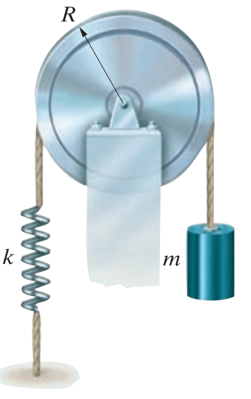
Problem 21.1 In Active Example 21.1, suppose that the pulley has radius $R = 100$ mm and its moment of inertia is $I = 0.005$ kg-m^2. The mass $m = 2$ kg, and the spring constant is $k = 200$ N/m. If the mass is displaced downward from its equilibrium position and released, what are the period and frequency of the resulting vibration?



Solution: From Active Example 21.1 we have

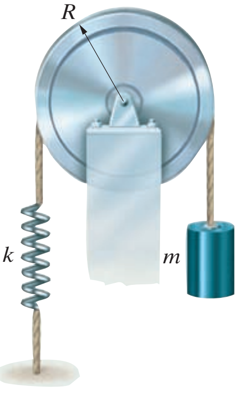
$$\omega = \sqrt{\frac{k}{m + \dfrac{I}{R^2}}} = \sqrt{\frac{(200\ \text{N/m})}{(2\ \text{kg}) + \dfrac{(0.005\ \text{kg-m}^2)}{(0.1\ \text{m})^2}}} = 8.94\ \text{rad/s}$$

Thus

$$\tau = \frac{2\pi}{\omega} = \frac{2\pi}{8.94\ \text{rad/s}} = 0.702\ \text{s}, \quad f = \frac{1}{\tau} = 1.42\ \text{Hz}.$$

$$\boxed{\tau = 0.702\ \text{s}, \quad f = 1.42\ \text{Hz}.}$$

Problem 21.2 In Active Example 21.1, suppose that the pulley has radius $R = 4$ cm and its moment of inertia is $I = 0.06$ kg-m^2. The suspended object weighs 5 N, and the spring constant is $k = 10$ N/m. The system is initially at rest in its equilibrium position. At $t = 0$, the suspended object is given a downward velocity of 1 m/s. Determine the position of the suspended object relative to its equilibrium position as a function of time.



Solution: From Active Example 21.1 we have

$$\omega = \sqrt{\frac{k}{m + \dfrac{I}{R^2}}} = \sqrt{\frac{(10\ \text{N/m})}{\left(\dfrac{5\ \text{N}}{9.81\ \text{m/s}^2}\right) + \dfrac{(0.06\ \text{kg-m}^2)}{(0.04\ \text{m})^2}}} = 0.51\ \text{rad/s}$$

The general solution is

$$x = A\sin\omega t + B\cos\omega t, \quad v = A\omega\cos\omega t - B\omega\sin\omega t.$$

Putting in the initial conditions, we have

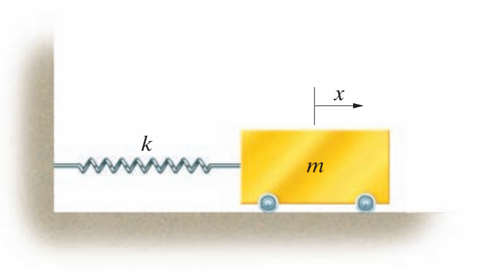
$$x(t = 0) = B = 0 \Rightarrow B = 0,$$

$$v(t = 0) = A\omega = (1\ \text{ft/s}) \Rightarrow A = \frac{1\ \text{m/s}}{0.51\ \text{rad/s}} = 1.96\ \text{m}.$$

Thus the equation is

$$\boxed{x = 1.96 \sin 0.51t\ (\text{m}).}$$

Problem 21.3 The mass $m = 4$ kg. The spring is unstretched when $x = 0$. The period of vibration of the mass is measured and determined to be 0.5 s. The mass is displaced to the position $x = 0.1$ m and released from rest at $t = 0$. Determine its position at $t = 0.4$ s.



Solution: Knowing the period, we can find the natural frequency

$$\omega = \frac{2\pi}{\tau} = \frac{2\pi}{0.5 \text{ s}} = 12.6 \text{ rad/s}.$$

The general solution is

$$x = A \sin \omega t + B \cos \omega t, \quad v = A\omega \cos \omega t - B\omega \sin \omega t.$$

Putting in the initial conditions, we have

$x(t = 0) = B = 0.1 \text{ m} \Rightarrow B = 0.1 \text{ m},$

$v(t = 0) = A\omega = 0 \Rightarrow A = 0.$

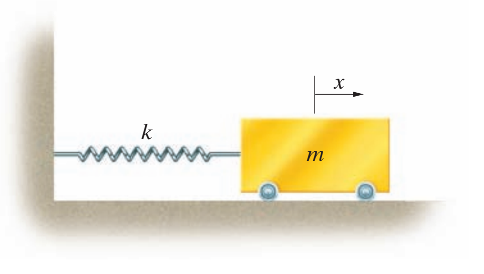
Thus the equation is

$$x = (0.1 \text{ m}) \cos(12.6 \text{ rad/s } t)$$

At the time $t = 0.4$ s, we find

$$\boxed{x = 0.0309 \text{ m.}}$$

Problem 21.4 The mass $m = 4$ kg. The spring is unstretched when $x = 0$. The frequency of vibration of the mass is measured and determined to be 6 Hz. The mass is displaced to the position $x = 0.1$ m and given a velocity $dx/dt = 5$ m/s at $t = 0$. Determine the amplitude of the resulting vibration.



Solution: Knowing the frequency, we can find the natural frequency

$$\omega = 2\pi f = 2\pi(6 \text{ Hz}) = 37.7 \text{ rad/s}.$$

The general solution is

$$x = A \sin \omega t + B \cos \omega t, \quad v = A\omega \cos \omega t - B\omega \sin \omega t.$$

Putting in the initial conditions, we have

$x(t = 0) = B = 0.1 \text{ m} \Rightarrow B = 0.1 \text{ m},$

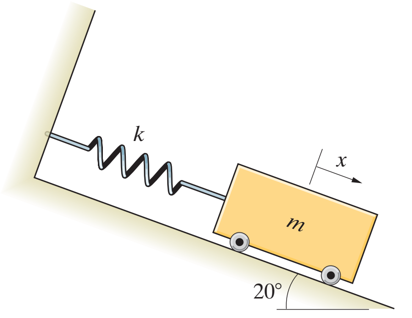
$v(t = 0) = A\omega = 5 \text{ m/s} \Rightarrow A = \dfrac{5 \text{ m/s}}{37.7 \text{ rad/s}} = 0.133 \text{ m}.$

The amplitude of the motion is given by

$\sqrt{A^2 + B^2} = \sqrt{(0.1 \text{ m})^2 + (0.133 \text{ m})^2} = 0.166 \text{ m}.$

$\boxed{\text{Amplitude} = 0.166 \text{ m.}}$

Problem 21.5 The mass $m = 4$ kg and the spring constant is $k = 64$ N/m. For vibration of the spring-mass oscillator relative to its equilibrium position, determine (a) the frequency in Hz and (b) the period.



Solution: Since the vibration is around the equilibrium position, we have

(a) $$\boxed{\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{64 \text{ N/m}}{4 \text{ kg}}} = 4 \text{ rad/s}\left(\frac{1 \text{ cycle}}{2\pi \text{ rad}}\right) = 0.637 \text{ Hz}}$$

(b) $$\boxed{\tau = \frac{2\pi}{\omega} = \frac{2\pi}{4 \text{ rad/s}} = 1.57 \text{ s}}$$

Problem 21.6 The mass $m = 4$ kg and the spring constant is $k = 64$ N/m. The spring is unstretched when $x = 0$. At $t = 0$, $x = 0$ and the mass has a velocity of 2 m/s down the inclined surface. What is the value of x at $t = 0.8$ s?

Solution: The equation of motion is

$$m\frac{d^2x}{dt^2} + kx = mg\sin 20^\circ \Rightarrow \frac{d^2x}{dt^2} + \left(\frac{k}{m}\right)x = g\sin 20^\circ$$

Putting in the numbers we have

$$\frac{d^2x}{dt^2} + (16\text{s}^{-2})x = 3.36 \text{ m/s}^2$$

The solution to this nonhomogeneous equation is

$$x = A\sin([4 \text{ s}^{-1}]t) + B\cos([4 \text{ s}^{-1}]t) + 0.210 \text{ m}$$

Using the initial conditions we have

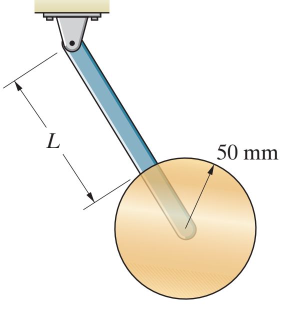
$$\left.\begin{aligned} 0 &= B + 0.210 \text{ m} \\ 2 \text{ m/s} &= A(4 \text{ s}^{-1}) \end{aligned}\right\} \Rightarrow A = 0.5 \text{ m}, \; B = -0.210 \text{ m}$$

The motion is

$$x = (0.5 \text{ m})\sin([4 \text{ s}^{-1}]t) - (0.210 \text{ m})\cos([4 \text{ s}^{-1}]t) + 0.210 \text{ m}$$

At $t = 0.8$ s we have $\boxed{x = 0.390 \text{ m}}$

Problem 21.7 Suppose that in a mechanical design course you are asked to design a pendulum clock, and you begin with the pendulum. The mass of the disk is 2 kg. Determine the length L of the bar so that the period of small oscillations of the pendulum is 1 s. For this preliminary estimate, neglect the mass of the bar.



Solution: Given $m = 2$ kg, $r = 0.05$ m

For small angles the equation of motion is

$$\left(\frac{1}{2}mr^2 + m[r+L]^2\right)\frac{d^2\theta}{dt^2} + mg(L+r)\theta = 0$$

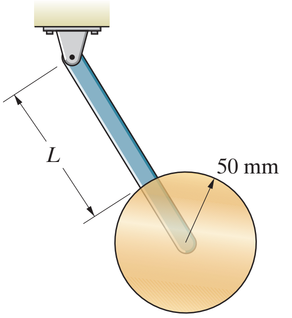
$$\Rightarrow \frac{d^2\theta}{dt^2} + \left(\frac{2g[L+r]}{r^2 + [r+L]^2}\right)\theta = 0$$

The period is $\tau = 2\pi\sqrt{\dfrac{r^2 + 2(r+L)^2}{2g(L+r)}}$

Set $\tau = 1$ s and solve to find $\boxed{L = 0.193 \text{ m}}$

Problem 21.8 The mass of the disk is 2 kg and the mass of the slender bar is 0.4 kg. Determine the length L of the bar so that the period of small oscillations of the pendulum is 1 s.

Strategy: Draw a graph of the value of the period for a range of lengths L to estimate the value of L corresponding to a period of 1 s.



Solution: We have

$$m_d = 2 \text{ kg}, \ m_b = 0.4 \text{ kg}, \ r = 0.05 \text{ m}, \ \tau = 1 \text{ s}.$$

The moment of inertia of the system about the pivot point is

$$I = \frac{1}{3}m_b L^2 + \frac{1}{2}m_d r^2 + m_d(L+r)^2.$$

The equation of motion for small amplitudes is

$$\left(\frac{1}{3}m_b L^2 + \frac{1}{2}m_d r^2 + m_d(L+r)^2\right)\ddot{\theta} + \left(m_b\frac{L}{2} + m_d[L+r]\right)g\theta = 0$$

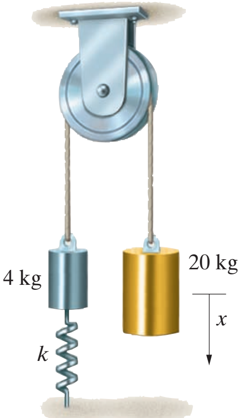
Thus, the period is given by

$$\tau = 2\pi\sqrt{\frac{\frac{1}{3}m_b L^2 + \frac{1}{2}m_d r^2 + m_d(L+r)^2}{\left(m_b + \frac{L}{2} + m_d[L+r]\right)g}}$$

This is a complicated equation to solve. You can draw the graph and get an approximate solution, or you can use a root solver in your calculator or on your computer.

Using a root solver, we find $\boxed{L = 0.203 \text{ m.}}$

Problem 21.9 The spring constant is $k = 785$ N/m. The spring is unstretched when $x = 0$. Neglect the mass of the pulley, that is, assume that the tension in the rope is the same on both sides of the pulley. The system is released from rest with $x = 0$. Determine x as a function of time.



Solution: We have the equations

$$T - (4 \text{ kg})(9.81 \text{ m/s}^2) - (785 \text{ N/m})x = (4 \text{ kg})\ddot{x}$$

$$T - (20 \text{ kg})(9.81 \text{ m/s}^2) = -(20 \text{ kg})\ddot{x}$$

If we eliminate the tension T from these equations, we find

$$(24 \text{ kg})\ddot{x} + (785 \text{ N/m})x = (16 \text{ kg})(9.81 \text{ m/s}^2)$$

$$\ddot{x} + (5.72 \text{ rad/s})^2 x = 6.54 \text{ m/s}^2.$$

The solution of this equation is

$$x = A\sin\omega t + B\cos\omega t + (0.200 \text{ m}), \quad v = A\omega\cos\omega t - B\omega\sin\omega t.$$

Using the initial conditions, we have

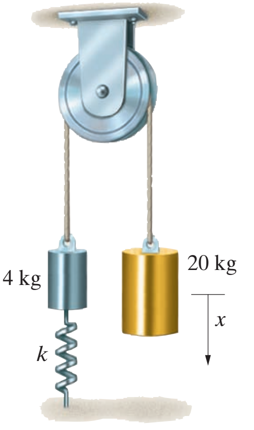
$$x(t=0) = B + (0.200 \text{ m}) \Rightarrow B = -0.200 \text{ m},$$

$$v(t=0) = A\omega = 0 \Rightarrow A = 0.$$

Thus the equation is

$$\boxed{x = (0.200 \text{ m})(1 - \cos[5.72 \text{ rad/s } t]).}$$

Problem 21.10 The spring constant is $k = 785$ N/m. The spring is unstretched with $x = 0$. The radius of the pulley is 125 mm, and moment of inertia about its axis is $I = 0.05$ kg-m^2. The system is released from rest with $x = 0$. Determine x as a function of time.



Solution: Let T_1 be the tension in the rope on the left side, and T_2 be the tension in the rope on the right side. We have the equations

$$T_1 - (4\text{ kg})(9.81\text{ m/s}^2) - (785\text{ N/m})x = (4\text{ kg})\ddot{x}$$

$$T_2 - (20\text{ kg})(9.81\text{ m/s}^2) = -(20\text{ kg})\ddot{x}$$

$$(T_2 - T_1)(0.125\text{ m}) = (0.05\text{ kg-m}^2)\frac{\ddot{x}}{(0.125\text{ m})}$$

If we eliminate the tensions T_1 and T_2 from these equations, we find

$$(27.2\text{ kg})\ddot{x} + (785\text{ N/m})x = (16\text{ kg})(9.81\text{ m/s}^2)$$

$$\ddot{x} + (5.37\text{ rad/s})^2 x = 5.77\text{ m/s}^2.$$

The solution of this equation is

$$x = A\sin\omega t + B\cos\omega t + (0.200\text{ m}),\;\; v = A\omega\cos\omega t - B\omega\sin\omega t.$$

Using the initial conditions, we have

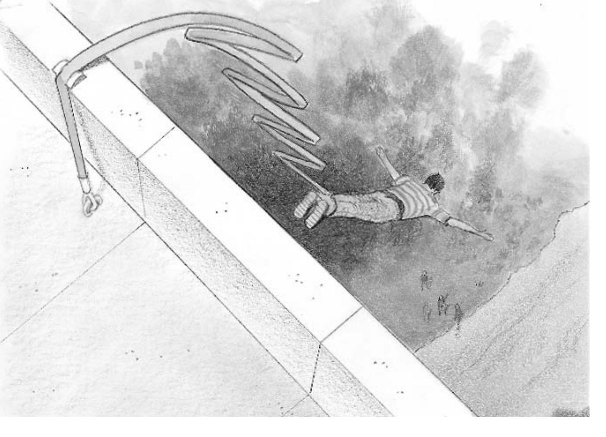
$$x(t = 0) = B + (0.200\text{ m}) \Rightarrow B = -0.200\text{ m},$$

$$v(t = 0) = A\omega = 0 \Rightarrow A = 0.$$

Thus the equation is

$$\boxed{x = (0.200\text{ m})(1 - \cos[5.37\text{ rad/s }t]).}$$

Problem 21.11 A "bungee jumper" who weighs 711.7 N leaps from a bridge above a river. The bungee cord has an unstretched length of 18.3 m, and it stretches an additional 12.2 m before the jumper rebounds. Model the cord as a linear spring. When his motion has nearly stopped, what are the period and frequency of his vertical oscillations? (You can't model the cord as a linear spring during the early part of his motion. Why not?)



Solution: Use energy to find the spring constant

$$T_1 = 0,\;\; V_1 = 0,\;\; T_2 = 0,\;\; V_2 = -(711.7\text{ N})(30.5\text{ m}) + \frac{1}{2}k(12.2\text{ m})^2$$

$$T_1 + V_1 = T_2 + V_2 \;\Rightarrow\; k = 291.9\text{ N/m}$$

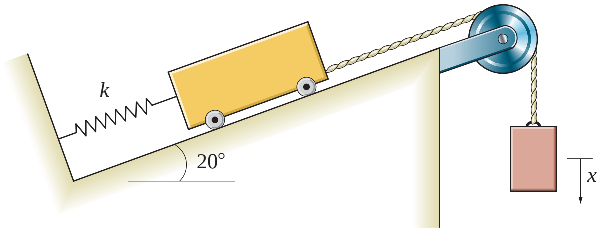
$$\boxed{\begin{aligned} f &= \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{291.9\text{ N/m}}{(711.7\text{ N})/(9.81\text{ m/s}^2)}} = 0.319\text{ Hz},\\ \tau &= \frac{1}{f} = 3.13\text{ s}\end{aligned}}$$

The cord cannot be modeled as a linear spring during the early motion because it is slack and does not support a compressive load.

Problem 21.12 The spring constant is $k = 800$ N/m, and the spring is unstretched when $x = 0$. The mass of each object is 30 kg. The inclined surface is smooth. Neglect the mass of the pulley. The system is released from rest with $x = 0$.

(a) Determine the frequency and period of the resulting vibration.

(b) What is the value of x at $t = 4$ s?



Solution: The equations of motion are

$$T - mg\sin\theta - kx = m\ddot{x}, \quad T - mg = -m\ddot{x}.$$

If we eliminate T, we find

$$2m\ddot{x} + kx = mg(1 - \sin\theta), \quad \ddot{x} + \frac{k}{2m}x = \frac{1}{2}g(1 - \sin\theta),$$

$$\ddot{x} + \frac{1}{2}\left(\frac{800 \text{ N/m}}{30 \text{ kg}}\right)x = \frac{1}{2}(9.81 \text{ m/s}^2)(1 - \sin 20^\circ),$$

$$\ddot{x} + (3.65 \text{ rad/s})^2 x = 3.23 \text{ m/s}^2.$$

(a) The natural frequency, frequency, and period are

$$\omega = 3.65 \text{ rad/s}, \quad f = \frac{\omega}{2\pi} = 0.581 \text{ Hz}, \quad \tau = \frac{1}{f} = 1.72 \text{ s}.$$

$$\boxed{\begin{aligned} f &= 0.581 \text{ s}, \\ \tau &= 1.72 \text{ s}. \end{aligned}}$$

(b) The solution to the differential equation is

$$x = A\sin\omega t + B\cos\omega t + 0.242 \text{ m}, \quad v = A\omega\cos\omega t - B\omega\sin\omega t.$$

Putting in the initial conditions, we have

$$x(t = 0) = B + 0.242 \text{ m} = 0 \Rightarrow B = -0.242 \text{ m},$$

$$v(t = 0) = A\omega = 0 \Rightarrow A = 0.$$

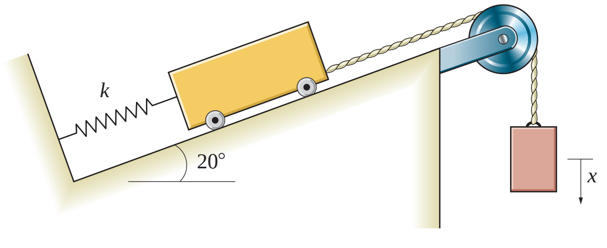
Thus the equation is $x = (0.242 \text{ m})(1 - \cos[3.65 \text{ rad/s } t])$

At $t = 4$ s we have $\boxed{x = 0.351 \text{ m.}}$

Problem 21.13 The spring constant is $k = 800$ N/m, and the spring is unstretched when $x = 0$. The mass of each object is 30 kg. The inclined surface is smooth. The radius of the pulley is 120 mm and its moment of inertia is $I = 0.03$ kg-m^2. At $t = 0$, $x = 0$ and $dx/dt = 1$ m/s.

