Question 2

a) $$T_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & \sin(\alpha) \\ 0 & -\sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

$$T_y = \begin{bmatrix} \cos(\alpha) & 0 & -\sin(\alpha) \\ 0 & 1 & 0 \\ \sin(\alpha) & 0 & \cos(\alpha) \end{bmatrix}$$

$$T_z = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b) i) From $A \rightarrow C$, so T_{CA}

$$T_{CA} = T_{CB} \cdot T_{BA}$$

$$T_{CB} = T_y(d\beta) = \begin{bmatrix} \cos(d\beta) & 0 & -\sin(d\beta) \\ 0 & 1 & 0 \\ \sin(d\beta) & 0 & \cos(d\beta) \end{bmatrix}$$

$$T_{BA} = T_x(d\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(d\alpha) & \sin(d\alpha) \\ 0 & -\sin(d\alpha) & \cos(d\alpha) \end{bmatrix}$$

$$\boxed{T_{CA} = T_y(d\beta) \cdot T_x(d\alpha)}$$

ii) Ω^C_{CA} is asked

$$\Omega^C_{CA} = \Omega^C_{CB} + \Omega^C_{BA}$$

$$\Omega^B_{CB} = \dot\beta \cdot Y_B \rightarrow \Omega^C_{CB} = T_{CB} \cdot \Omega^B_{CB}$$

$$\Omega^C_{CB} = \dot\beta \cdot \begin{bmatrix} \cos(d\beta) & 0 & -\sin(d\beta) \\ 0 & 1 & 0 \\ \sin(d\beta) & 0 & \cos(d\beta) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \dot\beta \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \dot\beta \cdot Y_C \qquad (Y_B = Y_C)$$

$$\Omega^A_{BA} = \dot\alpha \cdot X_A \rightarrow \Omega^C_{BA} = T_{CA} \cdot \Omega^A_{BA}$$

$$T_{CA} = \begin{bmatrix} \cos(d\beta) & \sin(d\beta) \cdot \sin(d\alpha) & -\sin(d\beta) \cdot \cos(d\alpha) \\ 0 & \cos(d\alpha) & \sin(d\alpha) \\ \sin(d\beta) & -\cos(d\beta) \cdot \sin(d\alpha) & \cos(d\beta) \cdot \cos(d\alpha) \end{bmatrix}$$

$$\Omega^C_{BA} = \dot\alpha T_{CA} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \dot\alpha \cdot \begin{bmatrix} \cos(d\beta) \\ 0 \\ \sin(d\beta) \end{bmatrix} = \dot\alpha \cdot X_A \Rightarrow$$

$$(X_A = \cos(d\beta) \cdot X_c + \sin(d\beta) \cdot Z_c)$$

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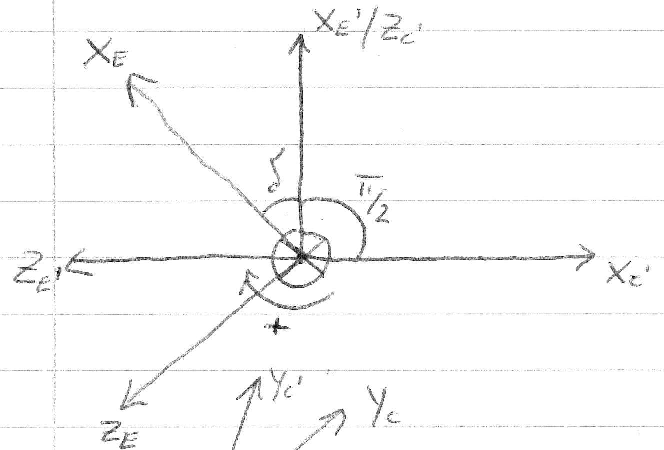
so $$\boxed{\Omega^{C}_{CA} = \dot{\beta}\,\bar{Y}_C + \dot{\alpha}\left(\cos(\partial\beta)\cdot\bar{X}_C + \sin(\partial\beta)\cdot\bar{Z}_C\right)}$$

c) $\bar{R} = -R\cdot\bar{Z}_E$

$$\frac{d\bar{R}}{dt} = -\frac{dR}{dt}\cdot\bar{Z}_E - R\cdot\frac{d\bar{Z}_E}{dt} = -\dot{R}\cdot\bar{Z}_E - R\left(\Omega^{E}_{EC}\times\bar{Z}_E\right)$$

