. It also implies that the rolling motion is very strongly damped. (We will see this again, when examining the aperiodic roll in chapter 10.)

We can also investigate the actual effects of the rolling motion. To do this, we examine a chord at a distance y from the fuselage. This chord will have an additional vertical velocity of py. The change in angle of attack of this chord thus is

$$\Delta \alpha = \frac{py}{V} = \frac{pb}{2V} \frac{y}{b/2}. \tag{2.2}$$

So a chord that is far away from the fuselage will experience a big change in angle of attack. The change in lift is therefore biggest for these chords. These chords also have a relatively big distance to the CG of the aircraft. For this reason, they will significantly effect the resulting moment.

Other parts of the aircraft may also influence C_{l_p} slightly. However, their influence is very small, compared to the effects of the wings. The contributions of the other parts are therefore neglected.

2.3 Yawing moments

To find C_{n_p}, we again examine a rolling aircraft. The rolling of the aircraft has two important effects.

First, we look at the vertical tailplane. As was discussed earlier, this tailplane will move. It thus has an effective angle of attack, and therefore a horizontal force. This horizontal force causes the aircraft to yaw. A positive rolling motion causes a positive yawing moment. The vertical tailplane thus has a positive contribution to C_{n_p}. (So $\left(C_{n_p}\right)_v > 0$.)

But now let's look at the wings. Let's suppose that the aircraft is rolling to the right. For the right wing, it then appears as if the flow comes (partially) from below. The lift is per definition perpendicular to the direction of the incoming flow. The lift vector is thus tilted forward. Part of this lift causes the aircraft to yaw to the left. The opposite happens for the left wing: The lift vector is tilted backward. Again, this causes a yawing moment to the left. (This effect is known as **adverse yaw**.) So we conclude that, due to the wings, a positive rolling motion results in a negative yawing moment. We thus have $\left(C_{n_p}\right)_w < 0$.

For most normal flights, the effects of the vertical stabilizer are a bit bigger than the effects of the wing. We thus have $C_{n_p} > 0$. However, high roll rates and/or high angles of attack increase the effect of the wings. In this case, we will most likely have $C_{n_p} < 0$.

3 Yaw rate stability derivatives

3.1 Horizontal forces

In this part, we'll try to find the stability derivatives C_{Y_r}, C_{l_r} and C_{n_r}. We start with the not very important coefficient C_{Y_r}. Let's examine a yawing aircraft. Due to the yawing moment, the vertical tailplane moves horizontally. Because of this, its effective angle of attack will change by

$$\Delta\alpha_v = \frac{r l_v}{V} = \frac{rb}{2V}\frac{l_v}{b/2}. \tag{3.1}$$

The contribution of the tailplane to C_{Y_r} is now given by

$$(C_{Y_r})_v = 2C_{Y_{v_\alpha}}\left(\frac{V_v}{V}\right)^2\frac{S_v l_v}{Sb}. \tag{3.2}$$

The contribution is positive, so $(C_{Y_r})_v > 0$. Next to the vertical tailplane, there are also other parts influencing C_{Y_r}. Most parts have a negative contribution to C_{Y_r}. However, none of these contributions are as big as $(C_{Y_r})_v$. The stability derivative C_{Y_r} is therefore still positive. It is only slightly smaller than $(C_{Y_r})_v$.

3.2 Rolling moments

We will now examine C_{l_r}. There are two important contributions to C_{l_r}. They come from the vertical tailplane and the wings.

First we examine the vertical tailplane. We just saw that a yawing motion causes a horizontal force on the vertical tailplane. This horizontal force causes a moment

$$(C_{l_r})_v = (C_{Y_r})_v\left(\frac{z_v - z_{cg}}{b}\cos\alpha_0 - \frac{x_v - x_{cg}}{b}\sin\alpha_0\right). \tag{3.3}$$

A positive yawing motion results in a positive moment. We thus have $(C_{l_r})_v > 0$.

Now let's examine the wings. Because of the yawing motion, one wing will move faster, while the other wing will move slower. This causes the lift on one wing to increase, while it will decrease on the other wing. This results in a rolling moment.