(a) Determine the frequency and period of the resulting vibration.

(b) What is the value of x at $t = 4$ s?



Solution: Let T_1 be the tension in the rope on the left of the pulley, and T_2 be the tension in the rope on the right of the pulley. The equations of motion are

$$T_1 - mg\sin\theta - kx = m\ddot{x}, \; T_2 - mg = -m\ddot{x}, \; (T_2 - T_1)r = I\frac{\ddot{x}}{r}.$$

If we eliminate T_1 and T_2, we find

$$\left(2m + \frac{I}{r^2}\right)\ddot{x} + kx = mg(1 - \sin\theta),$$

$$\ddot{x} + \frac{kr^2}{2mr^2 + I}x = \frac{mgr^2}{2mr^2 + I}(1 - \sin\theta), \quad \ddot{x} + (3.59 \text{ rad/s})^2 x = 3.12 \text{ m/s}^2.$$

(a) The natural frequency, frequency, and period are

$$\omega = 3.59 \text{ rad/s}, \quad f = \frac{\omega}{2\pi} = 0.571 \text{ Hz}, \quad \tau = \frac{1}{f} = 1.75 \text{ s}.$$

$$\boxed{\begin{aligned} f &= 0.571 \text{ s}, \\ \tau &= 1.75 \text{ s}. \end{aligned}}$$

(b) The solution to the differential equation is

$$x = A\sin\omega t + B\cos\omega t + 0.242 \text{ m}, \quad v = A\omega\cos\omega t - B\omega\sin\omega t.$$

Putting in the initial conditions, we have

$$x(t = 0) = B + 0.242 \text{ m} = 0 \Rightarrow B = -0.242 \text{ m},$$

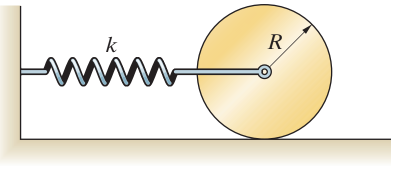
$$v(t = 0) = A\omega = (1 \text{ m/s}) \Rightarrow A = \frac{1 \text{ m/s}}{3.59 \text{ rad/s}} = 0.279 \text{ m}.$$

Thus the equation is

$$x = (0.242 \text{ m})(1 - \cos[359 \text{ rad/s} t]) + (0.279 \text{ m})\sin[359 \text{ rad/s } t]$$

At $t = 4$ s we have $\boxed{x = 0.567 \text{ m.}}$

Problem 21.14 The 89 N disk rolls on the horizontal surface. Its radius is $R = 152.4$ mm. Determine the spring constant k so that the frequency of vibration of the system relative to its equilibrium position is $f = 1$ Hz.



Solution: Given $m = \dfrac{89\text{ N}}{9.81\text{ m/s}^2}$, $R = 0.152$ m

The equations of motion

$$\left.\begin{aligned} -kx + F &= m\frac{d^2x}{dt^2} \\ FR &= -\left(\frac{1}{2}mR^2\right)\frac{1}{R}\frac{d^2x}{dt^2} \end{aligned}\right\} \Rightarrow \frac{d^2x}{dt^2} + \left(\frac{2k}{3m}\right)x = 0$$

We require that $f = \dfrac{1}{2\pi}\sqrt{\dfrac{2k}{3m}} = 1\text{ Hz} \Rightarrow \boxed{k = 537\text{ N/m}}$

Problem 21.15 The 89 N disk rolls on the horizontal surface. Its radius is $R =$ 152.4 mm. The spring constant is $k = 218.9$ N/m. At $t = 0$, the spring is unstretched and the disk has a clockwise angular velocity of 2 rad/s. What is the amplitude of the resulting vibrations of the center of the disk?

Solution: See the solution to 21.14

$$\frac{d^2x}{dt^2} + \left(\frac{2k}{3m}\right)x = 0 \Rightarrow \frac{d^2x}{dt^2} + (16.09\text{ rad/s})^2 x = 0$$

The solution is

$$x = A\cos([16.09\text{ rad/s}]t) + B\sin([16.09\text{ rad/s}]t)$$

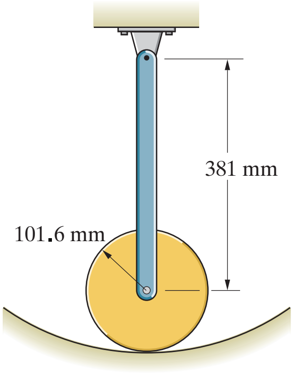
Using the initial conditions

$$\left.\begin{aligned} 0 &= A \\ (2\text{ rad/s})(0.152\text{ m}) &= B(16.09\text{ rad/s}) \end{aligned}\right\} \Rightarrow A = 0,\ B = 0.076\text{ m}$$

Thus $x = (0.076\text{ m})\sin([16.09\text{ rad/s}]t)$

The amplitude is $\boxed{B = 0.076\text{ m}}$

Problem 21.16 The 8.9 N bar is pinned to the 22.2 N disk. The disk rolls on the circular surface. What is the frequency of small vibrations of the system relative to its vertical equilibrium position?



Solution: Use energy methods.

$$T = \frac{1}{2}\left(\frac{1}{3}\left[\frac{8.9\text{ N}}{9.81\text{ m/s}^2}\right][0.381\text{ m}]^2\right)\omega_{\text{bar}}^2 + \frac{1}{2}\left(\frac{22.2\text{ N}}{9.81\text{ m/s}^2}\right)(0.381\text{ m})^2\,\omega_{\text{bar}}^2$$

$$+\frac{1}{2}\left(\frac{1}{2}\left[\frac{22.2\text{ N}}{9.81\text{ m/s}^2}\right][0.102\text{ m}]^2\right)\left(\frac{0.381}{0.102}\right)^2\omega_{\text{bar}}^2 = (0.268\text{ N-m-s}^2)\omega_{\text{bar}}^2$$

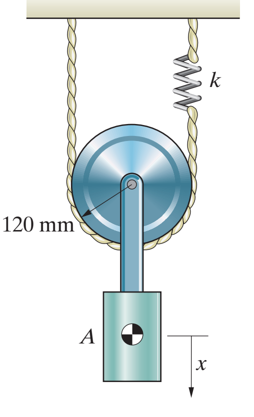
$$V = -(8.9\text{ N})(0.191\text{ m})\cos\theta - (22.2\text{ N})(0.381\text{ m})\cos\theta = -(10.17\text{ N-m})\cos\theta$$

Differentiating and linearizing we find

$$(0.537\text{ N-m-s}^2)\frac{d^2\theta}{dt^2} + (10.17\text{ N-m})\theta = 0 \;\Rightarrow\; \frac{d^2\theta}{dt^2} + (4.35\text{ rad/s})^2\theta = 0$$

$$\boxed{f = \frac{4.35\text{ rad/s}}{2\pi\text{ rad}} = 0.692\text{ Hz}}$$

Problem 21.17 The mass of the suspended object A is 4 kg. The mass of the pulley is 2 kg and its moment of inertia is 0.018 N-m^2. For vibration of the system relative to its equilibrium position, determine (a) the frequency in Hz and (b) the period.



Solution: Use energy methods

$$T = \frac{1}{2}(6\text{ kg})v^2 + \frac{1}{2}(0.018\text{ kg-m}^2)\left(\frac{v}{0.12\text{ m}}\right)^2 = (3.625\text{ kg})v^2$$

$$V = -(6\text{ kg})(9.81\text{ m/s}^2)x + \frac{1}{2}(150\text{ N/m})(2x)^2$$

$$= (300\text{ N/m})x^2 - (58.9\text{ N})x$$

Differentiating we have

$$(7.25\text{ kg})\frac{d^2x}{dt^2} + (600\text{ N/m})x = 58.9\text{ N}$$

$$\Rightarrow \frac{d^2x}{dt^2} + (9.10\text{ rad/s})^2x = 8.12\text{ m/s}^2$$

(a) $\boxed{f = \frac{9.10\text{ rad/s}}{2\pi\text{ rad}} = 1.45\text{ Hz}}$

(b) $\boxed{\tau = \frac{1}{f} = 0.691\text{ s}}$

Problem 21.18 The mass of the suspended object A is 4 kg. The mass of the pulley is 2 kg and its moment of inertia is 0.018 N-m^2. The spring is unstretched when $x = 0$. At $t = 0$, the system is released from rest with $x = 0$. What is the velocity of the object A at $t = 1$ s?

Solution: See the solution to 21.17

The motion is given by

$$x = A\cos([9.10 \text{ rad/s}]t) + B\sin([9.10 \text{ rad/s}]t) + 0.0981 \text{ m}$$

$$v = \frac{dx}{dt} = (9.10 \text{ rad/s})\{-A\sin([9.10 \text{ rad/s}]t) + B\cos([9.10 \text{ rad/s}]t)\}$$

Use the initial conditions

$$\left.\begin{array}{r} A + 0.0981 \text{ m} = 0 \\ B(0.910 \text{ rad/s}) = 0 \end{array}\right\} \Rightarrow A = -0.0981 \text{ m}, \; B = 0$$

Thus we have

$$x = (0.0981 \text{ m})(1 - \cos([9.10 \text{ rad/s}]t))$$

$$v = \frac{dx}{dt} = -(0.892 \text{ m/s})\sin([9.10 \text{ rad/s}]t)$$

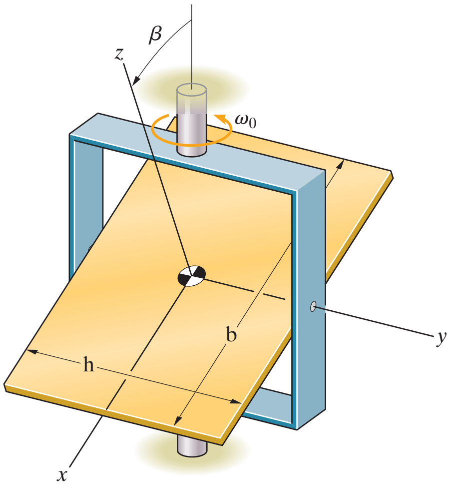
At $t = 1$ s, $\boxed{v = 0.287 \text{ m/s}}$

Problem 21.19 The thin rectangular plate is attached to the rectangular frame by pins. The frame rotates with constant angular velocity $\omega_0 = 6$ rad/s.The angle β between the z axis of the body-fixed coordinate system and the vertical is governed by the equation

$$\frac{d^2\beta}{dt^2} = -\omega_0^2 \sin\beta\cos\beta.$$

Determine the frequency of small vibrations of the plate relative to its horizontal position.

Strategy: By writing $\sin\beta$ and $\cos\beta$ in terms of their Taylor series and assuming that β is small, show that the equation governing β can be expressed in the form of Eq. (21.5).



Solution:

$$\frac{d^2\beta}{dt^2} + \omega_0^2\sin\beta\cos\beta = 0 \;\Rightarrow\; \frac{d^2\beta}{dt^2} + \frac{1}{2}\omega_0^2\sin 2\beta = 0$$

Linearizing we have

$$\frac{d^2\beta}{dt^2} + \frac{1}{2}\omega_0^2 2\beta = 0 \;\Rightarrow\; \frac{d^2\beta}{dt^2} + \omega_0^2\beta = 0$$

$$\omega = \omega_0 = 6 \text{ rad/s} \;\Rightarrow\; \boxed{f = \frac{6 \text{ rad/s}}{2\pi \text{ rad}} = 0.954 \text{ Hz}}$$

Problem 21.20 Consider the system described in Problem 21.19. At $t = 0$, the angle $\beta = 0.01$ rad and $d\beta/dt = 0$. Determine β as a function of time.

Solution: See 21.19

The equation of motion is

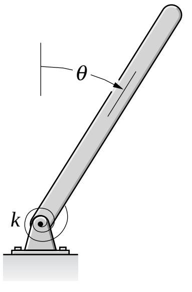
$$\frac{d^2\beta}{dt^2} + (6 \text{ rad/s})^2\beta = 0$$

The solution is

$$\boxed{\beta = (0.01 \text{ rad})\cos([6.00 \text{ rad/s}]t)}$$

Problem 21.21 A slender bar of mass m and length l is pinned to a fixed support as shown. A torsional spring of constant k attached to the bar at the support is unstretched when the bar is vertical. Show that the equation governing small vibrations of the bar from its vertical equilibrium position is

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0, \quad \text{where } \omega^2 = \frac{(k - \frac{1}{2}mgl)}{\frac{1}{3}ml^2}.$$



Solution: The system is conservative. The pivot is a fixed point. The moment of inertia about the fixed point is $I = mL^2/3$. The kinetic energy of the motion of the bar is

$$T = \frac{1}{2}I\left(\frac{d\theta}{dt}\right)^2.$$

The potential energy is the sum of the energy in the spring and the gravitational energy associated with the change in height of the center of mass of the bar,

$$V = \frac{1}{2}k\theta^2 - \frac{mgL}{2}(1 - \cos\theta).$$

For a conservative system,

$$T + V = \text{const.} = \frac{1}{2}I\left(\frac{d\theta}{dt}\right)^2 + \frac{1}{2}k\theta^2 - \frac{mgL}{2}(1 - \cos\theta).$$

Take the time derivative and reduce:

$$\left(\frac{d\theta}{dt}\right)\left[\frac{mL^2}{3}\left(\frac{d^2\theta}{dt^2}\right) + k\theta - \frac{mgL}{2}\sin\theta\right] = 0.$$

Ignore the possible solution $\dfrac{d\theta}{dt} = 0$, from which

$$\frac{mL^2}{3}\left(\frac{d^2\theta}{dt^2}\right) + k\theta - \frac{mgL}{2}\sin\theta = 0.$$

For small amplitude vibrations $\sin\theta \to \theta$, and the canonical form (see Eq. (21.4)) of the equation of motion is

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0, \quad \text{where } \boxed{\omega^2 = \frac{k - \frac{mgL}{2}}{\frac{1}{3}mL^2}}$$

Problem 21.22 The initial conditions of the slender bar in Problem 21.21 are

$$t = 0 \begin{cases} \theta = 0 \\ \dfrac{d\theta}{dt} = \dot{\theta}_0. \end{cases}$$

(a) If $k > \frac{1}{2}mgl$, show that θ is given as a function of time by

$$\theta = \frac{\dot{\theta}_0}{\omega}\sin\omega t, \quad \text{where } \omega^2 = \frac{(k - \frac{1}{2}mgl)}{\frac{1}{3}ml^2}.$$

(b) If $k < \frac{1}{2}mgl$, show that θ is given as a function of time by

$$\theta = \frac{\dot{\theta}_0}{2h}(e^{ht} - e^{-ht}), \quad \text{where } h^2 = \frac{(\frac{1}{2}mgl - k)}{\frac{1}{3}ml^2}.$$

Strategy: To do part (b), seek a solution of the equation of motion of the form $x = Ce^{\lambda t}$, where C and λ are constants.

Solution: Write the equation of motion in the form

$$\frac{d^2\theta}{dt^2} + p^2\theta = 0, \quad \text{where } p = \sqrt{\frac{k - \dfrac{mgL}{2}}{\dfrac{1}{3}mL^2}}.$$

$$\text{Define} \quad \omega = \sqrt{\frac{\left(k - \dfrac{mgL}{2}\right)}{\left(\dfrac{1}{3}mL^2\right)}} \text{ if } k > \frac{mgL}{2},$$

$$\text{and} \quad h = \sqrt{\frac{\left(\dfrac{1}{2}mgL - k\right)}{\left(\dfrac{1}{3}mL^2\right)}},$$

if $k < \dfrac{mgL}{2}$, from which $p = \omega$ if $k > \dfrac{mgL}{2}$, and $p = ih$, if $k < \dfrac{mgL}{2}$, where $i = \sqrt{-1}$. Assume a solution of the form $\theta = A\sin pt + B\cos pt$. The time derivative is $\dfrac{d\theta}{dt} = pA\cos pt - pB\sin pt$. Apply the initial conditions at $t = 0$, to obtain $B = 0$, and $A = \dfrac{\dot{\theta}_0}{p}$, from which the solution is $\theta = \dfrac{\dot{\theta}_0}{p}\sin pt$.

(a) Substitute: if $k > \dfrac{mgL}{2}$, $\boxed{\theta = \dfrac{\dot{\theta}_0}{\omega}\sin\omega t}$.

(b) If $k < \dfrac{mgL}{2}$, $\theta = \dfrac{\dot{\theta}_0}{ih}\sin(iht)$. From the definition of the hyperbolic sine,

$$\frac{\sinh(ht)}{h} = \frac{\sin(iht)}{ih} = \frac{1}{2h}\left(e^{ht} - e^{-ht}\right),$$

from which the solution is $\boxed{\theta = \dfrac{\dot{\theta}_0}{2h}(e^{ht} - e^{-ht})}$. [*Check*: An alternate solution for part (b) based on the suggested strategy is:

For $\quad k < \dfrac{mgL}{2}$

write the equation of motion in the form

$$\frac{d^2\theta}{dt^2} - h^2\theta = 0,$$

and assume a general solution of the form

$$\theta = Ce^{\lambda t} + De^{-\lambda t}.$$

Substitute into the equation of motion to obtain $(\lambda^2 - h^2)\theta = 0$, from which $\lambda = \pm h$, and the solution is $\theta = Ce^{ht} + De^{-ht}$, where the positive sign is taken without loss of generality. The time derivative is

$$\frac{d\theta}{dt} = hCe^{ht} - hDe^{-ht}.$$

Apply the initial conditions at $t = 0$ to obtain the two equations: $0 = C + D$, and $\dot{\theta}_0 = hC - hD$. Solve:

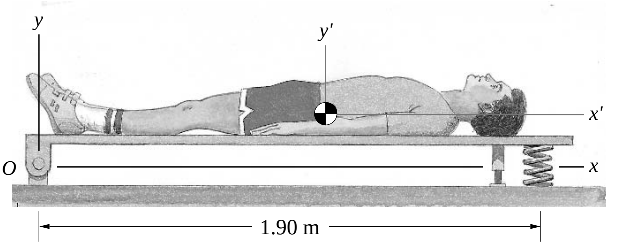
$$C = \frac{\dot{\theta}_0}{2h},$$

$$D = -\frac{\dot{\theta}_0}{2h},$$

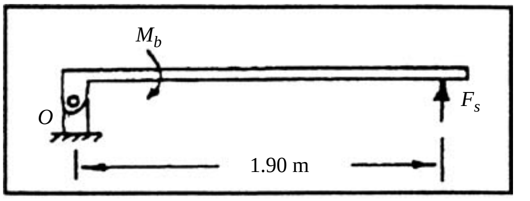
from which the solution is

$$\theta = \frac{\dot{\theta}_0}{2h}(e^{ht} - e^{-ht}). \quad \textit{check}]$$

Problem 21.23 Engineers use the device shown to measure an astronaut's moment of inertia. The horizontal board is pinned at O and supported by the linear spring with constant $k = 12$ kN/m. When the astronaut is not present, the frequency of small vibrations of the board about O is measured and determined to be 6.0 Hz. When the astronaut is lying on the board as shown, the frequency of small vibrations of the board about O is 2.8 Hz. What is the astronaut's moment of inertia about the z axis?

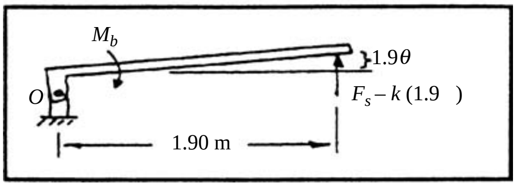


Solution: *When the astronaut is not present*: Let F_s be the spring force and M_b be the moment about 0 due to the board's weight when the system is in equilibrium. The moment about 0 equals zero, $\sum M_{(\text{pt }0)} = (1.9)F_s - M_b = 0(1)$. When the system is in motion and displaced by a small counterclockwise angle θ, the spring force decreases to $F_s - k(1.9\theta)$: The equation of angular motion about 0 is



$$\sum M_{(\text{pt}0)} = (1.9)[F_s - k(1.9\theta)] - M_b = I_b \frac{d^2\theta}{dt^2},$$

where I_b is the moment of inertial of the board about the z axis. Using Equation (1), we can write the equation of angular motion as



$$\frac{d^2\theta}{dt^2} + \omega_1^2\theta = 0, \quad \text{where } \omega_1^2 = \frac{k(1.9)^2}{I_b} = \frac{(12{,}000)(1.9)^2}{I_b}.$$

We know that $f_1 = \omega_1/2\pi = 6$ Hz, so $\omega_1 = 12\pi = 37.7$ rad/s and we can solve for I_b: $I_b = 30.48$ kg-m^2.