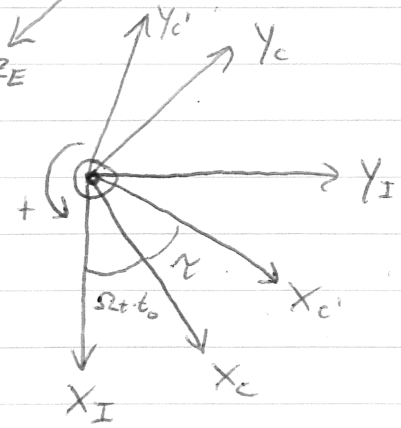
$$\Omega^{E}_{EC} = \Omega^{E}_{EC'} + \Omega^{E}_{C'C}$$

$$\Omega^{C}_{C'C} = \dot{\tau}\cdot\bar{Z}_C \Rightarrow \Omega^{E}_{C'C} = T_{EC}\cdot\Omega^{C}_{C'C}$$
$$\Omega^{C'}_{EC'} = \dot{\delta}\cdot\bar{Y}_C \Rightarrow \Omega^{E}_{EC'} = T_{EC'}\cdot\Omega^{C'}_{EC'}$$



$\otimes$ = $Y_{C'}$ going into the paper from $C \to E$

thus δ and $\frac{\pi}{2}$ should be −



$\odot$ = Z_C coming out of the paper from $C \to C'$

thus τ should be +

$$T_{EC} = T_{EC'}\cdot T_{C'C}$$
$$T_{EC'} = T_Y\left(-\delta-\frac{\pi}{2}\right)$$
$$T_{C'C} = T_Z(\tau)$$

$$T_{C'C} = T_Z(\tau) = \begin{bmatrix}\cos(\tau) & \sin(\tau) & 0\\ -\sin(\tau) & \cos(\tau) & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$T_{EC'} = T_Y\left(-\delta-\frac{\pi}{2}\right) = \begin{bmatrix}\cos(-\delta-\frac{\pi}{2}) & 0 & -\sin(-\delta-\frac{\pi}{2})\\ 0 & 1 & 0\\ \sin(-\delta-\frac{\pi}{2}) & 0 & \cos(-\delta-\frac{\pi}{2})\end{bmatrix} = \begin{bmatrix}-\sin(\delta) & 0 & \cos(\delta)\\ 0 & 1 & 0\\ -\cos(\delta) & 0 & -\sin(\delta)\end{bmatrix}$$

$$T_{EC} = \begin{bmatrix}-\sin(\delta)\cdot\cos(\tau) & -\sin(\delta)\cdot\sin(\tau) & \cos(\delta)\\ -\sin(\tau) & \cos(\tau) & 0\\ -\cos(\delta)\cdot\cos(\tau) & -\cos(\delta)\cdot\sin(\tau) & -\sin(\delta)\end{bmatrix}$$

$$\Omega^{E}_{c'c} = \dot{\tau} \cdot \begin{bmatrix} -\sin(\delta)\cdot\cos(\tau) & -\sin(\delta)\cdot\sin(\tau) & \cos(\delta) \\ -\sin(\delta) & \cos(\tau) & 0 \\ -\cos(\delta)\cdot\cos(\tau) & -\cos(\delta)\cdot\sin(\tau) & -\sin(\delta) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \dot{\tau} \cdot \begin{bmatrix} \cos(\delta) \\ 0 \\ -\sin(\delta) \end{bmatrix}$$