Sadly, it's rather hard to find an equation for the moment caused by the wings. So we won't examine that any further. However, it is important to remember that a positive yawing motion causes a positive rolling moment. We thus have $(C_{l_r})_w > 0$. The total coefficient C_{l_r} is then, of course, also positive.

3.3 Yawing moments

Finally, we examine C_{n_r}. The most important contribution comes from the vertical tailplane. We know that a yawing motion causes a horizontal force on the vertical tailplane. This force is such that it damps the yawing motion. The contribution $(C_{n_r})_v$ is thus very highly negative. In fact, it is given by

$$(C_{n_r})_v = -(C_{Y_r})_v\frac{l_v}{b} = -2C_{Y_{v_\alpha}}\left(\frac{V_v}{V}\right)^2\frac{S_v l_v^2}{Sb^2}. \tag{3.4}$$

The vertical tailplane is about the only part seriously effecting the coefficient C_{n_r}. Sometimes also the fuselage effects it. This effect is also negative. (So $(C_{n_r})_f < 0$.) The coefficient C_{n_r} itself is thus also very strongly negative. This implies that the yawing motion is highly damped.

4 Other lateral stability derivatives

4.1 Aileron deflections

Let's consider the ailerons. The aileron deflection δ_a is defined as

$$\delta_a = \delta_{a_{right}} - \delta_{a_{left}}. \tag{4.1}$$

A deflection of the ailerons causes almost no change in horizontal forces. We thus have $C_{Y_{\delta_a}} = 0$. The so-called **aileron effectiveness** $C_{l_{\delta_a}}$ is, of course, not negligible. (Causing moments about the X axis is what ailerons are for.) The coefficient $C_{n_{\delta_a}}$ usually isn't negligible either. By the way, the moments caused by an aileron deflection are given by

$$L = C_{l_{\delta_a}} \delta_a \frac{1}{2} \rho V^2 S b \qquad \text{and} \qquad N = C_{n_{\delta_a}} \delta_a \frac{1}{2} \rho V^2 S b. \tag{4.2}$$

$C_{l_{\delta_a}}$ is negative. A positive aileron deflection causes a negative rolling moment. $C_{n_{\delta_a}}$ is, however, positive. So a positive aileron deflection causes positive yaw.

4.2 Rudder deflections

The rudder stability derivatives are $C_{Y_{\delta_r}}$, $C_{l_{\delta_r}}$ and $C_{n_{\delta_r}}$. The forces and moments caused by a rudder deflection are given by

$$Y = C_{Y_{\delta_r}} \delta_r \frac{1}{2} \rho V^2 S, \qquad L = C_{l_{\delta_r}} \delta_r \frac{1}{2} \rho V^2 S b. \quad \text{and} \quad N = C_{n_{\delta_r}} \delta_r \frac{1}{2} \rho V^2 S b. \tag{4.3}$$

The coefficient $C_{Y_{\delta_r}}$ is given by

$$C_{Y_{\delta_r}} = C_{Y_{v_\delta}} \left(\frac{V_v}{V} \right)^2 \frac{S_v}{S}. \tag{4.4}$$

The coefficient $C_{l_{\delta_r}}$ is then given by

$$C_{l_{\delta_r}} = C_{Y_{\delta_r}} \left(\frac{z_v - z_{cg}}{b} \cos \alpha_0 - \frac{x_v - x_{cg}}{b} \sin \alpha_0 \right). \tag{4.5}$$

$C_{l_{\delta_r}}$ is positive. This means that a positive rudder deflection causes a positive rolling moment. This effect is generally not desirable. Especially if $z_v - z_{cg}$ is big, measures are often taken to reduce this effect.

Finally, the **rudder effectiveness** $C_{n_{\delta_r}}$ is given by

$$C_{n_{\delta_r}} = -C_{Y_{\delta_r}} \frac{l_v}{b}. \tag{4.6}$$

This coefficient is negative. A positive rudder deflection thus causes a negative yawing moment.

4.3 Spoiler deflections

The last things we examine are the spoilers. Spoilers are often used in high-speed aircraft to provide roll control. A spoiler deflection δ_s on the left wing is defined to be positive. Due to this definition, we have $C_{l_{\delta_s}} < 0$ and $C_{n_{\delta_s}} < 0$. Of these two, the latter is the most important.