When the astronaut is present: Let F_s be the spring force and M_{ba} be the moment about 0 due to the weight of the board and astronaut when the system is in equilibrium. The moment about 0 equals zero, $\sum M_{(pt0)} = (1.9)F_s - M_{ba} = 0(2)$. When the system is in motion and displaced by a small counterclockwise angle θ, the spring force decreases to $F_s - k(1.9\theta)$: The equation of angular motion about 0 is

$$\sum M_{(\text{pt}0)} = (1.9)[F_s - k(1.9\theta)] - M_{ba} = (I_b + I_a)\frac{d^2\theta}{dt}^2,$$

where I_a is the moment of inertia of the astronaut about the z axis. Using Equation (2), we can write the equation of angular motion as

$$\frac{d^2\theta}{dt^2} + \omega_2^2\theta = 0, \quad \text{where } \omega_2^2 = \frac{k(1.9)^2}{I_b + I_a} = \frac{(12{,}000)(1.9)^2}{I_b + I_a}.$$

In this case $f_2 = \omega_2/2\pi = 2.8$ Hz, so $\omega_2 = 2.8(2\pi) = 17.59$ rad/s. Since we know I_b, we can determine I_a, obtaining $I_a = 109.48$ kg-m^2.

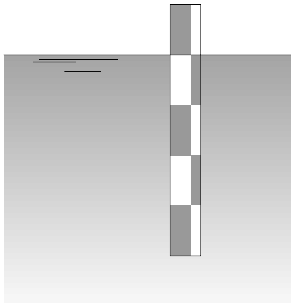
Problem 21.24 In Problem 21.23, the astronaut's center of mass is at $x = 1.01$ m, $y = 0.16$ m, and his mass is 81.6 kg. What is his moment of inertia about the z' axis through his center of mass?

Solution: From the solution of Problem 21.23, his moment of inertial about the z axis is $I_z = 109.48$ kg-m^2. From the parallel-axis theorem,

$$I_{z'} = I_z - (d_x^2 + d_y^2)m = 109.48 - [(1.01)^2 + (0.16)^2](81.6)$$

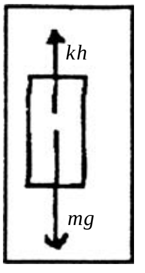
$$= 24.2 \text{ kg-m}^2.$$

Problem 21.25* A floating sonobuoy (sound-measuring device) is in equilibrium in the vertical position shown. (Its center of mass is low enough that it is stable in this position.) The device is a 10-kg cylinder 1 m in length and 125 mm in diameter. The water density is 1025 kg/m^3, and the buoyancy force supporting the buoy equals the weight of the water that would occupy the volume of the part of the cylinder below the surface. If you push the sonobuoy slightly deeper and release it, what is the frequency of the resulting vertical vibrations?



Solution: Choose a coordinate system with y positive downward. Denote the volume beneath the surface by $V = \pi R^2 d$, where $R = 0.0625$ m. The density of the water is $\rho = 1025$ kg/m^3. The weight of the displaced water is $W = \rho V g$, from which the buoyancy force is $F = \rho V g = \pi \rho R^2 g d$. By definition, the spring constant is

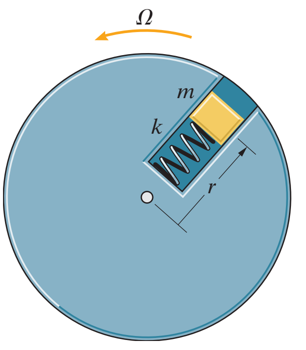
$$k = \frac{\partial F}{\partial d} = \pi \rho R^2 g = 123.4 \text{ N/m}.$$



If h is a positive change in the immersion depth from equilibrium, the force on the sonobuoy is $\sum F_y = -kh + mg$, where the negative sign is taken because the "spring force" kh opposes the positive motion h. From Newton's second law, $m\frac{d^2h}{dt^2} = mg - kh$. The canonical form (see Eq. (21.4)) is $\frac{d^2h}{dt^2} + \omega^2 h = g$, where $\omega = \sqrt{\frac{k}{m}} = 3.513$ rad/s. The frequency is $\boxed{f = \frac{\omega}{2\pi} = 0.5591 \text{ Hz}}$.

Problem 21.26 The disk rotates *in the horizontal plane* with constant angular velocity $\Omega = 12$ rad/s. The mass $m = 2$ kg slides in a smooth slot in the disk and is attached to a spring with constant $k = 860$ N/m. The radial position of the mass when the spring is unstretched is $r = 0.2$ m.

(a) Determine the "equilibrium" position of the mass, the value of r at which it will remain stationary relative to the center of the disk.
(b) What is the frequency of vibration of the mass relative to its equilibrium position?



Strategy: Apply Newton's second law to the mass in terms of polar coordinates.

Solution: Using polar coordinates, Newton's second law in the r direction is

$$\Sigma F_r : -k(r - r_0) = m(\ddot{r} - r\Omega^2) \Rightarrow \ddot{r} + \left(\frac{k}{m} - \Omega^2\right) r = \frac{k}{m} r_0$$

(a) The "equilibrium" position occurs when $\ddot{r} = 0$

$$r_{eq} = \frac{\frac{k}{m} r_0}{\frac{k}{m} - \Omega^2} = \frac{k r_0}{k - m\Omega^2} = \frac{(860\text{ N/m})(0.2\text{ m})}{(860\text{ N/m}) - (2\text{ kg})(12\text{ rad/s})^2} = 0.301\text{ m.}$$

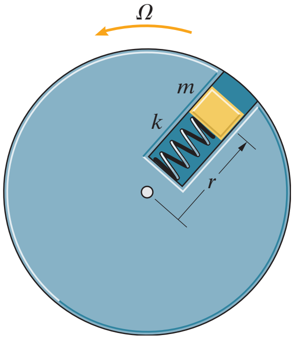
$$\boxed{r_{eq} = 0.301\text{ m.}}$$

(b) The frequency of vibration is found

$$\omega = \sqrt{\frac{k}{m} - \Omega^2} = \sqrt{\frac{860\text{ N/m}}{2\text{ kg}} - (12\text{ rad/s})^2} = 16.9\text{ rad/s}, \quad f = \frac{\omega}{2\pi} = 2.69\text{ Hz.}$$

$$\boxed{f = 2.69\text{ Hz.}}$$

Problem 21.27 The disk rotates *in the horizontal plane* with constant angular velocity $\Omega = 12$ rad/s. The mass $m = 2$ kg slides in a smooth slot in the disk and is attached to a spring with constant $k = 860$ N/m. The radial position of the mass when the spring is unstretched is $r = 0.2$ m. At $t = 0$, the mass is in the position $r = 0.4$ m and $dr/dt = 0$. Determine the position r as a function of time.



Solution: Using polar coordinates, Newton's second law in the r direction is

$$\Sigma F_r : -k(r - r_0) = m(\ddot{r} - r\Omega^2),$$

$$\ddot{r} + \left(\frac{k}{m} - \Omega^2\right) r = \frac{k}{m} r_0$$

$$\ddot{r} + (16.9\text{ rad/s})^2 r = 86\text{ m/s}^2.$$

The solution is

$$r = A\sin\omega t + B\cos\omega t + 0.301\text{ m}, \quad \frac{dr}{dt} = A\omega\cos\omega t - B\omega\sin\omega t.$$

Putting in the initial conditions, we have

$$r(t = 0) = B + 0.301\text{ m} = 0.4\text{ m} \Rightarrow B = 0.0993\text{ m},$$

$$\frac{dr}{dt}(t = 0) = A\omega = 0 \Rightarrow A = 0.$$

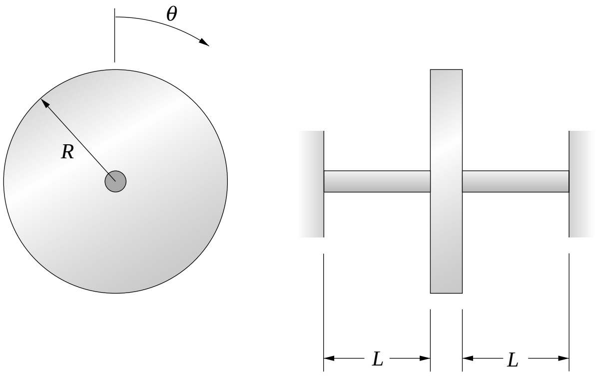
Thus the equation is

$$\boxed{r = (0.0993\text{ m})\cos[16.9\text{ rad/s } t] + (0.301\text{ m}).}$$

Problem 21.28 A homogeneous 44.5 N disk with radius $R = 0.31$ m is attached to two identical cylindrical steel bars of length $L = 0.31$m. The relation between the moment M exerted on the disk by one of the bars and the angle of rotation, θ, of the disk is

$$M = \frac{GJ}{L}\theta,$$

where J is the polar moment of inertia of the cross section of the bar and $G = 8.14 \times 10^{10}$ N/m^2 is the shear modulus of the steel. Determine the required radius of the bars if the frequency of rotational vibrations of the disk is to be 10 Hz.



Solution: The moment exerted by two bars on the disk is $M = k\theta$, where the spring constant is

$$k = \frac{\partial M}{\partial \theta} = \frac{2GJ}{L}.$$

The polar moment of the cross section of a bar is

$$J = \frac{\pi r^4}{2},$$

from which $k = \dfrac{\pi r^4 G}{L}$.

From the equation of angular motion,

$$I\frac{d^2\theta}{dt^2} = -k\theta.$$

The moment of inertia of the disk is

$$I = \frac{W}{2g}R_d^2,$$

from which

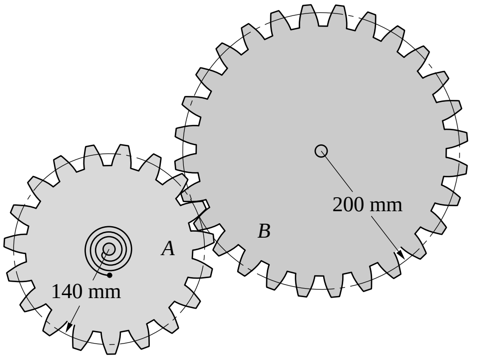
$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0, \quad \text{where } \omega = \sqrt{\frac{2\pi G g r^4}{W L R_d^2}} = \left(\frac{r^2}{R_d}\right)\sqrt{\frac{2\pi G g}{WL}}.$$

Solve: $r = \sqrt{R_d\sqrt{\dfrac{WL}{2\pi G g}}\omega} = \sqrt{R_d\sqrt{\dfrac{2\pi WL}{Gg}}f}$.

Substitute numerical values: $R_d = 0.31$ m, $L = 0.31$ m, $W = 445$ N, $G = 8.14 \times 10^{10}$ N/m^2, $g = 9.81$ m/s^2, from which

$\boxed{r = 0.01 \text{ m} = 10 \text{ mm}.}$

Problem 21.29 The moments of inertia of gears A and B are $I_A = 0.025$ kg-m^2 and $I_B = 0.100$ kg-m^2. Gear A is connected to a torsional spring with constant $k = 10$ N-m/rad. What is the frequency of small angular vibrations of the gears?



Solution: The system is conservative. Denote the rotation velocities of A and B by $\dot{\theta}_A$, and $\dot{\theta}_B$ respectively. The kinetic energy of the gears is $T = \frac{1}{2}I_A\dot{\theta}_A^2 + \frac{1}{2}I_B\dot{\theta}_B^2$. The potential energy of the torsional spring is $V = \frac{1}{2}k\theta_A^2$. $T + V = \text{const.} = \frac{1}{2}I_A\dot{\theta}_A^2 + \frac{1}{2}I_B\dot{\theta}_B^2 + \frac{1}{2}k\theta_A^2$. From kinematics, $\dot{\theta}_B = -\left(\frac{R_A}{R_B}\right)\dot{\theta}_A$. Substitute, define

$$M = \left(I_A + \left(\frac{R_A}{R_B}\right)^2 I_B\right) = 0.074 \text{ kg-m}^2,$$

and take the time derivative:

$$\left(\frac{d\theta_A}{dt}\right)\left(M\left(\frac{d^2\theta_A}{dt^2}\right) + k\theta_A\right) = 0.$$

Ignore the possible solution $\left(\frac{d\theta_A}{dt}\right) = 0$, from which

$$\frac{d^2\theta_A}{dt^2} + \omega^2\theta_A = 0, \quad \text{where } \omega = \sqrt{\frac{k}{M}} = 11.62 \text{ rad/s}.$$

The frequency is $\boxed{f = \frac{\omega}{2\pi} = 1.850 \text{ Hz}}$.

Problem 21.30 At $t = 0$, the torsional spring in Problem 21.29 is unstretched and gear B has a counterclockwise angular velocity of 2 rad/s. Determine the counterclockwise angular position of gear B relative to its equilibrium position as a function of time.

Solution: It is convenient to express the motion in terms of gear A, since the equation of motion of gear A is given in the solution to Problem 21.29: $\frac{d^2\theta_A}{dt^2} + \omega^2\theta_A = 0$, where

$$M = \left(I_A + \left(\frac{R_A}{R_B}\right)^2 I_B\right) = 0.074 \text{ kg-m}^2,$$

$$\omega = \sqrt{\frac{k}{M}} = 11.62 \text{ rad/s}.$$

Assume a solution of the form $\theta_A = A\sin\omega t + B\cos\omega t$. Apply the initial conditions,

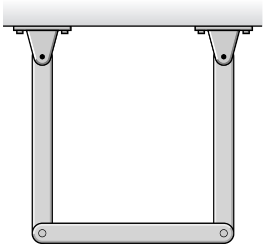
$$\theta_A = 0, \theta_A = -\left(\frac{R_B}{R_A}\right)\theta_B = -2.857 \text{ rad/s},$$

from which $B = 0$, $A = \frac{\theta_A}{\omega} = -0.2458$, and

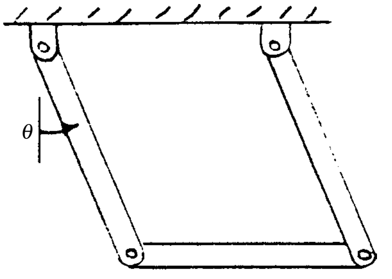
$$\theta_A = -0.2458\sin(11.62t) \text{ rad},$$

and $\boxed{\theta_B = -\left(\frac{R_A}{R_B}\right)\theta_A = 0.172\sin(11.6t) \text{ rad}}$

Problem 21.31 Each 2-kg slender bar is 1 m in length. What are the period and frequency of small vibrations of the system?



Solution: The total energy is



$$T + V = \frac{1}{2}\left(\frac{1}{3}mL^2\right)\left(\frac{d\theta}{dt}\right)^2 + \frac{1}{2}\left(\frac{1}{3}ml^2\right)\left(\frac{d\theta}{dt}\right)^2$$

$$+ \frac{1}{2}m\left(L\frac{d\theta}{dt}\right)^2 - mg\frac{L}{2}\cos\theta - mg\frac{L}{2}\cos\theta - mgL\cos\theta$$

$$= \frac{5}{6}mL^2\left(\frac{d\theta}{dt}\right)^2 - 2mgL\cos\theta.$$

$$\frac{d}{dt}(T+V) = \frac{5}{3}mL^2\frac{d\theta}{dt}\frac{d^2\theta}{dt^2} + 2mgL\sin\theta\frac{d\theta}{dt} = 0,$$

so the (linearized) equation of motion is

$$\frac{d^2\theta}{dt^2} + \frac{6g}{5L}\theta = 0.$$

Therefore

$$\omega = \sqrt{\frac{6g}{5L}} = \sqrt{\frac{6(9.81)}{5(1)}} = 3.43 \text{ rad/s},$$

so $\tau = \dfrac{2\pi}{\omega} = 1.83$ s, $f = 0.546$ Hz.

Problem 21.32* The masses of the slender bar and the homogeneous disk are m and m_d, respectively. The spring is unstretched when $\theta = 0$. Assume that the disk rolls on the horizontal surface.

(a) Show that the motion of the system is governed by the equation

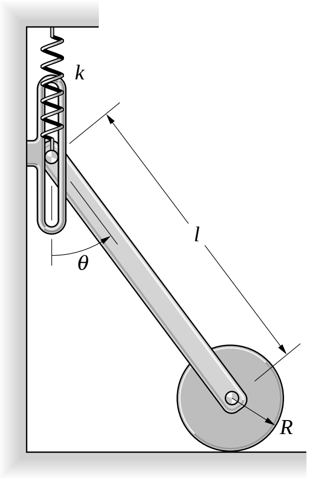
$$\left(\frac{1}{3}+\frac{3m_d}{2m}\cos^2\theta\right)\frac{d^2\theta}{dt^2}-\frac{3m_d}{2m}\sin\theta\cos\theta\left(\frac{d\theta}{dt}\right)^2$$

$$-\frac{g}{2l}\sin\theta+\frac{k}{m}(1-\cos\theta)\sin\theta = 0.$$

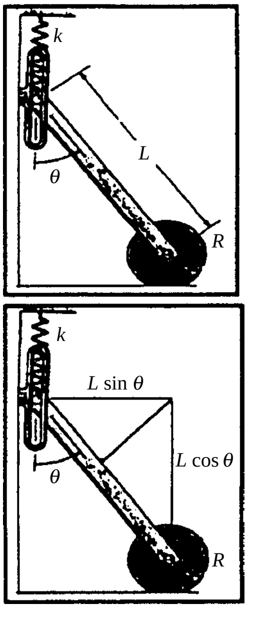
(b) If the system is in equilibrium at the angle $\theta = \theta_e$ and $\tilde{\theta} = \theta - \theta_e$ show that the equation governing small vibrations relative to the equilibrium position is

$$\left(\frac{1}{3}+\frac{3m_d}{2m}\cos^2\theta_e\right)\frac{d^2\tilde{\theta}}{dt^2}+\left[\frac{k}{m}(\cos\theta_e-\cos^2\theta_e\right.$$

$$\left.+\sin^2\theta_e)-\frac{g}{2l}\cos\theta_e\right]\tilde{\theta} = 0.$$



Solution: (See Example 21.2.) The system is conservative.



(a) The kinetic energy is

$$T = \tfrac{1}{2}I\dot{\theta}_2 + \tfrac{1}{2}mv^2 + \tfrac{1}{2}I_d\dot{\theta}_d^2 + \tfrac{1}{2}m_dv_d^2,$$

where $I = \dfrac{mL^2}{12}$ is the moment of inertia of the bar about its center of mass, v is the velocity of the center of mass of the bar, $I_d = \dfrac{mR^2}{2}$ is the polar moment of inertia of the disk, and v_d is the velocity of the center of mass of the disk. The height of the center of mass of the bar is $h = \dfrac{L\cos\theta}{2}$, and the stretch of the spring is $S = L(1-\cos\theta)$, from which the potential energy is

$$V = \frac{mgL}{2}\cos\theta + \frac{1}{2}kL^2(1-\cos\theta)^2.$$

The system is conservative,

$$T + V = \text{const.}\quad \dot{\theta}l_d = \frac{v}{R} = \frac{L\cos\theta}{R}\dot{\theta}.$$

Choose a coordinate system with the origin at the pivot point and the x axis parallel to the lower surface. The instantaneous center of rotation of the bar is located at $(L\sin\theta, L\cos\theta)$. The center of mass of the bar is located at $\left(\frac{L}{2}\sin\theta, \frac{L}{2}\cos\theta\right)$. The distance from the center of mass to the center of rotation is

$$r = \sqrt{\left(L-\frac{L}{2}\right)^2\sin^2\theta + \left(L-\frac{L}{2}\right)^2\cos^2\theta} = \frac{L}{2},$$

from which $v = \dfrac{L}{2}\dot{\theta}$. The velocity of the center of mass of the disk is $v_d = (L\cos\theta)\dot{\theta}$. The angular velocity of the disk is $\dot{\theta}l_d = \dfrac{v}{R} = \dfrac{L\cos\theta}{R}\dot{\theta}$.