<u>so</u>

$$\Omega^{E}_{c'c} = \dot{\tau} \left(\cos(\delta)\cdot \overline{X_E} - \sin(\delta)\cdot \overline{Z_E} \right)$$

$$\Omega^{E}_{Ec'} = \dot{\delta} \begin{bmatrix} -\sin(\delta) & 0 & \cos(\delta) \\ 0 & 1 & 0 \\ -\cos(\delta) & 0 & -\sin(\delta) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \dot{\delta} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

<u>so</u> $\Omega^{E}_{Ec'} = \dot{\delta} \cdot \overline{Y_E}$

<u>thus</u> $\Omega^{E}_{Ec} = \dot{\delta}\, \overline{Y_E} + \dot{\tau}\left(\cos(\delta)\cdot \overline{X_E} - \sin(\delta)\cdot \overline{Z_E} \right)$

$$\Omega^{E}_{Ec} \times \overline{Z_E} = \begin{bmatrix} \dot{\tau}\cdot\cos(\delta) \\ \dot{\delta} \\ \dot{\tau}\cdot -\sin(\delta) \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \dot{\delta} \\ -\dot{\tau}\cdot\cos(\delta) \\ 0 \end{bmatrix}$$

<u>thus</u> $\dfrac{d\overline{R}}{dt} = -\dot{R}\cdot \overline{Z_E} - R\left(\dot{\delta}\cdot \overline{X_E} - \dot{\tau}\cdot\cos(\delta)\, \overline{Y_E} \right)$

<u>or</u>

$$\boxed{\frac{d\overline{R}}{dt} = \begin{bmatrix} -R\dot{\delta} \\ R\cdot\dot{\tau}\cdot\cos(\delta) \\ -\dot{R} \end{bmatrix} \begin{bmatrix} X_E \\ Y_E \\ Z_E \end{bmatrix}}$$

Question 3 a) i) $\dot{\alpha} \approx q - \frac{L}{m \cdot V}$

$$\left.\frac{d\dot{\alpha}}{dq}\right|_0 = 1 = A_q$$

$$\left.\frac{d\dot{\alpha}}{dV}\right|_0 = \frac{L_0}{m_0 \cdot V_0^2} = A_V$$

$$\left.\frac{d\dot{\alpha}}{dL}\right|_0 = \frac{-1}{m_0 \cdot V_0} = A_L$$

$$\Delta\dot{\alpha} = A_q \cdot \Delta q + A_V \cdot \Delta V + A_L \cdot \Delta L$$

<u>though</u> $L = f(\alpha, M) \Rightarrow \Delta L = ?$

$L = C_L \cdot \frac{1}{2} \rho \cdot V^2 \cdot S_0 = C_L \cdot q \cdot S_0$

$C_L = f(M, \alpha)$

<u>thus</u> $\Delta L = \frac{dL}{dM} \cdot \Delta M + \frac{dL}{d\alpha} \cdot \Delta\alpha$

<u>though</u> $M = f(V, a) \Rightarrow \Delta M = ?$

$M = \frac{V}{a} \Rightarrow \Delta M = \Delta\left(\frac{V}{a}\right)$ assuming $a = \text{constant} \Rightarrow$

<u>thus</u> $\Delta M = \frac{1}{a_0} \cdot \Delta V = \frac{M_0}{V_0} \cdot \Delta V$

$$\frac{dL}{dM} = \frac{d(C_L)}{dM} \cdot q \cdot S_0 + \frac{d(q)}{dM} \cdot C_L \cdot S_0$$

$$\frac{dL}{d\alpha} = \frac{d(C_L)}{d\alpha} \cdot q_0 \cdot S_0$$

$$\frac{d(q)}{dM} = \frac{1}{2} \cdot \frac{d(\rho V^2)}{dM} = \frac{1}{2} \cdot \frac{d(\rho M^2 a^2)}{dM} = \rho_0 \cdot M_0 \cdot a_0^2 = \frac{2 q_0}{M_0}$$

<u>so</u> $$\Delta L = \left(\frac{d(C_L)}{dM} \cdot q_0 \cdot S_0 + \frac{2 q_0}{M_0} \cdot C_{L_0} \cdot S_0\right) \frac{M_0}{V_0} \cdot \Delta V + \frac{d(C_L)}{d\alpha} \cdot q_0 \cdot S_0 \cdot \Delta\alpha$$

$$\Delta L = \left(\frac{dC_L}{dM} \cdot \frac{M_0}{V_0} + \frac{2 C_{L_0}}{V_0}\right) \cdot q_0 \cdot S_0 \cdot \Delta V + \frac{dC_L}{d\alpha} \cdot q_0 \cdot S_0 \cdot \Delta\alpha$$

<u>thus</u> $$\Delta\dot{\alpha} = \Delta q + \frac{L_0}{m_0 V_0^2} \cdot \Delta V - \frac{1}{m_0 V_0} \cdot \left(\left(\frac{dC_L}{dM} \cdot \frac{M_0}{V_0} + \frac{2 C_L}{V_0}\right) \cdot q_0 \cdot S_0 \cdot \Delta V + \frac{dC_L}{d\alpha} \cdot q_0 \cdot S_0 \cdot \Delta\alpha\right)$$

<u>or</u>

$$\boxed{\Delta\dot{\alpha} = \Delta q + \left(\frac{L_0}{m_0 V_0^2} - \frac{q_0 \cdot S_0}{m_0 \cdot V_0} \cdot \left(\frac{dC_L}{dM} \cdot \frac{M_0}{V_0} + \frac{2 C_{L_0}}{V_0}\right)\right) \Delta V - \frac{q_0 \cdot S_0}{m_0 \cdot V_0} \cdot \frac{dC_L}{d\alpha} \cdot \Delta\alpha}$$

ii) $\dot{q} = \dfrac{M_y}{I_{yy}} + \dfrac{I_{xz}}{I_{yy}}(r^2 - p^2) + \dfrac{I_{zz} - I_{xx}}{I_{yy}} \cdot p \cdot r$

$$\left.\frac{d\dot{q}}{dr}\right|_0 = 2\frac{I_{xz}}{I_{yy}} \cdot r_0 + \frac{I_{zz} - I_{xx}}{I_{yy}} \cdot p_0 = A_r$$

$$\left.\frac{d\dot{q}}{dp}\right|_0 = -2\frac{I_{xz}}{I_{yy}} \cdot p_0 + \frac{I_{zz} - I_{xx}}{I_{yy}} \cdot r_0 = A_p$$

$$\left.\frac{d\dot{q}}{dM_y}\right|_0 = \frac{1}{I_{yy}} = A_{M_y}$$

$\Delta\dot{q} = A_r \cdot \Delta r + A_p \cdot \Delta p + A_{M_y} \cdot \Delta M_y$

though $M_y = f(\mathcal{M}, M_{T,y}) \Rightarrow$
$\Delta M_y = ?$

thus $\Delta M_y = \Delta\mathcal{M} + \Delta M_{T,y}$

though $\mathcal{M} = f(M, \alpha)$

thus $\Delta\mathcal{M} = \dfrac{d\mathcal{M}}{dM}\Delta M + \dfrac{d\mathcal{M}}{d\alpha} \cdot \Delta\alpha$

using $\Delta M = \dfrac{M_0}{V_0}\Delta V$ (see Q39i)

$$\mathcal{M} = C_m \cdot \tfrac{1}{2} \cdot \rho \cdot V^2 \cdot S_0 \cdot \bar{c}_0 = C_m \cdot q \cdot S_0 \cdot \bar{c}_0$$

$$\frac{d\mathcal{M}}{d\alpha} = \frac{dC_m}{d\alpha} \cdot q_0 \cdot S_0 \cdot \bar{c}_0$$

$$\frac{d\mathcal{M}}{dM} = \frac{d(C_m)}{dM} \cdot q \cdot S_0 \cdot \bar{c}_0 + \frac{dq}{dM} \cdot C_{m_0} \cdot S_0 \cdot \bar{c}_0$$