Substitute and reduce:

$$\frac{1}{2}\left(\frac{1}{3}+\frac{3m_d}{2m}\cos^2\right)\dot{\theta}^2 + \frac{g}{2L}\cos\theta + \frac{k}{2m}(1-\cos\theta)^2 = \text{const.}$$

Take the time derivative:

$$\dot{\theta}\left[\left(\frac{1}{3}+\frac{3m_d}{2m}\cos^2\theta\right)\frac{d^2\theta}{dt^2}-\frac{3m_d}{2m}\sin\theta\cos\theta\left(\frac{d\theta}{dt}\right)^2\right.$$

$$\left.-\frac{g}{2L}\sin\theta+\frac{k}{m}(1-\cos\theta)\sin\theta\right] = 0,$$

from which $$\left(\frac{1}{3}+\frac{3m_d}{2m}\cos^2\theta\right)\frac{d^2\theta}{dt^2}-\frac{3m_d}{2m}\sin\theta\cos\theta\left(\frac{d\theta}{dt}\right)^2$$

$$-\frac{g}{2L}\sin\theta+\frac{k}{m}(1-\cos\theta)\sin\theta = 0$$

(b) The non-homogenous term is $\frac{g}{2L}$, as can be seen by dividing the equation of motion by $\sin\theta \neq 0$,

$$\frac{\left(\frac{1}{3}+\frac{3m_d}{2m}\cos^2\theta\right)}{\sin\theta}\frac{d^2\theta}{dt^2}-\frac{3m_d}{2m}\cos\theta\left(\frac{d\theta}{dt}\right)^2$$

$$+\frac{k}{m}(1-\cos\theta)=\frac{g}{2L}.$$

Since the non-homogenous term is independent of time and angle, the equilibrium point can be found by setting the acceleration and the velocity terms to zero, $\cos\theta_e = 1-\frac{mg}{2Lk}$. [*Check*: This is identical to Eq. (21.15) in Example 21.2, as expected. *check*.]. Denote $\tilde{\theta}=\theta-\theta_e$. For small angles:

$$\cos\theta=\cos\tilde{\theta}\cos\theta_e-\sin\tilde{\theta}\sin\theta_e \to \cos\theta_e-\tilde{\theta}\sin\theta_e.$$

$$\cos^2\theta \to \cos^2\theta_e - 2\tilde{\theta}\sin\theta_e\cos\theta_e.$$

$$\sin\theta=\sin\tilde{\theta}\cos\theta_e+\cos\tilde{\theta}\sin\theta_e \to \tilde{\theta}\cos\theta_e+\sin^2\theta_e.$$

$$(1-\cos\theta)\sin\theta \to (1-\cos\theta_e)\sin\theta_e+\tilde{\theta}(\cos\theta_e-\cos\theta_e$$

$$+\sin^2\theta_e).$$

$$\sin\theta\cos\theta \to \tilde{\theta}(\cos^2\theta_e-\sin^2\theta_e).$$

Substitute and reduce:

$$(1)\left(\frac{1}{3}+\frac{3m_d}{2m}\cos^2\theta\right)\frac{d^2\theta}{dt^2}\to\left(\frac{1}{3}+\frac{3m_d}{m}\tilde{\theta}\sin\theta_e\cos\theta_e\right)$$

$$\times\frac{d^2\tilde{\theta}}{dt^2}\to\left(\frac{1}{3}+\frac{3m_d}{2m}\cos^2\theta_e\right)\frac{d^2\tilde{\theta}}{dt^2}.$$

$$(2)=-\frac{3m_d}{2m}\cos\theta\sin\theta\left(\frac{d\theta}{dt}\right)^2\to-\frac{3m_d}{2m}\tilde{\theta}\sin\theta_e\cos\theta_e$$

$$\times\left(\frac{d\tilde{\theta}}{dt}\right)^2\to 0.$$

$$(3)\quad -\frac{g}{2L}\sin\theta \qquad \to -\frac{g}{2L}\tilde{\theta}\cos\theta_e-\frac{g}{2L}\sin\theta_e.$$

$$(4)\quad \frac{k}{m}(1-\cos\theta)\sin\theta \to \frac{k}{m}(1-\cos\theta_e)\sin\theta_e$$

$$+\frac{k}{m}\tilde{\theta}(\cos\theta_e-\cos^2\theta_e+\sin^2\theta_e),$$

where the terms $\tilde{\theta}\frac{d^2\tilde{\theta}}{dt^2}\to 0$, $\tilde{\theta}\left(\frac{d\tilde{\theta}}{dt}\right)^2\to 0$, and terms in $\tilde{\theta}^2$ have been dropped.

Collect terms in (1) to (4) and substitute into the equations of motion:

$$\frac{\left(\frac{1}{3}+\frac{3m_d}{2m}\cos^2\theta_e\right)}{\sin\theta_e}\frac{d^2\tilde{\theta}}{dt^2}+\left[\frac{k}{m}\frac{(\cos\theta_e-\cos^2\theta_e+\sin^2\theta_e)}{\sin\theta_e}\right.$$

$$\left.-\frac{g}{2L}\frac{\cos\theta_e}{\sin\theta_e}\right]=\left[\frac{g}{2L}-\frac{k}{m}(1-\cos\theta_e)\right].$$

The term on the right $\frac{g}{2L}-\frac{k}{m}(1-\cos\theta_e)=0$, as shown by substituting the value $\cos\theta_e=1-\frac{mg}{2Lk}$, from which

$$\left(\frac{1}{3}+\frac{3m_d}{2m}\cos^2\theta_e\right)\frac{d^2\tilde{\theta}}{dt^2}$$

$$+\left[\frac{k}{m}(\cos\theta_e-\cos^2\theta_e+\sin^2\theta_e)-\frac{g}{2L}\cos\theta_e\right]\tilde{\theta}=0$$

is the equation of motion for small amplitude oscillations about the equilibrium point.

Problem 21.33* The masses of the bar and disk in Problem 21.32 are $m = 2$ kg and $m_d = 4$ kg, respectively. The dimensions $l = 1$ m and $R = 0.28$ m, and the spring constant is $k = 70$ N/m.

(a) Determine the angle θ_e at which the system is in equilibrium.
(b) The system is at rest in the equilibrium position, and the disk is given a clockwise angular velocity of 0.1 rad/s. Determine θ as a function of time.

Solution:

(a) From the solution to Problem 21.32, the static equilibrium angle is

$$\theta_e = \cos^{-1}\left(1 - \frac{mg}{2kL}\right) = 30.7^\circ = 0.5358 \text{ rad.}$$

(b) The canonical form (see Eq. (21.4)) of the equation of motion is $\dfrac{d^2\tilde{\theta}}{dt^2} + \omega^2\tilde{\theta} = 0$, where

$$\omega = \sqrt{\frac{\dfrac{k}{m}(\cos\theta_e - \cos^2\theta_e + \sin^2\theta_e) - \dfrac{g}{2L}\cos\theta_e}{\left(\dfrac{1}{3} + \dfrac{3m_d}{2m}\cos^2\theta_e\right)}}$$

$$= 1.891 \text{ rad/s.}$$

Assume a solution of the form

$$\tilde{\theta} = \theta - \theta_e = A\sin\omega t + B\cos\omega t,$$

from which

$$\theta = \theta_e + A\sin\omega t + B\cos\omega t.$$

From the solution to Problem 21.32, the angular velocity of the disk is

$$\dot{\theta}_d = \frac{v_d}{R} = \frac{(L\cos\theta_e)}{R}\dot{\theta}.$$

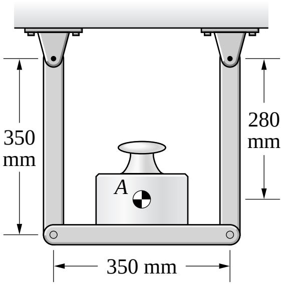
The initial conditions are

$$t = 0, \theta = \theta_e, \dot{\theta} = \frac{R\dot{\theta}_d}{L\cos\theta_e} = \frac{(0.1)(0.28)}{0.86} = 0.03256 \text{ rad/s},$$

from which $B = 0$, $A = 0.03256/\omega = 0.01722$, from which the solution is

$$\theta = 0.5358 + 0.01722\sin(1.891t).$$

Problem 21.34 The mass of each slender bar is 1 kg. If the frequency of small vibrations of the system is 0.935 Hz, what is the mass of the object A?



Solution: The system is conservative. Denote $L = 0.350$ m, $L_A = 0.280$ m, $m = 1$ kg, and M the mass of A. The moments of inertia about the fixed point is the same for the two vertical bars. The kinetic energy is

$$T = \tfrac{1}{2}I\dot{\theta}^2 + \tfrac{1}{2}I\dot{\theta}^2 + \tfrac{1}{2}mv^2 + \tfrac{1}{2}Mv_A^2,$$

where v is the velocity of the center of mass of the lower bar and v_A is the velocity of the center of mass of A, from which $T = \frac{1}{2}(2I\dot{\theta}^2 + mv^2 + Mv_A^2)$. The potential energy is

$$V = \frac{mgL}{2}(1 - \cos\theta) + \frac{mgL}{2}(1 - \cos\theta) + mgL(1 - \cos\theta)$$

$$+ MgL_A(1 - \cos\theta),$$

$$V = (MgL_A + 2mgL)(1 - \cos\theta).$$

$$T + V = \text{const.} = \tfrac{1}{2}(2I\dot{\theta}^2 + mv^2 + Mv_A^2) + (MgL_A + 2mgL)$$

$$\times (1 - \cos\theta).$$

From kinematics, $v = L\cos\theta(\dot{\theta})$, and $v_A = L_A\cos\theta(\dot{\theta})$. Substitute:

$$\tfrac{1}{2}(2I + (mL^2 + ML_A^2)\cos^2\theta)\dot{\theta}^2 + (MgL_A + 2mgL)(1 - \cos\theta)$$

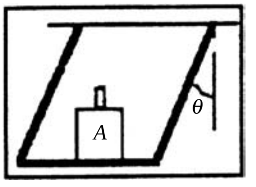
$$= \text{const.}$$

For small angles: $\cos^2\theta \to 1$, $(1 - \cos\theta) \to \dfrac{\theta^2}{2}$, from which

$$\frac{1}{2}(2I + mL^2 + ML_A^2)\dot{\theta}^2 + \left(\frac{MgL_A}{2} + mgL\right)\theta^2 = \text{const.}$$

Take the time derivative:

$$\dot{\theta}\left[(2I + mL^2 + ML_A^2)\frac{d^2\theta}{dt^2} + (MgL_A + 2mgL)\theta\right] = 0.$$



Ignore the possible solution $\dot{\theta} = 0$, from which

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0, \quad \text{where } \omega = \sqrt{\frac{2mgL + MgL_A}{2I + mL^2 + ML_A^2}}.$$

The moment of inertia of a slender bar about one end (the fixed point) is $I = \dfrac{mL^2}{3}$, from which

$$\omega = \sqrt{\frac{g(2 + \eta)}{\frac{5}{3}L + \eta L_A}}, \quad \text{where } \eta = \frac{ML_A}{mL}.$$

The frequency is

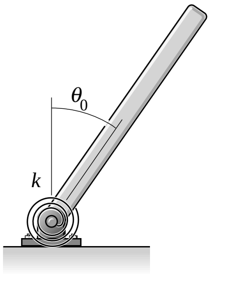
$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{3g(2 + \eta)}{5L + 3\eta L_A}} = 0.935 \text{ Hz}.$$

Solve:

$$\eta = \frac{5L\omega^2 - 6g}{3(g - L_A\omega^2)} = 3.502,$$

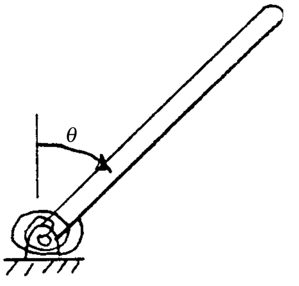
from which $\boxed{M = \dfrac{mL}{L_A}(3.502) = 4.38 \text{ kg}}$

Problem 21.35* The 4-kg slender bar is 2 m in length. It is held in equilibrium in the position $\theta_0 = 35^\circ$ by a torsional spring with constant k. The spring is unstretched when the bar is vertical. Determine the period and frequency of small vibrations of the bar relative to the equilibrium position shown.



Solution: The total energy is

$$T + v = \frac{1}{2}\left(\frac{1}{3}mL^2\right)\left(\frac{d\theta}{dt}\right)^2 + \frac{1}{2}k\theta^2 + mg\frac{L}{2}\cos\theta.$$



$$\frac{d}{dt}(T+v) = \frac{1}{3}mL^2\frac{d\theta}{dt}\frac{d^2\theta}{dt^2} + k\theta\frac{d\theta}{dt} - mg\frac{L}{2}\sin\theta\frac{d\theta}{dt} = 0,$$

so the equation of motion is

$$\frac{d^2\theta}{dt^2} + \frac{3k}{mL^2}\theta - \frac{3g}{2L}\sin\theta = 0. \quad \textbf{(1)}$$

Let $\theta_0 = 35^\circ$ be the equilibrium position:

$$\frac{3k}{mL^2}\theta_0 - \frac{3g}{2L}\sin\theta_0 = 0. \quad \textbf{(2)}$$

Solving for k,

$$k = \frac{mgL}{2}\frac{\sin\theta_0}{\theta_0} = \frac{(4)(9.81)(2)}{2}\frac{\sin 35^\circ}{(35\pi/180)} = 36.8 \text{ N-m}.$$

Let $\tilde{\theta} = \theta - \theta_0$. Then

$$\sin\theta = \sin(\theta_0 + \tilde{\theta}) = \sin\theta_0 + (\cos\theta_0)\tilde{\theta} + \cdots$$

Using this expression and Eq. (2), Eq. (1) (linearized) is

$$\frac{d^2\tilde{\theta}}{dt^2} + \left(\frac{3k}{mL^2} - \frac{3g}{2L}\cos\theta_0\right)\tilde{\theta} = 0.$$

Therefore $\omega = \sqrt{\dfrac{3k}{mL^2} - \dfrac{3g}{2L}\cos\theta_0}$

$$= \sqrt{\frac{(3)(36.8)}{(4)(2)^2} - \frac{(3)(9.81)}{(2)(2)}\cos 35^\circ}$$

$$= 0.939 \text{ rad/s},$$

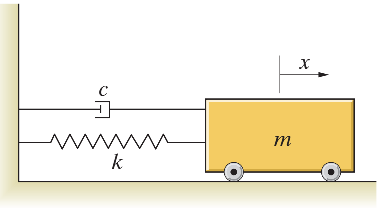
so $\tau = \dfrac{2\pi}{\omega} = 6.69$ s,

$$f = \frac{1}{\tau} = 0.149 \text{ Hz}.$$

Problem 21.36 The mass $m = 2$ kg, the spring constant is $k = 72$ N/m, and the damping constant is $c = 8$ N-s/m. The spring is unstretched when $x = 0$. The mass is displaced to the position $x = 1$ m and released from rest.

(a) If the damping is subcritical, what is the frequency of the resulting damped vibrations?
(b) What is the value of x at $t = 1$ s?

(See Active Example 21.3.)



Solution: The equation of motion is

$$m\ddot{x} + c\dot{x} + kx = 0,\ \ \ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0,\ \ \ddot{x} + \frac{8}{2}\dot{x} + \frac{72}{2}x = 0,$$

$$\ddot{x} + 2(2)\dot{x} + (6)^2 x = 0.$$

We recognize $\omega = 6$, $d = 2$, $d < \omega \Rightarrow$ yes, the motion is subcritical.

(a) $\omega_d = \sqrt{\omega^2 - d^2} = \sqrt{6^2 - 2^2} = 5.66$ rad/s, $f = \dfrac{\omega}{2\pi}$, $\boxed{f = 0.900 \text{ Hz.}}$

(b) The solution to the differential equation is

$$x = e^{-dt}(A \sin \omega_d t + B \cos \omega_d t)$$

$$\frac{dx}{dt} = e^{-dt}([A\omega_d - Bd] \cos \omega_d t - [B\omega_d + Ad] \sin \omega_d t)$$

Putting in the initial conditions we have

$$x(t = 0) = B = 1 \text{ m},$$

$$\frac{dx}{dt}(t = 0) = (A\omega_d - Bd) = 0 \Rightarrow A = \frac{d}{\omega_d}B = \frac{2}{5.66}(1 \text{ m}) = 0.354 \text{ m}.$$

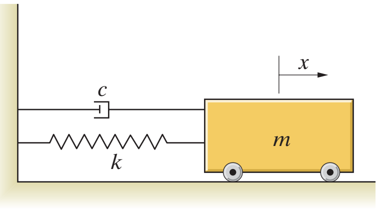
The equation of motion is now $x = e^{-2t}(0.354 \sin[5.66t] + \cos[5.66t])$.

At $t = 1$ s, we have $\boxed{x = 0.0816 \text{ m.}}$

Problem 21.37 The mass $m = 2$ kg, the spring constant is $k = 72$ N/m, and the damping constant is $c = 32$ N-s/m. The spring is unstretched with $x = 0$. The mass is displaced to the position $x = 1$ m and released from rest.

(a) If the damping is subcritical, what is the frequency of the resulting damped vibrations?
(b) What is the value of x at $t = 1$ s?

(See Active Example 21.3.)



Solution: The equation of motion is

$$m\ddot{x} + c\dot{x} + kx = 0, \quad \ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0, \quad \ddot{x} + \frac{32}{2}\dot{x} + \frac{72}{2}x = 0,$$

$$\ddot{x} + 2(8)\dot{x} + (6)^2 x = 0.$$

(a) We recognize $\omega = 6,\ d = 8,\ d > \omega \Rightarrow$ $\boxed{\text{Damping is supercritical.}}$

(b) We have $h = \sqrt{d^2 - \omega^2} = \sqrt{8^2 - 6^2} = 5.29$ rad/s.

The solution to the differential equation is

$$x = Ce^{-(d-h)t} + De^{-(d+h)t}, \quad \frac{dx}{dt} = -(d-h)Ce^{-(d-h)t} - (d+h)De^{-(d+h)t}.$$

Putting in the initial conditions, we have

$$\left.\begin{aligned} x(t=0) &= C + D = 1 \text{ m}, \\ \frac{dx}{dt}(t=0) &= -(d-h)C - (d+h)D = 0 \end{aligned}\right\} \Rightarrow C = \frac{d+h}{2h}, \quad D = \frac{h-d}{2h}$$

$C = 1.26$ ft, $D = -0.256$ m.

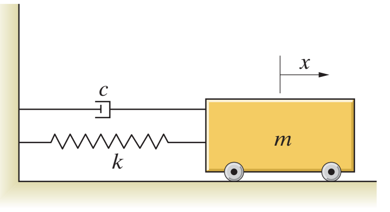
The general solution is then $x = (1.26)e^{-2.71t} - (0.256)e^{-13.3t}$.

At $t = 1$ s we have $\boxed{x = 0.0837 \text{ m.}}$

Problem 21.38 The mass $m = 4$ kg, the spring constant is $k = 72$ N/m. The spring is unstretched when $x = 0$.

(a) What value of the damping constant c causes the system to be critically damped?
(b) Suppose that c has the value determined in part (a). At $t = 0$, $x = 1$ m and $dx/dt = 4$ m/s. What is the value of x at $t = 1$ s?

(See Active Example 21.3.)



Solution: The equation of motion is

$$m\ddot{x} + c\dot{x} + kx = 0, \;\; \ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0, \;\; \ddot{x} + \frac{c}{4}\dot{x} + \frac{72}{4}x = 0,$$

$$\ddot{x} + 2\left(\frac{c}{8}\right)\dot{x} + (4.24)^2 x = 0.$$

(a) The system is critically damped when $d = \dfrac{c}{8} = \omega = 4.24 \Rightarrow \boxed{c = 33.9 \text{ N-s/m.}}$

(b) The solution to the differential equation is

$$x = Ce^{-dt} + Dte^{-dt}, \;\; \frac{dx}{dt} = (D - Cd - Ddt)e^{-dt}.$$

Putting in the initial conditions, we have

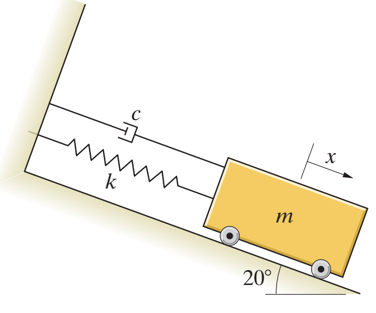
$$x(t = 0) = C = 1 \text{ m},$$

$$\frac{dx}{dt}(t = 0) = D - Cd = 4 \text{ m/s} \Rightarrow D = (1)(4.24) + 4 = 8.24 \text{ m}.$$

The general solution is then $x = (1 \text{ m})e^{-4.24t} + (8.24 \text{ m})te^{-4.24t}$.

At $t = 1$ s we have $\boxed{x = 0.133 \text{ m.}}$

Problem 21.39 The mass $m = 2$ kg, the spring constant is $k = 8$ N/m, and the damping coefficient is $c = 12$ N-s/m. The spring is unstretched when $x = 0$. At $t = 0$, the mass is released from rest with $x = 0$. Determine the value of x at $t = 2$ s.



Solution: The equation of motion is

$$(2 \text{ kg})\frac{d^2x}{dt^2} + (12 \text{ N-s/m})\frac{dx}{dt} + (8 \text{ N/m})x = (2 \text{ kg})(9.81 \text{ m/s}^2)\sin 20^\circ$$

$$\frac{d^2x}{dt^2} + 2(3 \text{ rad/s})\frac{dx}{dt} + (2 \text{ rad/s})^2 x = 3.36 \text{ m/s}^2$$

We identify $\omega = 2$ rad/s, $d = 3$ rad/s, $h = \sqrt{d^2 - \omega^2} = 2.24$ rad/s.

Since $d > \omega$, we have a supercritical case. The solution is

$$x = Ce^{-(d-h)t} + De^{-(d+h)t} + 0.839 \text{ m}$$

$$v = \frac{dx}{dt} = -C(d - h)e^{-(d-h)t} - D(d + h)e^{-(d+h)t}$$

Using the initial conditions we have

$$\left.\begin{aligned} 0 &= C + D + 0.839 \text{ m} \\ 0 &= -C(d - h) - D(d + h) \end{aligned}\right\} \Rightarrow C = -0.982 \text{ m}, \;\; D = 0.143 \text{ m}$$

The motion is

$$x = -(0.982 \text{ m})e^{-(0.764 \text{ rad/s})t} + (0.143 \text{ m})e^{-(5.24 \text{ rad/s})t} + 0.839 \text{ m}$$

At $t = 2$ s $\boxed{x = 0.626 \text{ m}}$

Problem 21.40 The mass $m = 2.19$ kg, the spring constant is $k = 7.3$ N/m, and the damping coefficient is $c = 11.67$ N-s/m. The spring is unstretched when $x = 0$. At $t = 0$, the mass is released from rest with $x = 0$. Determine the value of x at $t = 2$ s.

Solution: The equation of motion is

$$(2.19\ \text{kg})\frac{d^2x}{dt^2} + (11.67\ \text{N-s/m})\ \frac{dx}{dt} + (7.3\ \text{N/m})x$$

$$= (2.19\ \text{kg})(9.81\ \text{m/s}^2)\sin 20^\circ$$

$$\frac{d^2x}{dt^2} + 2(2.67\ \text{rad/s})\frac{dx}{dt} + (1.826\ \text{rad/s})^2 x = 3.36\ \text{m/s}^2$$

We identify

$$\omega = 1.83\ \text{rad/s},\ \ d = 2.67\ \text{rad/s},\ \ h = \sqrt{d^2 - \omega^2} = 1.94\ \text{rad/s}$$

Since $d > \omega$, we have the supercritical case. The solution is

$$x = Ce^{-(d-h)t} + De^{-(d+h)t} + 1.01\ \text{m}$$

$$v = \frac{dx}{dt} = -C(d-h)e^{-(d-h)t} - D(d+h)e^{-(d+h)t}$$

Using the initial conditions we find

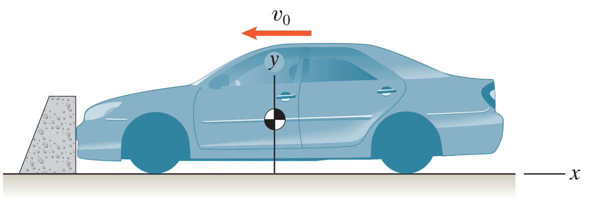
$$\left.\begin{array}{l} 0 = C + D + 1.01\ \text{m} \\ 0 = -C(d-h) - D(d+h) \end{array}\right\} \Rightarrow\ C = -1.19\ \text{m},\ \ D = 0.187\ \text{m}$$

Thus the solution is

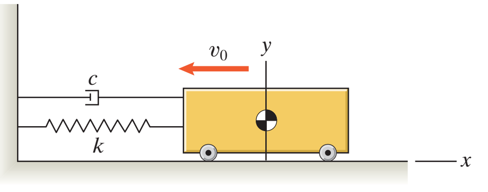
$$x = -(1.19\ \text{m})e^{-(0.723\ \text{rad/s})t} + (0.187\ \text{m})e^{-(4.61\ \text{rad/s})t} + 1.01\ \text{m}$$

At $t = 2$ s, $\boxed{x = 0.725\ \text{m}}$

Problem 21.41 A 79.8 kg test car moving with velocity $v_0 = 7.33$ m/s collides with a rigid barrier at $t = 0$. As a result of the behavior of its energy-absorbing bumper, the response of the car to the collision can be simulated by the damped spring-mass oscillator shown with $k = 8000$ N/m and $c = 3000$ N-s/m. Assume that the mass is moving to the left with velocity $v_0 = 7.33$ m/s and the spring is unstretched at $t = 0$. Determine the car's position (a) at $t = 0.04$ s and (b) at $t = 0.08$ s.



Car colliding with a rigid barrier



Simulation model

Solution: The equation of motion is

$$m\ddot{x} + c\dot{x} + kx = 0,$$

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$$

$$\ddot{x} + \frac{3000}{79.8}\dot{x} + \frac{8000}{79.8}x = 0, \quad \ddot{x} + 2(18.8)\dot{x} + (10.0)^2 x = 0$$

We recognize that $\omega = 10, \ d = 18.8 \ d > \omega \Rightarrow$ Supercritical damping.

We have $h = \sqrt{d^2 - \omega^2} = 15.9$ rad/s, $v_0 = 7.33$ m/s.

The solution to the differential equation is

$$x = Ce^{-(d-h)t} + De^{-(d+h)t}, \ \frac{dx}{dt} = -(d-h)Ce^{-(d-h)t} - (d+h)De^{-(d+h)t}.$$

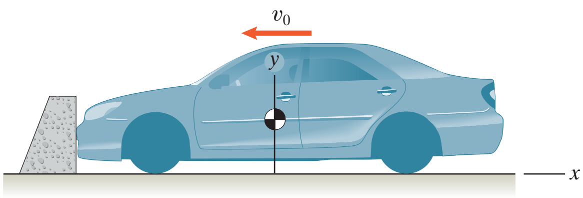
Putting in the initial conditions, we have

$$\left.\begin{array}{l} x(t=0) = C + D = 0 \\ \dfrac{dx}{dt}(t=0) = -(d-h)C - (d+h)D = -v_0 \end{array}\right\} \Rightarrow C = -D = -\frac{v_0}{2h} = -0.231 \text{ m}.$$

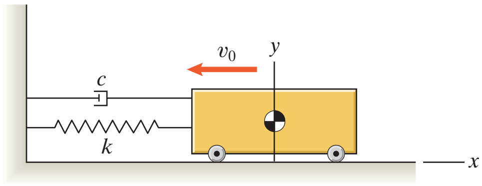
The general solution is then $x = (-0.231)(e^{-2.89t} - e^{-34.7t})$

At the 2 specified times we have $\boxed{\text{(a) } x = -0.148 \text{ m}, \ \text{(b) } x = -0.169 \text{ m}}$

Problem 21.42 A 79.8 kg test car moving with velocity $v_0 = 7.33$ m/s collides with a rigid barrier at $t = 0$. As a result of the behavior of its energy-absorbing bumper, the response of the car to the collision can be simulated by the damped spring-mass oscillator shown with $k = 8000$ N/m and $c = 3000$ N-s/m. Assume that the mass is moving to the left with velocity $v_0 = 7.33$ m/s and the spring is unstretched at $t = 0$. Determine the car's deceleration (a) immediately after it contacts the barrier; (b) at $t = 0.04$ s and (c) at $t = 0.08$ s.



Car colliding with a rigid barrier



Simulation model

Solution: From Problem 21.41 we know that the motion is given by

$$x = (-0.231)(e^{-2.89t} - e^{-34.7t}),$$

$$\frac{dx}{dt} = (-0.231)([-2.89]e^{-2.89t} - [-34.7]e^{-34.7t}),$$

$$\frac{d^2x}{dt^2} = (-0.231)([-2.89]^2 e^{-2.89t} - [-34.7]^2 e^{-34.7t})$$

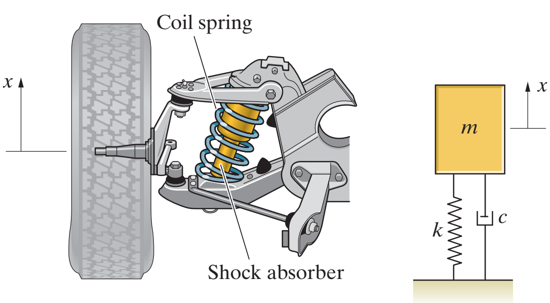
Thus the deceleration a is given by

$$a = -\frac{d^2x}{dt^2} = -(0.587)e^{-2.89t} + (84.5)e^{-34.7t}$$

Putting in the required times, we find

$$\boxed{\begin{aligned} &\text{(a) } a = 276 \text{ m/s}^2, \\ &\text{(b) } a = 67.6 \text{ m/s}^2, \\ &\text{(c) } a = 15.8 \text{ m/s}^2. \end{aligned}}$$

Problem 21.43 The motion of the car's suspension shown in Problem 21.42 can be modeled by the damped spring–mass oscillator in Fig. 21.9 with $m = 36$ kg, $k = 22$ kN/m, and $c = 2.2$ kN-s/m. Assume that no external forces act on the tire and wheel. At $t = 0$, the spring is unstretched and the tire and wheel are given a velocity $dx/dt = 10$ m/s. Determine the position x as a function of time.



Solution: Calculating ω and d, we obtain

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{22{,}000}{36}} = 24.72 \text{ rad/s}$$

and $$d = \frac{c}{2m} = \frac{2200}{2(36)} = 30.56 \text{ rad/s}.$$

Since $d > \omega$, the motion is supercritically damped and Equation (21.24) is the solution, where

$$h = \sqrt{d^2 - \omega^2} = 17.96 \text{ rad/s}.$$

Equation (21.24) is

$$x = Ce^{-(d-h)t} + De^{-(d+h)t},$$

or $$x = Ce^{-12.6t} + De^{-48.5t}.$$

The time derivative is

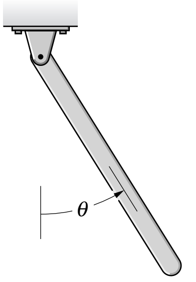
$$\frac{dx}{dt} = -12.6Ce^{-12.6t} - 48.5De^{-48.5t}.$$

At $t = 0$, $x = 0$ and $dx/dt = 10$ m/s: Hence, $0 = C + D$ and $10 = -12.6C - 48.5D$. Solving for C and D, we obtain $C = 0.278$ m, $D = -0.278$ m. The solution is

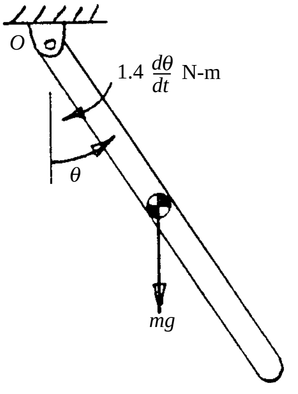
$$x = 0.278(e^{-12.6t} - e^{-48.5t}) \text{ m}.$$

Problem 21.44 The 4-kg slender bar is 2 m in length. Aerodynamic drag on the bar and friction at the support exert a resisting moment about the pin support of magnitude $1.4(d\theta/dt)$ N-m, where $d\theta/dt$ is the angular velocity in rad/s.

(a) What are the period and frequency of small vibrations of the bar?
(b) How long does it take for the amplitude of vibration to decrease to one-half of its initial value?



Solution:



(a) $\sum M_0 = I_0\alpha$:

$$-1.4\frac{d\theta}{dt} - mg\frac{L}{2}\sin\theta = \frac{1}{3}mL^2\alpha.$$

The (linearized) equation of motion is

$$\frac{d^2\theta}{dt^2} + \frac{4.2}{mL^2}\frac{d\theta}{dt} + \frac{3g}{2L}\theta = 0.$$

This is of the form of Eq. (21.16) with

$$d = \frac{4.2}{2mL^2} = \frac{4.2}{2(4)(2)^2} = 0.131 \text{ rad/s},$$

$$\omega = \sqrt{\frac{3g}{2L}} = \sqrt{\frac{3(9.81)}{2(2)}} = 2.71 \text{ rad/s}.$$

From Eq. (21.18), $\omega_d = \sqrt{\omega^2 - d^2} = 2.71$ rad/s, and from Eqs. (21.20),

$$\tau_d = \frac{2\pi}{\omega_d} = 2.32 \text{ s},$$

$$f_d = \frac{1}{\tau_d} = 0.431 \text{ Hz}.$$

(b) Setting $e^{-dt} = e^{-0.131t} = 0.5$, we obtain $t = 5.28$ s.

Problem 21.45 The bar described in Problem 21.44 is given a displacement $\theta = 2^\circ$ and released from rest at $t = 0$. What is the value of θ (in degrees) at $t = 2$ s?

Solution: From the solution of Problem 21.44, the damping is subcritical with $d = 0.131$ rad/s, $\omega = 2.71$ rad/s. From Eq. (21.19),

$$\theta = e^{-0.131t}(A\sin 2.71t + B\cos 2.71t),$$

so $$\frac{d\theta}{dt} = -0.131e^{-0.131t}(A\sin 2.71t + B\cos 2.71t)$$

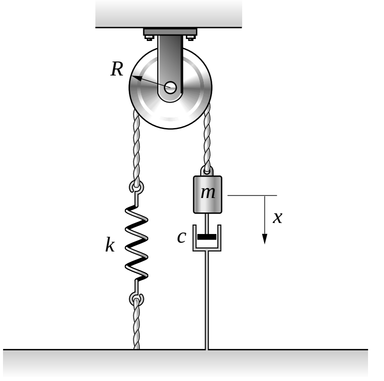
$$+ e^{-0.131t}(2.71A\cos 2.71t - 2.71B\sin 2.71t).$$

At $t = 0$, $\theta = 2^\circ$ and $d\theta/dt = 0$. Substituting these conditions, $2^\circ = B$, $0 = -0.131B + 2.71A$, we see that $B = 2^\circ$, $A = 0.0969^\circ$, so

$$\theta = e^{-0.131t}(0.0969^\circ \sin 2.71t + 2^\circ \cos 2.71t).$$

At $t = 2$ s, we obtain $\theta = 0.942^\circ$.

Problem 21.46 The radius of the pulley is $R = 100$ mm and its moment of inertia is $I = 0.1$ kg-m^2. The mass $m = 5$ kg, and the spring constant is $k = 135$ N/m. The cable does not slip relative to the pulley. The coordinate x measures the displacement of the mass relative to the position in which the spring is unstretched. Determine x as a function of time if $c = 60$ N-s/m and the system is released from rest with $x = 0$.



Solution: Denote the angular rotation of the pulley by θ. The moment on the pulley is $\sum M = R(kx) - RF$, where F is the force acting on the right side of the pulley. From the equation of angular motion for the pulley,

$$I\frac{d^2\theta}{dt^2} = Rkx - RF,$$

from which $F = -\frac{I}{R}\frac{d^2\theta}{dt^2} + kx$.

The force on the mass is $-F + f + mg$, where the friction force $f = -c\frac{dx}{dt}$ acts in opposition to the velocity of the mass. From Newton's second law for the mass,

$$m\frac{d^2x}{dt^2} = -F - c\frac{dx}{dt} + mg = \frac{I}{R}\frac{d^2\theta}{dt^2} - kx - e\frac{dx}{dt} + mg.$$

From kinematics, $\theta = -\frac{x}{R}$, from which the equation of motion for the mass is

$$\left(\frac{I}{R^2} + m\right)\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = mg.$$

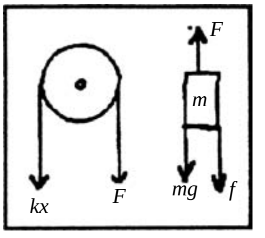
The canonical form (see Eq. (21.16)) of the equation of motion is

$$\frac{d^2x}{dt^2} + 2d\frac{dx}{dt} + \omega^2 x = \frac{R^2mg}{I + R^2m},$$

where $d = \frac{cR^2}{2(I + R^2m)} = 2$ rad/s, $\omega^2 = \frac{kR^2}{(I + R^2m)} = 9$ (rad/s)^2.

The damping is sub-critical, since $d^2 < \omega^2$. The solution is the sum of the solution to the homogenous equation of motion, of the form $x_c = e^{-dt}(A\sin\omega_d t + B\cos\omega_d t)$, where $\omega_d = \sqrt{\omega^2 - d^2} = 2.236$ rad/s, and the solution to the non-homogenous equation, of the form

$$x_p = \frac{mgR^2}{(I + R^2m)\omega^2} = \frac{mg}{k} = 0.3633.$$



(The particular solution x_p is obtained by setting the acceleration and velocity to zero and solving, since the non-homogenous term mg is not a function of time or position.) The solution is

$$x = x_c + x_p = e^{-dt}(A\sin\omega_d t + B\cos\omega_d t) + \frac{mg}{k}.$$

Apply the initial conditions: at $t = 0$, $x = 0$, $\frac{dx}{dt} = 0$, from which $0 = B + \frac{mg}{k}$, and $0 = -d[x_c]_{t=0} + \omega_d A = -dB + \omega_d A$, from which $B = -0.3633$, $A = \frac{dB}{\omega_d} = -0.3250$, and

$$x(t) = e^{-dt}\left(-\frac{dmg}{k\omega_d}\sin\omega_d - \left(\frac{mg}{k}\right)\cos\omega_d t\right) + \left(\frac{mg}{k}\right)$$

$$\boxed{\begin{aligned} x(t) = e^{-2t}(&-0.325\sin(2.236\ t) - 0.363\cos(2.236\ t)) \\ &+ 0.3633\ \text{(m)} \end{aligned}}$$

Problem 21.47 For the system described in Problem 21.46, determine x as a function of time if $c = 120$ N-s/m and the system is released from rest with $x = 0$.

Solution: From the solution to Problem 21.46 the canonical form of the equation of motion is

$$\frac{d^2x}{dt^2} + 2d\frac{dx}{dt} + \omega^2 x = \frac{R^2 mg}{I + R^2 m},$$

where $d = \dfrac{cR^2}{2(I + R^2 m)} = 4$ rad/s,

and $\omega^2 = \dfrac{kR^2}{(I + R^2 m)} = 9$ (rad/s)^2.

The system is supercritically damped, since $d^2 > \omega^2$. The homogenous solution is of the form (see Eq. (21.24)) $x_c = Ce^{-(d-h)t} + De^{-(d+h)t}$, where

$h = \sqrt{d^2 - \omega^2} = 2.646$ rad/s, $(d - h) = 1.354$, $(d + h) = 6.646$.

The particular solution is $x_p = \dfrac{mg}{k} = 0.3633$. The solution is

$$x(t) = x_c + x_{p_c} = Ce^{-(d-h)t} + De^{-(d+h)t} + \frac{mg}{k}.$$

Apply the initial conditions, at $t = 0$, $x = 0$, $\dfrac{dx}{dt} = 0$, from which $0 = C + D + \dfrac{mg}{k}$, and $0 = -(d - h)C - (d + h)D$. Solve:

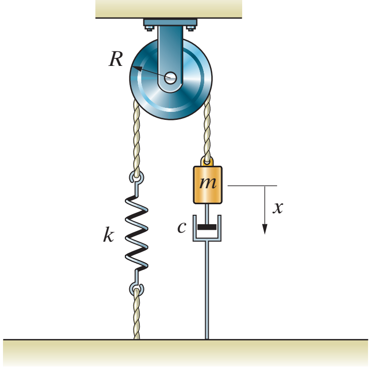
$$C = -\frac{(d + h)}{2h}\left(\frac{mg}{k}\right),$$

and $D = \dfrac{(d - h)}{2h}\left(\dfrac{mg}{k}\right)$,

from which

$$\boxed{\begin{aligned} x(t) &= \frac{mg}{k}\left(1 - \frac{(d + h)}{2h}e^{-(d-h)t} + \frac{(d - h)}{2h}e^{-(d+h)t}\right) \\ &= 0.3633(1 - 1.256e^{-1.354t} + 0.2559e^{-6.646t}) \text{ (m)} \end{aligned}}.$$

Problem 21.48 For the system described in Problem 21.46, choose the value of c so that the system is critically damped, and determine x as a function of time if the system is released from rest with $x = 0$.



Solution: From the solution to Problem 21.46, the canonical form of the equation of motion is

$$\frac{d^2x}{dt^2} + 2d\frac{dx}{dt} + \omega^2 x = \frac{R^2 mg}{I + R^2 m},$$

where $d = \dfrac{cR^2}{2(I + R^2 m)}$, and $\omega^2 = \dfrac{kR^2}{(I + R^2 m)} = 9$ (rad/s)^2.