$\dfrac{dq}{dM} = \dfrac{2q_0}{M_0}$ (See Q39i)

so $\Delta\mathcal{M} = \dfrac{dC_m}{dM} \cdot q_0 \cdot S_0 \cdot \bar{c}_0 \cdot \dfrac{M_0}{V_0} \cdot \Delta V + \dfrac{2q_0}{M_0} \cdot C_m \cdot S_0 \cdot \bar{c}_0 \cdot \dfrac{M_0}{V_0} \cdot \Delta V + \dfrac{dC_m}{d\alpha} \cdot q_0 \cdot S_0 \cdot \bar{c}_0 \cdot \Delta\alpha$

$$\Delta\mathcal{M} = \left(\frac{dC_m}{dM} \cdot \frac{M_0}{V_0} + \frac{2C_m}{V_0}\right) \cdot q_0 \cdot S_0 \cdot \bar{c}_0 \cdot \Delta V + \frac{dC_m}{d\alpha} \cdot q_0 \cdot S_0 \cdot \bar{c}_0 \cdot \Delta\alpha$$

thus

$$\Delta\dot{q} = \left(2\frac{I_{xz}}{I_{yy}} \cdot r_0 + \frac{I_{zz} - I_{xx}}{I_{yy}} \cdot p_0\right) \cdot \Delta r + \left(-2\frac{I_{xz}}{I_{yy}} \cdot p_0 + \frac{I_{zz} - I_{xx}}{I_{yy}} \cdot r_0\right) \cdot \Delta p$$
$$+ \frac{1}{I_{yy}} \cdot \Delta M_{T,y} + \frac{dC_m}{d\alpha} \cdot \frac{q_0 \cdot S_0 \cdot \bar{c}_0}{I_{yy}} \cdot \Delta\alpha + \left(\frac{dC_m}{dM} \cdot \frac{M_0}{V_0} + \frac{2C_m}{V_0}\right) \cdot \frac{q_0 \cdot S_0 \cdot \bar{c}_0}{I_{yy}} \cdot \Delta V$$

with $\Delta\dot{x} = A \cdot \Delta x + B \cdot \Delta u$ this becomes:

$$\Delta \dot{q} = \begin{bmatrix} \left(\frac{dC_m}{dM}\cdot\frac{M_0}{V_0} + \frac{2C_m}{V_0}\right)\cdot\frac{q_0\cdot S_0\cdot \bar{c}_0}{I_{yy}} \\ -2\frac{I_{xz}}{I_{yy}}\cdot p_0 + \frac{I_{zz}-I_{xx}}{I_{yy}}\cdot r_0 \\ +2\frac{I_{xz}}{I_{yy}}\cdot r_0 + \frac{I_{zz}-I_{xx}}{I_{yy}}\cdot p_0 \\ \frac{dC_m}{d\alpha}\cdot\frac{q_0\cdot S_0\cdot \bar{c}_0}{I_{yy}} \end{bmatrix}^T \cdot \begin{bmatrix} \Delta V \\ \Delta p \\ \Delta r \\ \Delta \alpha \end{bmatrix} + \frac{1}{I_{yy}}\cdot \Delta M_{T_y}$$

b) Credits to "Studentje 1989" for providing the elaborations of Q3b

In state-space form we get:

$$\begin{bmatrix} \Delta \dot{q} \\ \Delta \dot{\alpha} \end{bmatrix} = \begin{bmatrix} 0 & C_{m_\alpha}\cdot\frac{q_0\cdot S_{ref}\, c_{ref}}{I_{yy}} \\ 1 & -C_{L_\alpha}\cdot\frac{q_0\cdot S_{ref}}{m\cdot V_0} \end{bmatrix}\begin{bmatrix} \Delta q \\ \Delta \alpha \end{bmatrix} + \begin{bmatrix} \frac{1}{I_{yy}} \\ 0 \end{bmatrix}\cdot \Delta M_{T_y}$$