For critical damping, $d^2 = \omega^2$, from which $d = 3$ rad/s. The homogenous solution is (see Eq. (21.25)) $x_c = Ce^{-dt} + Dte^{-dt}$ and the particular solution is $x_p = \dfrac{mg}{k} = 0.3633$ m. The solution:

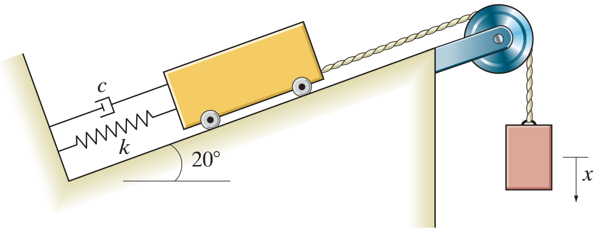
$$x(t) = x_c + x_p = Ce^{-dt} + Dte^{-dt} + \frac{mg}{k}.$$

Apply the initial conditions at $t = 0$, $x = 0$, $\dfrac{dx}{dt} = 0$, from which $C + \dfrac{mg}{k} = 0$, and $-dC + D = 0$. Solve: $C = -\dfrac{mg}{k} = -0.3633$, $D = dC = -1.09$. The solution is

$$x(t) = \frac{mg}{k}(1 - e^{-dt}(1 + dt)),$$

$$\boxed{x(t) = 0.363(1 - e^{-3t}(1 + 3t)) \text{ m}}$$

Problem 21.49 The spring constant is $k = 800$ N/m, and the spring is unstretched when $x = 0$. The mass of each object is 30 kg. The inclined surface is smooth. The radius of the pulley is 120 mm and it moment of inertia is $I = 0.03$ kg-m^2. Determine the frequency and period of vibration of the system relative to its equilibrium position if (a) $c = 0$, (b) $c = 250$ N-s/m.



Solution: Let T_1 be the tension in the rope on the left of the pulley, and T_2 be the tension in the rope on the right of the pulley. The equations of motion are

$$T_1 - mg\sin\theta - c\dot{x} - kx = m\ddot{x},\ T_2 - mg = -m\ddot{x},\ (T_2 - T_1)\,r = I\frac{\ddot{x}}{r}.$$

If we eliminate T_1 and T_2, we find

$$\left(2m + \frac{I}{r^2}\right)\ddot{x} + c\dot{x} + kx = mg(1 - \sin\theta),$$

$$\ddot{x} + \frac{cr^2}{2mr^2 + I}\dot{x} + \frac{kr^2}{2mr^2 + I}x = \frac{mgr^2}{2mr^2 + I}(1 - \sin\theta),$$

$$\ddot{x} + (0.016c)\dot{x} + (3.59\text{ rad/s})^2 x = 3.12\text{ m/s}^2.$$

(a) If we set $c = 0$, the natural frequency, frequency, and period are

$$\omega = 3.59\text{ rad/s},\ f = \frac{\omega}{2\pi} = 0.571\text{ Hz},\ \tau = \frac{1}{f} = 1.75\text{ s.}\quad \boxed{\begin{array}{l} f = 0.571\text{ s,} \\ \tau = 1.75\text{ s.}\end{array}}$$

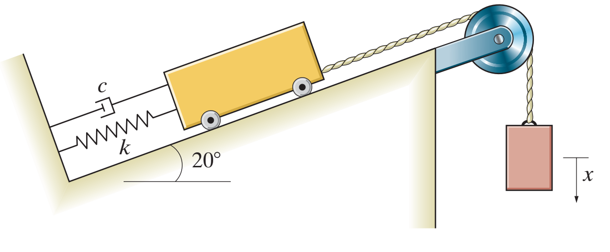
(b) If we set $c = 250$ N/m, then

$$\ddot{x} + 2(2.01)\dot{x} + (3.59\text{ rad/s})^2 x = 3.12\text{ m/s}^2.$$

We recognize $\omega = 3.59,\ d = 2.01,\ \omega_d = \sqrt{\omega^2 - d^2} = 2.97$ rad/s.

$$f = \frac{\omega_d}{2\pi},\quad \tau = \frac{1}{f},\quad \boxed{f = 0.473\text{ Hz},\ \tau = 2.11\text{ s.}}$$

Problem 21.50 The spring constant is $k = 800$ N/m, and the spring is unstretched when $x = 0$. The damping constant is $c = 250$ N-s/m. The mass of each object is 30 kg. The inclined surface is smooth. The radius of the pulley is 120 mm and it moment of inertia is $I = 0.03$ kg-m^2. At $t = 0$, $x = 0$ and $dx/dt = 1$ m/s. What is the value of x at $t = 2$ s?



Solution: From Problem 21.49 we know that the damping is sub-critical and the key parameters are

$$\omega = 3.59\text{ rad/s},\ d = 2.01\text{ rad/s},\ \omega_{\rm d} = 2.97\text{ rad/s}.$$

The equation of motion is $\ddot{x} + 2(2.01)\dot{x} + (3.59\text{ rad/s})^2 x = 3.12\text{ m/s}^2$.

The solution is

$$x = e^{-dt}(A\sin\omega_d t + B\cos\omega_d t) + 0.242$$

$$\frac{dx}{dt} = e^{-dt}([A\omega_d - Bd]\cos\omega_d t - [Ad + B\omega_d]\sin\omega_d t)$$

Putting in the initial conditions, we have

$$x(t = 0) = B + 0.242 = 0 \Rightarrow B = -0.242,$$

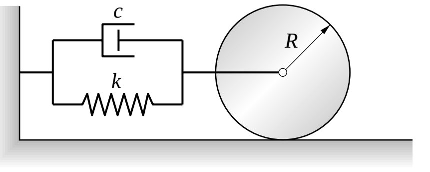
$$\frac{dx}{dt}(t = 0) = A\omega_d - Bd = 1 \Rightarrow A = 0.172$$

Thus the motion is governed by

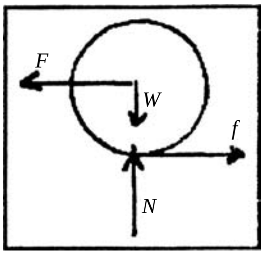
$$x = e^{-2.01t}(0.172\sin[2.97t] - 0.242\cos[2.97t]) + 0.242$$

At time $t = 2$ s we have $\boxed{x = 0.237\text{ m.}}$

Problem 21.51 The homogeneous disk weighs 445 N and its radius is $R = 0.31$ m. It rolls on the plane surface. The spring constant is $k = 1459.3$ N/m and the damping constant is $c = 43.8$ N-s/m. Determine the frequency of small vibrations of the disk relative to its equilibrium position.



Solution: Choose a coordinate system with the origin at the center of the disk and the positive x axis parallel to the floor. Denote the angle of rotation by θ. The *horizontal* forces acting on the disk are



$$\sum F = -kx - c\frac{dx}{dt} + f.$$

From Newton's second law,

$$m\frac{d^2x}{dt^2} = \sum F = -kx - c\frac{dx}{dt} + f.$$

The moment about the center of mass of the disk is $\sum M = Rf$. From the equation of angular motion, $I\frac{d^2\theta}{dt^2} = Rf$, from which $f = \frac{I}{R}\frac{d^2\theta}{dt^2}$, where the moment of inertia is $I = \frac{mR^2}{2} = 0.642$ kg-m^2. Substitute:

$$m\frac{d^2x}{dt^2} = -kx - c\frac{dx}{dt} + \frac{I}{R}\frac{d^2\theta}{dt^2}.$$

From kinematics, $\theta = -\frac{x}{R}$, from which the equation of motion is

$$\left(m + \frac{I}{R^2}\right)\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0.$$

The canonical form (see Eq. (21.16)) is

$$\frac{d^2x}{dt^2} + 2d\frac{dx}{dt} + \omega^2 x = 0,$$

where $d = \frac{cR^2}{2(mR^2 + I)} = \frac{c}{3m} = 0.3217$ rad/s

and $\omega^2 = \frac{kR^2}{(I + R^2m)} = \frac{2k}{3m} = 21.45$ (rad/s)^2.

The damping is sub-critical, since $d^2 < \omega^2$. The frequency is

$$\boxed{f_d = \frac{1}{2\pi}\sqrt{\omega^2 - d^2} = 0.7353 \text{ Hz}}$$

Problem 21.52 In Problem 21.51, the spring is unstretched at $t = 0$ and the disk has a clockwise angular velocity of 2 rad/s. What is the angular velocity of the disk when $t = 3$ s?

Solution: From the solution to Problem 21.51, the canonical form of the equation of motion is

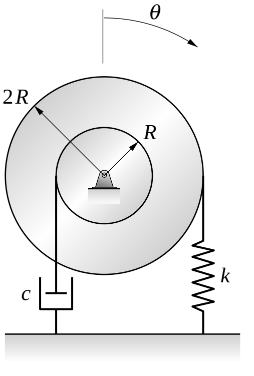
$$\frac{d^2x}{dt^2} + 2d\frac{dx}{dt} + \omega^2 x = 0,$$

where $d = \frac{c}{3m} = 0.322$ rad/s and $\omega^2 = \frac{2k}{3m} = 21.47$ (rad/s)^2.

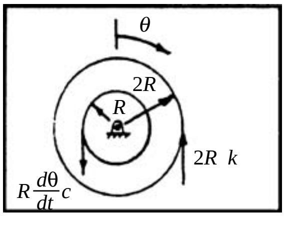
The system is sub-critically damped, so that the solution is of the form $x = e^{-dt}(A \sin \omega_d t + B \cos \omega_d t)$, where $\omega_d = \sqrt{\omega^2 - d^2} =$ 4.622 rad/s. Apply the initial conditions: $x_0 = 0$, and from kinematics, $\dot{\theta}_0 = \dot{x}_0/R = -2$ rad/s, from which $\dot{x}_0 = 0.61$ m/s, from which $B = 0$ and $A = \dot{x}_0/\omega_d = 0.132$. The solution is $x(t) = e^{-dt}\left(\frac{\dot{x}_0}{\omega_d}\right) \sin \omega_d t$, and $\dot{x}(t) = -dx + \dot{x}_0 e^{-dt} \cos \omega_d t$. At $t = 3$ s, $x = 0.048$ m and $\dot{x} = 0.0466$ m/s. From kinematics, $\theta(t) = -\frac{\dot{x}(t)}{R}$.

At $t = 3$ s, $\boxed{\dot{\theta} = -0.153 \text{ rad/s}}$ clockwise.

Problem 21.53 The moment of inertia of the stepped disk is I. Let θ be the angular displacement of the disk relative to its position when the spring is unstretched. Show that the equation governing θ is identical in form to Eq. (21.16), where $d = \frac{R^2 c}{2I}$ and $\omega^2 = \frac{4R^2 k}{I}$.



Solution: The sum of the moments about the center of the stepped disk is



$$\sum M = -R\left(Rc\frac{d\theta}{dt}\right) - 2R(2Rk\theta).$$

From the equation of angular motion

$$\sum M = I\frac{d^2\theta}{dt^2}.$$

The equation of motion is

$$I\frac{d^2\theta}{dt^2} + R^2 c\frac{d\theta}{dt} + 4R^2 k\theta = 0.$$

The canonical form is

$$\boxed{\frac{d^2\theta}{dt^2} + 2d\frac{d\theta}{dt} + \omega^2\theta = 0}, \text{ where } d = \frac{R^2 c}{2I}, \omega^2 = \frac{4R^2 k}{I}.$$

Problem 21.54 In Problem 21.53, the radius $R = 250$ mm, $k = 150$ N/m, and the moment of inertia of the disk is $I = 2$ kg-m^2.

(a) At what value of c will the system be critically damped?
(b) At $t = 0$, the spring is unstretched and the clockwise angular velocity of the disk is 10 rad/s. Determine θ as a function of time if the system is critically damped.
(c) Using the result of (b), determine the maximum resulting angular displacement of the disk and the time at which it occurs.

Solution: From the solution to Problem 21.53, the canonical form of the equation of motion is

$$\frac{d^2\theta}{dt^2} + 2d\frac{d\theta}{dt} + \omega^2\theta = 0,$$

where $d = \dfrac{R^2c}{2I}$ and $\omega^2 = \dfrac{4R^2k}{I}$.

(a) For critical damping, $d^2 = \omega^2$, from which

$\boxed{c = \frac{4}{R}\sqrt{kI} = 277 \text{ N-s/m}}$, and $d = 4.330$ rad/s.

(b) The solution is of the form (see Eq. (21.25)) $\theta = Ce^{-dt} + Dte^{-dt}$. Apply the initial conditions: $\theta_0 = 0$, $\dot{\theta}_0 = -10$ rad/s, from which $C = 0$, and $D = -10$. The solution is

$\boxed{\theta(t) = \dot{\theta}_0 te^{-dt} = -10te^{-4.330t} \text{ rad/s}}$.

(c) The maximum (or minimum) value of the angular displacement is obtained from

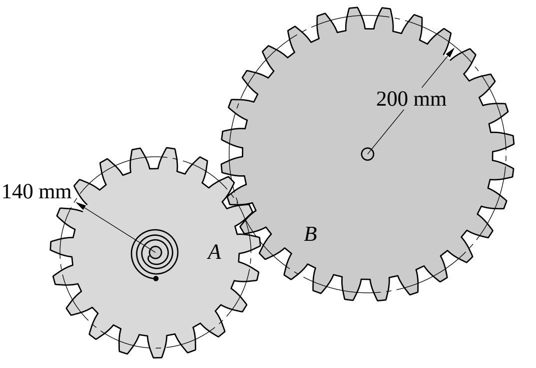
$$\frac{d\theta}{dt} = 0 = -10e^{-4.330t}(1 - 4.33t) = 0,$$

from which the maximum/minimum occurs at

$\boxed{t_{\max} = \frac{1}{4.330} = 0.231 \text{ s}}$.

The angle is $[\theta]_{t=t_{\max}} = -0.850$ rad *clockwise*.

Problem 21.55 The moments of inertia of gears A and B are $I_A = 0.025$ kg-m^2 and $I_B = 0.100$ kg-m^2. Gear A is connected to a torsional spring with constant $k = 10$ N-m/rad. The bearing supporting gear B incorporates a damping element that exerts a resisting moment on gear B of magnitude $2(d\theta_B/dt)$ N-m, where $d\theta_B/dt$ is the angular velocity of gear B in rad/s. What is the frequency of small angular vibrations of the gears?



Solution: The sum of the moments on gear A is $\sum M = -k\theta_A + R_A F$, where the moment exerted by the spring opposes the angular displacement θ_A. From the equation of angular motion,

$$I_A \frac{d^2\theta_A}{dt^2} = \sum M = -k\theta_A + R_A F,$$

from which $F = \left(\frac{I_A}{R_A}\right)\frac{d^2\theta_A}{dt^2} + \left(\frac{k}{R_A}\right)\theta_A.$

The sum of the moments acting on gear B is

$$\sum M = -2\frac{d\theta_B}{dt} + R_B F,$$

where the moment exerted by the damping element opposes the angular velocity of B. From the equation of angular motion applied to B,

$$I_B \frac{d^2\theta_B}{dt^2} = \sum M = -2\frac{d\theta_B}{dt} + R_B F.$$

Substitute the expression for F,

$$I_B \frac{d^2\theta_B}{dt^2} + 2\frac{d\theta_B}{dt} - \left(\frac{R_B}{R_A}\right)\left(I_A \frac{d^2\theta_A}{dt^2} + k\theta_A\right) = 0.$$

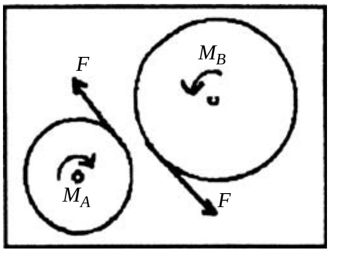
From kinematics,

$$\theta_A = -\left(\frac{R_B}{R_A}\right)\theta_B,$$

from which the equation of motion for gear B is

$$\left(I_B + \left(\frac{R_B}{R_A}\right)^2 I_A\right)\frac{d^2\theta_B}{dt^2} + 2\frac{d\theta_B}{dt} + \left(\frac{R_B}{R_A}\right)^2 k\theta_B = 0.$$

Define $M = I_B + \left(\frac{R_B}{R_A}\right)^2 I_A = 0.1510$ kg-m^2.



The canonical form of the equation of motion is

$$\frac{d^2\theta_B}{dt^2} + 2d\frac{d\theta_B}{dt} + \omega^2\theta_B = 0,$$

where $d = \frac{1}{M} = 6.622$ rad/s

and $\omega^2 = \left(\frac{R_B}{R_A}\right)^2 \frac{k}{M} = 135.1$ (rad/s)^2.

The system is sub critically damped, since $d^2 < \omega^2$, from which $\omega_d = \sqrt{\omega^2 - d^2} = 9.555$ rad/s, from which the frequency of small vibrations is

$$\boxed{f_d = \frac{\omega_d}{2\pi} = 1.521 \text{ Hz}}$$

Problem 21.56 At $t = 0$, the torsional spring in Problem 21.55 is unstretched and gear B has a counterclockwise angular velocity of 2 rad/s. Determine the counterclockwise angular position of gear B relative to its equilibrium position as a function of time.

Solution: From the solution to Problem 21.55, the canonical form of the equation of motion for gear B is

$$\frac{d^2\theta_B}{dt^2} + 2d\frac{d\theta_B}{dt} + \omega^2\theta_B = 0,$$

where $M = I_B + \left(\frac{R_B}{R_A}\right)^2 I_A = 0.1510$ kg-m^2,

$$d = \frac{1}{M} = 6.622 \text{ rad/s},$$

and $\omega^2 = \left(\frac{R_B}{R_A}\right)^2 \frac{k}{M} = 135.1$ (rad/s)^2.

The system is sub critically damped, since $d^2 < \omega^2$, from which $\omega_d = \sqrt{\omega^2 - d^2} = 9.555$ rad/s. The solution is of the form $\theta_B(t) = e^{-dt}(A\sin\omega_d t + B\cos\omega_d t)$. Apply the initial conditions, $[\theta_B]_{t=0} = 0$, $[\dot{\theta}_B]_{t=0} = 2$ rad/s, from which $B = 0$, and $A = \dfrac{2}{\omega_d} = 0.2093$, from which the solution is

$$\boxed{\theta_B(t) = e^{-6.62t}(0.209\sin(9.55t))}$$

Problem 21.57 For the case of critically damped motion, confirm that the expression $x = Ce^{-dt} + Dte^{-dt}$ is a solution of Eq. (21.16).

Solution: Eq. (21.16) is

$$\frac{d^2x}{dt^2} + 2d\frac{dx}{dt} + \omega^2 x = 0.$$

We show that the expression is a solution by substitution. The individual terms are:

(1) $x = e^{-dt}(C + Dt)$,

(2) $\dfrac{dx}{dt} = -dx + De^{-dt}$,

(3) $\dfrac{d^2x}{dt^2} = -d\dfrac{dx}{dt} - dDe^{-dt} = d^2x - 2dDe^{-dt}$.

Substitute:

$$\frac{d^2x}{dt^2} + 2d\frac{dx}{dt} + \omega^2 x = (d^2x - 2De^{-dt}) + 2d(-dx + De^{-dt})$$

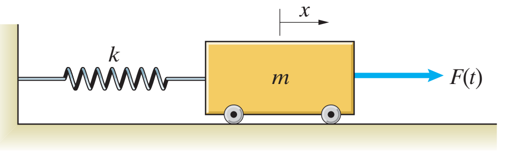
$$+ \omega^2 x = 0.$$

Reduce:

$$\frac{d^2x}{dt^2} + 2d\frac{dx}{dt} + \omega^2 x = (-d^2 + \omega^2)x = 0.$$

This is true if $d^2 = \omega^2$, which is the definition of a critically damped system. *Note*: Substitution leading to an identity shows that $x = Ce^{-dt} + Dte^{-dt}$ is *a solution*. It *does not* prove that it is *the only solution*.

Problem 21.58 The mass $m = 2$ kg and the spring constant is $k = 72$ N/m. The spring is unstretched when $x = 0$. The mass is initially stationary with the spring unstretched, and at $t = 0$ the force $F(t) = 10 \sin 4t$ N is applied to the mass. What is the position of the mass at $t = 2$ s?



Solution: The equation of motion is

$$m\ddot{x} + kx = F(t) \Rightarrow \ddot{x} + \frac{k}{m}x = \frac{F(t)}{m} \Rightarrow \ddot{x} + \frac{72}{2}x = \frac{10}{2}\sin 4t$$

$$\ddot{x} + (6)^2 x = 5 \sin 4t$$

We recognize the following:

$\omega = 6, \omega_0 = 4, d = 0, a_0 = 5, b_0 = 0.$

$$A_p = \frac{(6^2 - 4^2)5}{(6^2 - 4^2)^2} = 0.25, \; B_p = 0$$

Therefore, the complete solution (homogeneous plus particular) is

$$x = A \sin 6t + B \cos 6t + 0.25 \sin 4t, \; \frac{dx}{dt} = 6A \cos 6t - 6B \sin 6t + \cos 4t.$$

Putting in the initial conditions, we have

$$x(t = 0) = B = 0, \; \frac{dx}{dt}(t = 0) = 6A + 1 \Rightarrow A = -0.167.$$

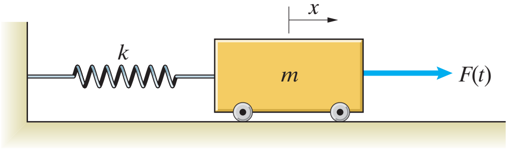
The complete solution is now

$$x = -0.167 \sin 6t + 0.25 \sin 4t$$

At $t = 6$ s we have

$$\boxed{x = 0.337 \text{ m.}}$$

Problem 21.59 The mass $m = 2$ kg and the spring constant is $k = 72$ N/m. The spring is unstretched when $x = 0$. At $t = 0$, $x = 1$ m, $dx/dt = 1$ m/s, and the force $F(t) = 10 \sin 4t + 10 \cos 4t$ N is applied to the mass. What is the position of the mass at $t = 2$ s?



Solution: The equation of motion is

$$m\ddot{x} + kx = F(t) \Rightarrow \ddot{x} + \frac{k}{m}x = \frac{F(t)}{m} \Rightarrow \ddot{x} + \frac{72}{2}x = \frac{10}{2}\sin 4t + \frac{10}{2}\cos 4t$$

$$\ddot{x} + (6)^2 x = 5 \sin 4t + 5 \cos 4t.$$

We recognize the following:

$\omega = 6, \omega_0 = 4, d = 0, a_0 = 5, b_0 = 5.$

$$A_p = \frac{(6^2 - 4^2)5}{(6^2 - 4^2)^2} = 0.25, \; B_p = \frac{(6^2 - 4^2)5}{(6^2 - 4^2)^2} = 0.25.$$

Therefore, the complete solution (homogeneous plus particular) is

$$x = A \sin 6t + B \cos 6t + 0.25 \sin 4t + 0.25 \cos 4t,$$

$$\frac{dx}{dt} = 6A \cos 6t - 6B \sin 6t + \cos 4t - \sin 4t.$$

Putting in the initial conditions, we have

$$x(t = 0) = B + 0.25 = 1 \Rightarrow B = 0.75$$

$$\frac{dx}{dt}(t = 0) = 6A + 1 = 1 \Rightarrow A = 0.$$

The complete solution is now

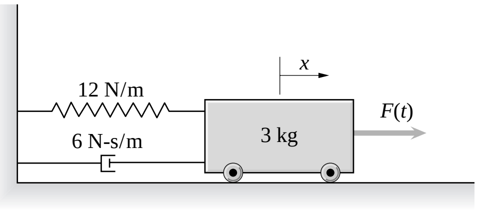
$$x = 0.75 \cos 6t + 0.25 \sin 4t + 0.25 \cos 4t.$$

At $t = 6$ s we have

$$\boxed{x = 0.844 \text{ m.}}$$

Problem 21.60 The damped spring–mass oscillator is initially stationary with the spring unstretched. At $t = 0$, a constant force $F(t) = 6$ N is applied to the mass.

(a) What is the steady-state (particular) solution?
(b) Determine the position of the mass as a function of time.



Solution: Writing Newton's second law for the mass, the equation of motion is

$$F(t) - c\frac{dx}{dt} - kx = m\frac{d^2x}{dt^2} \text{ or}$$

$$F(t) - 6\frac{dx}{dt} - 12x = 3\frac{d^2x}{dt^2},$$

which we can write as

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 4x = \frac{F(t)}{3}. \quad \textbf{(1)}$$

(a) If $F(t) = 6$ N, we seek a particular solution of the form $x_p = A_0$, a constant. Substituting it into Equation (1), we get $4x_p = \dfrac{F(t)}{3} = 2$ and obtain the particular solution: $x_p = 0.5$ m.

(b) Comparing equation (1) with Equation (21.26), we see that $d = 1$ rad/s and $\omega = 2$ rad/s. The system is subcritically damped and the homogeneous solution is given by Equation (21.19) with $\omega_d = \sqrt{\omega^2 - d^2} = 1.73$ rad/s. The general solution is

$$x = x_h + x_p = e^{-1}(A \sin 1.73t + B \cos 1.73t) + 0.5 \text{ m}.$$

The time derivative is

$$\frac{dx}{dt} = -e^{-1}(A \sin 1.73t + B \cos 1.73t)$$

$$+ e^{-t}(1.73A \cos 1.73t - 1.73t - 1.73B \sin 1.73t).$$

At $t = 0$, $x = 0$, and $dx/dt = 0 \quad 0 = B + 0.5$, and $0 = -B + 1.73A$. We see that $B = -0.5$ and $A = -0.289$ and the solution is

$$x = e^{-t}(-0.289 \sin 1.73t - 0.5 \cos 1.73t) + 0.5 \text{ m}.$$

Problem 21.61 The damped spring–mass oscillator shown in Problem 21.60 is initially stationary with the spring unstretched. At $t = 0$, a force $F(t) = 6 \cos 1.6t$ N is applied to the mass.

(a) What is the steady-state (particular) solution?
(b) Determine the position of the mass as a function of time.

Solution: Writing Newton's second law for the mass, the equation of motion can be written as

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 4x = \frac{F(t)}{3} = 2\cos 1.6t.$$

(a) Comparing this equation with Equation (21.26), we see that $d = 1$ rad/s, $\omega = 2$ rad/s, and the forcing function is $a(t) = 2\cos 1.6t$. This forcing function is of the form of Equation (21.27) with $a_0 = 0$, $b_0 = 2$ and $\omega_0 = 1.6$. Substituting these values into Equation (21.30), the particular solution is $x_p = 0.520 \sin 1.6t + 0.234 \cos 1.6t$ (m).

(b) The system is subcritically damped so the homogeneous solution is given by Equation (21.19) with $\omega_d = \sqrt{\omega^2 - d^2} = 1.73$ rad/s. The general solution is

$$x = x_k + x_p = e^{-t}(A \sin 1.73t + B \cos 1.73t) + 0.520 \sin 1.6t$$

$$+ 0.234 \cos 1.6t.$$

The time derivative is

$$\frac{dx}{dt} = -e^{-t}(A \sin 1.73t + B \cos 1.73t)$$

$$+ e^{-t}(1.73A \cos 1.73t - 1.73B \sin 1.73t)$$

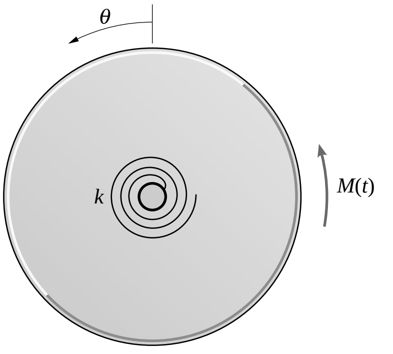
$$+ (1.6)(0.520) \cos 1.6t - (1.6)(0.234) \sin 1.6t$$

At $t = 0$, $x = 0$ and $dx/dt = 0 : 0 = B + 0.2340 = -B + 1.73A + (1.6)(0.520)$. Solving, we obtain $A = -0.615$ and $B = -0.234$, so the solution is

$$x = e^{-t}(-0.615 \sin 1.73t - 0.234 \cos 1.73t) + 0.520 \sin 1.6t$$

$$+ 0.234 \cos 1.6t \text{ m}.$$

Problem 21.62 The disk with moment of inertia $I = 3$ kg-m^2 rotates about a fixed shaft and is attached to a torsional spring with constant $k = 20$ N-m/rad. At $t = 0$, the angle $\theta = 0$, the angular velocity is $d\theta/dt = 4$ rad/s, and the disk is subjected to a couple $M(t) = 10 \sin 2t$ N-m. Determine θ as a function of time.



Solution: The equation of angular motion for the disk is

$$M(t) - k\theta = 1\frac{d^2\theta}{dt^2}: \text{ or } 10 \sin 2t - (20)\theta = 3\frac{d^2\theta}{dt^2},$$

or, rewriting in standard form, we have

$$\frac{d^2\theta}{dt^2} + \frac{20}{3}\theta = \frac{10}{3}\sin 2t$$

(a) Comparing this equation with equation (21.26), we see that $d = 0$, $\omega = \sqrt{20/3} = 2.58$ rad/s and the forcing function is $a(t) = \frac{10}{3}\sin 2t$. This forcing function is of the form of Equation (21.27), with $a_0 = 10/3$, $b_0 = 0$ and $\omega_0 = 2$. Substituting these values into Equation (21.30), the particular solution is $\theta_p = 1.25 \sin 2t$.

(b) The general solution is

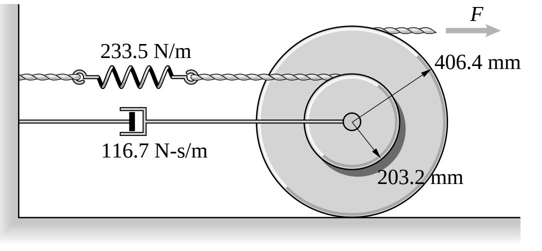
$$\theta = \theta_h + \theta_p = A \sin 2.58t + B \cos 2.58t + 1.25 \sin 2t$$

The time derivative is

$$\frac{d\theta}{dt} = 2.58A \cos 2.58t - 2.58B \sin 2.58t + 2.50 \cos 2t.$$

At $t = 0$, $\theta = 0$ and $d\theta/dt = 4$ rad/s, $0 = B$, and $4 = 2.58A + 2.50$. Solving, we obtain $A = 0.581$ and $B = 0$. The solution is $\theta = 0.581 \sin 2.58t + 1.25 \sin 2t$ rad.

Problem 21.63 The stepped disk weighs 89 N and its moment of inertia is $I = 0.81$ kg-m^2. It rolls on the horizontal surface. The disk is initially stationary with the spring unstretched, and at $t = 0$ a constant force $F = 44.5$ N is applied as shown. Determine the position of the center of the disk as a function of time.



Solution: The strategy is to apply the free body diagram to obtain equations for both θ and x, and then to eliminate one of these. An essential element in the strategy is the determination of the stretch of the spring. Denote $R = 0.203$ m, and the stretch of the spring by S. Choose a coordinate system with the positive x axis to the right. The sum of the moments about the center of the disk is $\sum M_C = RkS + 2Rf - 2RF$. From the equation of angular motion,

$$I\frac{d^2\theta}{dt^2} = \sum M_C = RkS + 2Rf - 2RF.$$

Solve for the reaction at the floor:

$$f = \frac{I}{2R}\frac{d^2\theta}{dt^2} - \frac{k}{2}S + F.$$

The sum of the horizontal forces:

$$\sum F_x = -kS - c\frac{dx}{dt} + F + f.$$

From Newton's second law:

$$m\frac{d^2x}{dt^2} = \sum F_x = -kS - c\frac{dx}{dt} + F + f.$$

Substitute for f and rearrange:

$$m\frac{d^2x}{dt^2} + \frac{I}{2R}\frac{d^2\theta}{dt^2} + c\frac{dx}{dt} + \frac{3}{2}kS = 2F.$$

From kinematics, the displacement of the center of the disk is $x = -2R\theta$. The stretch of the spring is the amount wrapped around the disk plus the translation of the disk, $S = -R\theta - 2R\theta = -3R\theta = \frac{3}{2}x$. Substitute:

$$\left(m + \frac{I}{(2R)^2}\right)\frac{d^2x}{dt^2} + c\frac{dx}{dt} + \left(\frac{3}{2}\right)^2 kx = 2F.$$

Define $M = m + \dfrac{I}{(2R)^2} = 14$ kg.

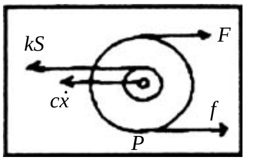
The canonical form of the equation of motion is

$$\frac{d^2x}{dt^2} + 2d\frac{dx}{dt} + \omega^2 x = a,$$

where $d = \dfrac{c}{2M} = 4.170$ rad/s,

$$\omega^2 = \left(\frac{3}{2}\right)^2\frac{k}{M} = 37.53 \text{ (rad/s)}^2,$$

and $a = \dfrac{2F}{M} = 6.35$ m/s^2.

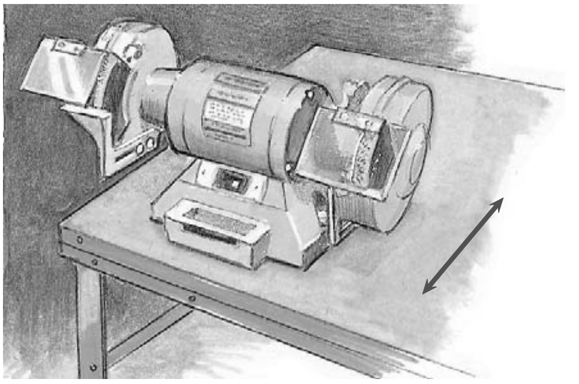


The particular solution is found by setting the acceleration and velocity to zero and solving: $x_p = \dfrac{a}{\omega^2} = \dfrac{8F}{9k} = 0.169$ m. Since $d^2 < \omega^2$, the system is sub-critically damped, so the homogenous solution is $x_h = e^{-dt}(A\sin\omega_d t + B\cos\omega_d t)$, where $\omega_d = \sqrt{\omega^2 - d^2} = 4.488$ rad/s. The complete solution is $x = e^{-dt}(A\sin\omega_d t + B\cos\omega_d t) + \dfrac{8F}{9k}$. Apply the initial conditions: at $t = 0$, $x_0 = 0$, $\dot{x}_0 = 0$, from which $B = -\dfrac{8F}{9k} = -0.169$, and $A = \dfrac{dB}{\omega_d} = -0.157$. Adopting $g = 9.81$ m/s^2 the solution is

$$x = \frac{8F}{9k}\left[1 - e^{dt}\left(\frac{d}{\omega_d}\sin\omega_d t + \cos\omega_d t\right)\right]$$

$$\boxed{= 0.169 - e^{-4.170t}(0.157\sin 4\,488t + 0.169\cos 4\,488t) \text{ m}},$$

Problem 21.64* An electric motor is bolted to a metal table. When the motor is on, it causes the tabletop to vibrate horizontally. Assume that the legs of the table behave like linear springs, and neglect damping. The total weight of the motor and the tabletop is 667 N. When the motor is not turned on, the frequency of horizontal vibration of the tabletop and motor is 5 Hz. When the motor is running at 600 rpm, the amplitude of the horizontal vibration is 0.25 mm. What is the magnitude of the oscillatory force exerted on the table by the motor at its 600-rpm running speed?



Solution: For $d = 0$, the canonical form (see Eq. (21.26)) of the equation of motion is

$$\frac{d^2x}{dt^2} + \omega^2 x = a(t),$$

where $\omega^2 = (2\pi f)^2 = (10\pi)^2 = 986.96 \text{ (rad/s)}^2$

and $a(t) = \dfrac{F(t)}{m} = \dfrac{gF(t)}{W}$.

The forcing frequency is

$$f_0 = \left(\frac{600}{60}\right) = 10 \text{ Hz},$$

from which $\omega_0 = (2\pi)10 = 62.83$ rad/s. Assume that $F(t)$ can be written in the form $F(t) = F_0 \sin\omega_0 t$. From Eq. (21.31), the amplitude of the oscillation is

$$x_0 = \frac{a_0}{\omega^2 - \omega_0^2} = \frac{gF_0}{W(\omega^2 - \omega_0^2)}.$$

Solve for the magnitude:

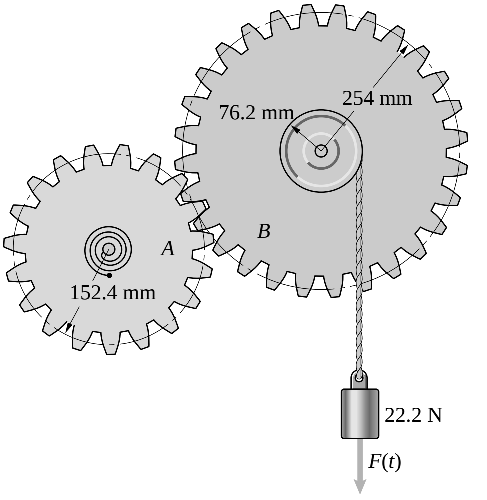
$$|F_0| = \frac{W}{g}|(\omega^2 - \omega_0^2)|x_0$$

Substitute numerical values:

$W = 667$ N, $g = 9.81$ m/s^2, $|(\omega^2 - \omega_0^2)| = 2960.9$ (rad/s)^2,

$x_0 = 0.25$ mm. $= 0.00025$ m, from which $\boxed{F_0 = 51.2 \text{ N}}$

Problem 21.65 The moments of inertia of gears A and B are $I_A = 0.019$ kg-m^2 and $I_B = 0.136$ kg-m^2. Gear A is connected to a torsional spring with constant $k = 2.71$ N-m/rad. The system is in equilibrium at $t = 0$ when it is subjected to an oscillatory force $F(t) = 17.8 \sin 3t$ N. What is the downward displacement of the 22.2 N weight as a function of time?



Solution: Choose a coordinate system with the x axis positive upward. The sum of the moments on gear A is $\sum M = -k\theta_A + R_A F$. From Newton's second law,

$$I_A \frac{d^2\theta_A}{dt^2} = \sum M = -k\theta_A + R_A F,$$

from which $F = \left(\frac{I_A}{R_A}\right)\frac{d^2\theta_A}{dt^2} + \left(\frac{k}{R_A}\right)\theta_A$.

The sum of the moments acting on gear B is $\sum M = R_B F - R_W F_d$. From the equation of angular motion applied to gear B,

$$I_B \frac{d^2\theta_B}{dt^2} = \sum M_B = R_B F - R_W F_d.$$

Substitute for F to obtain the equation of motion for gear B:

$$I_B \frac{d^2\theta_B}{dt^2} - \left(\frac{R_B}{R_A}\right) I_A \frac{d^2\theta_A}{dt^2} - \left(\frac{R_A}{R_B}\right) k\theta_A - R_W F_d = 0.$$

Solve:

$$F_d = \left(\frac{I_B}{R_W}\right)\frac{d^2\theta_B}{dt^2} - \left(\frac{R_B}{R_A R_W}\right)\frac{d^2\theta_A}{dt^2} - \left(\frac{R_B}{R_A R_W}\right)k\theta_A.$$

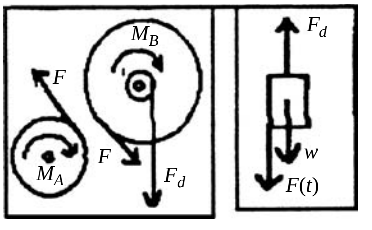
The sum of the forces on the weight are $\sum F = +F_d - W - F(t)$. From Newton's second law applied to the weight,

$$\left(\frac{W}{g}\right)\frac{d^2x}{dt^2} = F_d - W - F(t).$$

Substitute for F_d, and rearrange to obtain the equation of motion for the weight:

$$\frac{W}{g}\frac{d^2x}{dt^2} - \frac{I_B}{R_W}\frac{d^2\theta_B}{dt^2} + \frac{R_B I_A}{R_W R_A}\frac{d^2\theta_A}{dt^2} + \frac{R_B}{R_W R_A}k\theta_A$$

$$= -W - F(t).$$



From kinematics, $\theta_A = -\left(\frac{R_B}{R_A}\right)\theta_B$, and $x = -R_W\theta_B$, from which

$$\frac{d^2\theta_B}{dt^2} = -\frac{1}{R_W}\frac{d^2x}{dt^2}, \quad \frac{d^2\theta_A}{dt^2} = -\left(\frac{R_B}{R_A}\right)\frac{d^2\theta_B}{dt^2} = \frac{R_B}{R_W R_A}\frac{d^2x}{dt^2},$$

and $\theta_A = \frac{R_B}{R_W R_A}x$.