λ = eigenvalue $\qquad A\cdot\mu = \lambda\mu \Rightarrow (A-\lambda I)\cdot\mu = 0$

μ = eigen vector $\qquad$ non-trivial solution for $|A-\lambda I| = 0$

$$|A-\lambda I| = \begin{bmatrix} -\lambda & C_{m_\alpha}\cdot\frac{q_0\cdot S_{ref}\cdot c_{ref}}{I_{yy}} \\ 1 & -C_{L_\alpha}\cdot\frac{q_0\cdot S_{ref}}{m\cdot V_0} - \lambda \end{bmatrix} \xrightarrow{\text{determinant}} = \underbrace{\lambda^2 + C_{L_\alpha}\cdot\frac{q_0\cdot S_{ref}}{m\cdot V_0}\lambda - C_{m_\alpha}\frac{q_0\cdot S_{ref}\cdot c_{ref}}{I_{yy}}}_{=0}$$

so

$$\boxed{\lambda_{1,2} = \frac{-C_{L_\alpha}\cdot\frac{q_0\cdot S_{ref}}{m\cdot V_0} \pm \sqrt{\left(C_{L_\alpha}\cdot\frac{q_0\cdot S_{ref}}{m\cdot V_0}\right)^2 + 4\cdot C_{m_\alpha}\cdot\frac{q_0\cdot S_{ref}\cdot c_{ref}}{I_{yy}}}}{2}}$$

c) $\lambda_{1,2} = -1{,}9744 \cdot 10^{-2} \pm 1{,}5534 \cdot 10^{1} j = \xi + \eta j$

$$\zeta = \frac{-\xi_0}{\sqrt{\xi^2 + \eta^2}} = -\frac{-1{,}9744 \cdot 10^{-2}}{\sqrt{(-1{,}9744 \cdot 10^{-2})^2 + (1{,}5534 \cdot 10^{1})^2}} \approx 0{,}00127\ (-)$$

$$\omega_0 = \sqrt{\xi^2 + \eta^2}\,\frac{V}{b} = \sqrt{(-1{,}9744 \cdot 10^{-2})^2 + (1{,}5534 \cdot 10^{1})^2} \cdot \frac{V}{b} \approx 15{,}53\,\frac{V}{b}\ (\text{rad/s})$$

$$\omega_n = \omega_0 \cdot \sqrt{1-\zeta^2} = 15{,}53\,\frac{V}{b} \cdot \sqrt{1 - 0{,}00127^2} \approx 15{,}53\,\frac{V}{b}\ (\text{rad/s})$$

$$P = \frac{2\pi}{\omega_n} \approx \frac{2\pi}{\eta} \cdot \frac{b}{V} = \frac{b}{V} \cdot \frac{2\pi}{1{,}5534 \cdot 10^{1}} \approx \frac{b}{V} \cdot 0{,}4045\ (s)$$

$$\left.\begin{aligned} T_{1/2} &= \frac{\ln(1/2)}{\xi} \cdot \frac{b}{V} \\ T_2 &= \frac{\ln(2)}{\xi} \cdot \frac{b}{V} \end{aligned}\right\}$$

as $\xi > 0$, the motion is damped, thus $T_{1/2}$ can be used. This is logical as using T_2 would give a negative result ($\xi < 0$)

$$T_{1/2} = \frac{\ln(1/2)}{-1{,}9744 \cdot 10^{-2}} \cdot \frac{b}{V} \approx 35{,}1\,\frac{b}{V}\ (s)$$

$$\boxed{\begin{aligned} &\text{so}\quad \zeta \approx 0{,}00127\ (-) \\ &\phantom{\text{so}}\quad P \approx 0{,}4045 \cdot \tfrac{b}{V}\ (s) \\ &\phantom{\text{so}}\quad T_{1/2} \approx 35{,}1 \cdot \tfrac{b}{V}\ (s) \end{aligned}}$$

d) The mode under c) is stable because $\xi > 0$, thus the motion is damped.