Define $\eta = \frac{R_B}{R_W R_A} = 21.87\ \text{m}^{-1}$,

and $M = \frac{W}{g} + \frac{I_B}{(R_W)^2} + \left(\frac{R_B}{R_W R_A}\right)^2 I_A = 34.69$ kg,

from which the equation of motion for the weight *about the unstretched spring position* is:

$$M\frac{d^2x}{dt^2} + (k\eta^2)x = -W - F(t).$$

For $d = 0$, the canonical form (see Eq. (21.16)) of the equation of motion is

$$\frac{d^2x}{dt^2} + \omega^2 x = a(t), \quad \text{where } \omega^2 = \frac{\eta^2 k}{M} = 37.39 \text{ (rad/s)}^2.$$

[*Check*: This agrees with the result in the solution to Problem 21.53, as it should, since nothing has changed except for the absence of a damping element. *check*.] The non-homogenous terms are $a(t) = \frac{W}{M} + \frac{F(t)}{M}$. Since $\frac{W}{M}$ is not a function of t,

$$x_{pw} = -\frac{W}{\omega^2 M} = -\frac{W}{\eta^2 k} = -0.017 \text{ m},$$

which is the equilibrium point. Make the transformation $\tilde{x}_p = x_p - x_{pw}$. The equation of motion about the equilibrium point is

$$\frac{d^2\mathbf{x}}{dt^2} + \omega^2 \tilde{x} = -\frac{17.8 \sin 3t}{M}.$$

Assume a solution of the form $\tilde{x}_p = A_p \sin 3t + B_p \cos 3t$. Substitute:

$$(\omega^2 - 3^2)(A_p \sin 3t + B_p \cos 3t) = -\frac{3 \sin 3t}{M},$$

from which $B_p = 0$, and

$$A_p = -\frac{17.8}{M(\omega^2 - 3^2)} = -0.0181 \text{ m}.$$

The particular solution is

$$\tilde{x}_p = -\frac{17.8}{M(\omega^2 - 3^2)} \sin 3t = -0.0181 \sin 3t \text{ m}.$$

The solution to the homogenous equation is

$$x_h = A_h \sin \omega t + B_h \cos \omega t,$$

and the complete solution is

$$\tilde{x}(t) = A_h \sin \omega t + B_h \cos \omega t - \frac{17.8}{M(\omega^2 - 3^2)} \sin 3t.$$

Apply the initial conditions: at $t = 0$, $x_0 = 0$, $\dot{x}_0 = 0$, from which $0 = B$,

$$0 = \omega A_h - \frac{53.4}{M(\omega^2 - 3^2)},$$

from which $A_h = \frac{53.4}{\omega M(\omega^2 - 3^2)} = 0.00887$.

The complete solution for vibration about the equilibrium point is:

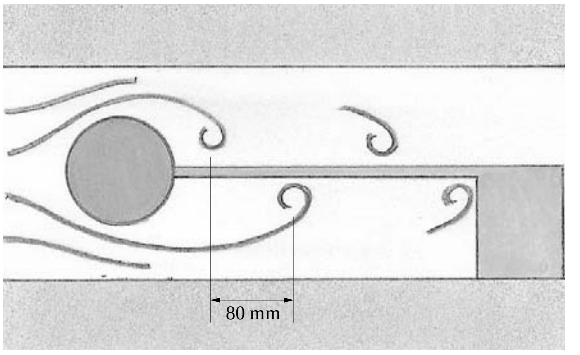
$$\tilde{x}(t) = \frac{17.8}{M(\omega^2 - \omega_0^2)} \left(\frac{3}{\omega} \sin \omega t - \sin 3t \right)$$

$$= 0.00887 \sin 6.114t - 0.0181 \sin 3t \text{ m}.$$

The downward travel is the negative of this:

$$\boxed{\tilde{x}_{\text{down}} = -0.00887 \sin 6.114t + 0.0181 \sin 3t \text{ m}}$$

Problem 21.66* A 1.5-kg cylinder is mounted on a sting in a wind tunnel with the cylinder axis transverse to the direction of flow. When there is no flow, a 10-N vertical force applied to the cylinder causes it to deflect 0.15 mm. When air flows in the wind tunnel, vortices subject the cylinder to alternating lateral forces. The velocity of the air is 5 m/s, the distance between vortices is 80 mm, and the magnitude of the lateral forces is 1 N. If you model the lateral forces by the oscillatory function $F(t) = (1.0)\sin\omega_0 t$ N, what is the amplitude of the steady-state lateral motion of the sphere?



Solution: The time interval between the appearance of an upper vortex and a lower vortex is $\delta_t = \dfrac{0.08}{5} = 0.016$ s, from which the period of a *sinusoidal-like* disturbance is $\tau = 2(\delta t) = 0.032$ s, from which $f_0 = \dfrac{1}{\tau} = 31.25$ Hz. [*Check*: Use the physical relationship between frequency, wavelength and velocity of propagation of a *small amplitude sinusoidal wave*, $\lambda f = v$. The wavelength of a traveling sinusoidal disturbance is the distance between two peaks or two troughs, or twice the distance between adjacent peaks and troughs, $\lambda = 2(0.08) = 0.16$ m, from which the frequency is $f_0 = \dfrac{v}{\lambda} = 31.25$ Hz. *check*] The circular frequency is $\omega_0 = 2\pi f_0 = 196.35$ rad/s. The spring constant of the sting is

$$k = \frac{F}{\delta} = \frac{10}{0.00015} = 66667 \text{ N/m}.$$

The natural frequency of the sting-cylinder system is

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{66667}{1.5}} = 33.55 \text{ Hz},$$

from which $\omega = 2\pi f = 210.82$ rad/s. For $d = 0$, the canonical form (see Eq. 21.16)) of the equation of motion is $\dfrac{d^2x}{dt^2} + \omega^2 x = a(t)$, where

$$a(t) = \frac{F}{m} = \frac{1}{15}\sin\omega_0 t = 0.6667\sin 196.3t.$$

From Eq. (21.31) the amplitude is

$$\boxed{E = \frac{a_0}{\omega^2 - \omega_0^2} = \frac{0.6667}{5891} = 1.132 \times 10^{-4} \text{ m}}$$

[*Note*: This is a small deflection (113 *microns*) but the associated aerodynamic forces may be significant to the tests (e.g. $F_{amp} = 7.5$ N), since the sting is stiff. Vortices may cause undesirable noise in sensitive static aerodynamic loads test measurements.]

Problem 21.67 Show that the amplitude of the particular solution given by Eq. (21.31) is a maximum when the frequency of the oscillatory forcing function is $\omega_0 = \sqrt{\omega^2 - 2d^2}$.

Solution: Eq. (21.31) is

$$E_p = \frac{\sqrt{a_0^2 + b_0^2}}{\sqrt{(\omega^2 - \omega_0^2)^2 + 4d^2\omega_0^2}}.$$

Since the numerator is a constant, rearrange:

$$\eta = \frac{E_p}{\sqrt{a_0^2 + b_0^2}} = [(\omega^2 - \omega_0^2)^2 + 4d^2\omega_0^2]^{-\frac{1}{2}}.$$

The maximum (or minimum) is found from

$$\frac{d\eta}{d\omega_0} = 0.$$

$$\frac{d\eta}{d\omega_0} = -\frac{1}{2}\frac{4[-(\omega^2 - \omega_0^2)(\omega_0) + 2d^2\omega_0]}{[(\omega^2 - \omega_0^2)^2 + 4d^2\omega_0^2]^{\frac{3}{2}}} = 0,$$

from which $-(\omega^2 - \omega_0^2)(\omega_0) + 2d^2\omega_0 = 0$. Rearrange, $(\omega_0^2 - \omega^2 + 2d^2)\omega_0 = 0$. Ignore the possible solution $\omega_0 = 0$, from which

$$\boxed{\omega_0 = \sqrt{\omega^2 - 2d^2}}.$$

Let $\tilde{\omega}_0 = \sqrt{\omega^2 - 2d^2}$ be the maximizing value. To show that η is indeed a maximum, take the second derivative:

$$\left[\frac{d^2\eta}{d\omega_0^2}\right]_{\omega_0} = \tilde{\omega}_0 = \frac{3}{4}\frac{[(4\omega_0)(\omega_0^2 - \omega^2 + 2d^2)]^2_{\omega_0=\tilde{\omega}_0}}{[(\omega^2 - \omega_0^2)^2 + 2d^2\omega_0^2]^{\frac{5}{2}}_{\omega_0=\tilde{\omega}_0}}$$

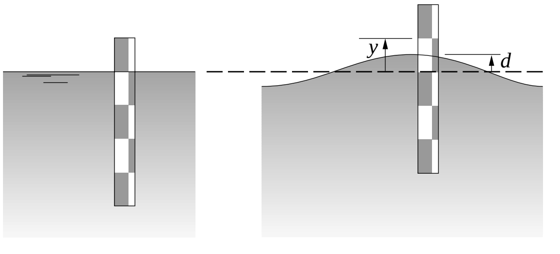
$$- \frac{1}{2}\frac{4[3\omega_0^2 - \omega^2 + 2d^2]_{\omega_0=\tilde{\omega}_0}}{[(\omega^2 - \omega_0^2)^2 + 2d^2\omega_0^2]^{\frac{3}{2}}_{\omega_0=\tilde{\omega}_0}},$$

from which $\left[\dfrac{d^2\eta}{dt^2}\right]_{\omega_0=\tilde{\omega}_0} = -\dfrac{4\omega_0^2}{[2\omega^2 d^2]^{\frac{3}{2}}} < 0,$

which demonstrates that it is a maximum.

Problem 21.68* A sonobuoy (sound-measuring device) floats in a standing-wave tank. The device is a cylinder of mass m and cross-sectional area A. The water density is ρ, and the buoyancy force supporting the buoy equals the weight of the water that would occupy the volume of the part of the cylinder below the surface. When the water in the tank is stationary, the buoy is in equilibrium in the vertical position shown at the left. Waves are then generated in the tank, causing the depth of the water at the sonobuoy's position *relative to its original depth* to be $d = d_0 \sin\omega_0 t$. Let y be the sonobuoy's vertical position relative to its original position. Show that the sonobuoy's vertical position is governed by the equation

$$\frac{d^2y}{dt^2} + \left(\frac{A\rho g}{m}\right)y = \left(\frac{A\rho g}{m}\right)d_0 \sin\omega_0 t.$$



Solution: The volume of the water displaced at equilibrium is $V = Ah$ where A is the cross-sectional area, and h is the equilibrium immersion depth. The weight of water displaced is $\rho V g = \rho g A h$, so that the buoyancy force is $F_b = \rho A g h$.

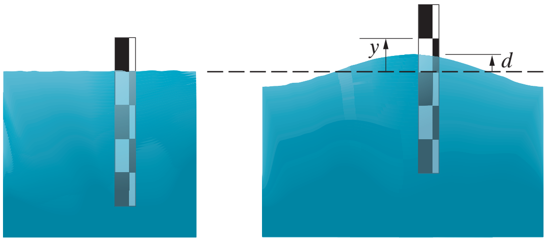
The sum of the vertical forces is $\sum F_y = \rho g A h - mg = 0$ at equilibrium, where m is the mass of the buoy. By definition, the spring constant is $\dfrac{\partial F_b}{\partial h} = k = \rho g A$. For any displacement δ of the immersion depth from the equilibrium depth, the net vertical force on the buoy is $\sum F_y = -\rho g A(h + \delta) + mg = -\rho A g \delta = -k\delta$, since h is the equilibrium immersion depth. As the waves are produced, $\delta = y - d$, where $d = d_0 \sin\omega_0 t$, from which $\sum F_y = -k(y - d)$. From Newton's second law,

$$m\frac{d^2y}{dt^2} = -k(y - d), \text{ from which } m\frac{d^2y}{dt^2} + ky = kd.$$

Substitute:

$$\boxed{\frac{d^2y}{dt^2} + \left(\frac{\rho g A}{m}\right)y = \left(\frac{\rho g A}{m}\right)d_0 \sin\omega_0 t}$$

Problem 21.69 Suppose that the mass of the sonobuoy in Problem 21.68 is $m = 10$ kg, its diameter is 125 mm, and the water density is $\rho = 1025$ kg/m^3. If $d = 0.1 \sin 2t$ m, what is the magnitude of the steady-state vertical vibrations of the sonobuoy?



Solution: From the solution to Problem 21.68,

$$\frac{d^2y}{dt^2} + \left(\frac{\rho g A}{m}\right) y = \left(\frac{\rho g A}{m}\right) d_0 \sin \omega_0 t.$$

The canonical form is

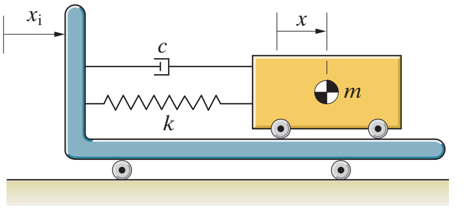
$$\frac{d^2y}{dt}^2 + \omega^2 y = a(t),$$

where $\omega^2 = \dfrac{\rho \pi d^2 g}{4\ m} = \dfrac{1025\pi(0.125^2)(9.81)}{4(10)} = 12.34\ (\text{rad/s})^2,$

and $a(t) = \omega^2 (0.1) \sin 2t$. From Eq. (21.31), the amplitude of the steady state vibrations is

$$\boxed{E_p = \frac{\omega^2(0.1)}{|(\omega^2 - 2^2)|} = 0.1480 \text{ m}}$$

Problem 21.70 The mass weighs 222 N. The spring constant is $k = 2919$ N/m, and $c = 146$ N-s/m. If the base is subjected to an oscillatory displacement x_i of amplitude 0.254 m and frequency $\omega_i = 15$ rad/s, what is the resulting steady-state amplitude of the displacement of the mass relative to the base?



Solution: The canonical form of the equation of motion is

$$\frac{d^2x}{dt^2} + 2d\frac{dt}{dt} + \omega^2 x = a(t),$$

where $d = \dfrac{c}{2m} = \dfrac{gc}{2W} = 3.217$ rad/s,

$$\omega^2 = \frac{kg}{W} = 128.7\ (\text{rad/s})^2,$$

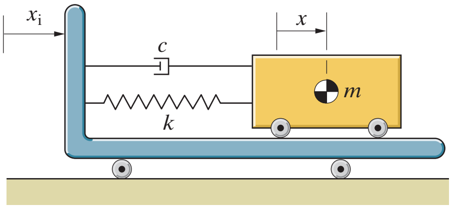
and (see Eq. (21.38),

$$a(t) = -\frac{d^2x_i}{dt^2} = x_i \omega_i^2 \sin(\omega_i t - \phi).$$

The displacement of the mass relative to its base is

$$\boxed{\begin{aligned} E_p &= \frac{\omega_i^2 x_i}{\sqrt{(\omega^2 - \omega_i^2)^2 + 4d^2\omega_i^2}} \\ &= \frac{(15^2)(0.254)}{\sqrt{(11.34^2 - 15^2)^2 + 4(3.217^2)(15^2)}} = 0.419 \text{ m.} \end{aligned}}$$

Problem 21.71 The mass in Fig. 21.21 is 100 kg. The spring constant is $k = 4$ N/m, and $c = 24$ N-s/m. The base is subjected to an oscillatory displacement of frequency $\omega_i = 0.2$ rad/s. The steady-state amplitude of the displacement of the mass relative to the base is measured and determined to be 200 mm. What is the amplitude of the displacement of the base? (See Example 21.7.)



Solution: From Example 21.7 and the solution to Problem 21.70, the canonical form of the equation of motion is

$$\frac{d^2x}{dt^2} + 2d\frac{dx}{dt} + \omega^2 x = a(t)$$

where $d = \dfrac{c}{2m} = 0.12$ rad/s, $\omega^2 = \dfrac{k}{m} = 0.04$ (rad/s)^2,

and (see Eq. (21.38))

$$a(t) = -\frac{d^2x_i}{dt^2} = x_i\omega_i^2 \sin(\omega_i t - \phi).$$

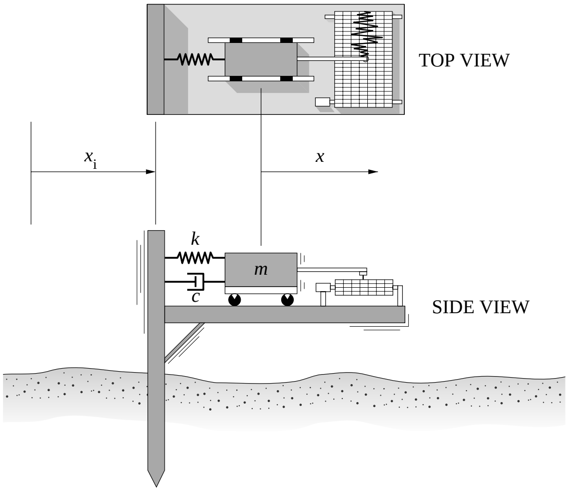
The displacement of the mass relative to its base is

$$E_p = 0.2 = \frac{\omega_i^2 x_i}{\sqrt{(\omega^2 - \omega_i^2)^2 + 4d^2\omega_i^2}}$$

$$= \frac{(0.2^2)x_i}{\sqrt{(0.2^2 - 0.2^2)^2 + 4(0.12^2)(0.2^2)}} = 0.8333x_i \text{ m},$$

from which $\boxed{x_i = \dfrac{0.2}{0.8333} = 0.24 \text{ m}}$

Problem 21.72 A team of engineering students builds the simple seismograph shown. The coordinate x_i measures the local horizontal ground motion. The coordinate x measures the position of the mass relative to the frame of the seismograph. The spring is unstretched when $x = 0$. The mass $m = 1$ kg, $k = 10$ N/m, and $c = 2$ N-s/m. Suppose that the seismograph is initially stationary and that at $t = 0$ it is subjected to an oscillatory ground motion $x_i = 10\sin 2t$ mm. What is the amplitude of the steady-state response of the mass? (See Example 21.7.)



Solution: From Example 21.7 and the solution to Problem 21.70, the canonical form of the equation of motion is

$$\frac{d^2x}{dt^2} + 2d\frac{dx}{dt} + \omega^2 x = a(t)$$

where $d = \dfrac{c}{2m} = 1$ rad/s, $\omega^2 = \dfrac{k}{m} = 10$ (rad/s)^2,

and (see Eq. (21.38))

$$a(t) = -\frac{d^2x_i}{dt^2} = x_i\omega_i^2 \sin\omega_i t \quad \text{where } x_i = 10 \text{ mm}, \omega_i = 2 \text{ rad/s}.$$

The amplitude of the steady state response of the mass relative to its base is

$$\boxed{E_p = \frac{\omega_i^2 x_i}{\sqrt{(\omega^2 - \omega_i^2)^2 + 4d^2\omega_i^2}} = \frac{(2^2)(10)}{\sqrt{(3.162^2 - 2^2)^2 + 4(1^2)(2^2)}} = 5.55 \text{ mm}}$$

Problem 21.73 In Problem 21.72, determine the position x of the mass relative to the base as a function of time. (See Example 21.7.)

Solution: From Example 21.7 and the solution to Problem 21.70, the canonical form of the equation of motion is

$$\frac{d^2x}{dt^2} + 2d\frac{dx}{dt} + \omega^2 x = a(t)$$

where $d = \frac{c}{2m} = 1$ rad/s, $\omega^2 = \frac{k}{m} = 10$ (rad/s)^2,

and $a(t) = -\frac{d^2x_i}{dt^2} = x_i\omega_i^2 \sin\omega_i t$

where $x_i = 10$ mm, $\omega_i = 2$ rad/s. From a comparison with Eq. (21.27) and Eq. (21.30), the particular solution is $x_p = A_p \sin\omega_i t + B_p \cos\omega_i t$, where

$$A_p = \frac{(\omega^2 - \omega_i^2)\omega_i^2 x_i}{(\omega^2 - \omega_i^2)^2 + 4d^2\omega_i^2} = 4.615.$$

$$B_p = -\frac{2d\omega_i^3 x_i}{(\omega^2 - \omega_i^2)^2 + 4d^2\omega_i^2} = -3.077.$$

[*Check*: Assume a solution of the form $x_p = A_p \sin\omega_i t + B_p \cos\omega_i t$. Substitute into the equation of motion:

$$[(\omega^2 - \omega_i^2)A_p - 2d\omega_i B_p]\sin\omega_i t$$

$$+ [(\omega^2 - \omega_i^2)B_p + 2d\omega_i Ap]\cos\omega_i t = x_i\omega_i^2 \sin\omega_i t.$$

Equate like coefficients:

$$(\omega^2 - \omega_i^2)A_p - 2d\omega_i B_p = x_i\omega_i^2,$$

and $2d\omega_i A_p + (\omega^2 - \omega_i^2)B_p = 0.$

Solve:

$$A_p = \frac{(\omega^2 - \omega_i^2)\omega_i^2 x_i}{(\omega^2 - \omega_i^2) + 4d^2\omega_i^2},$$

$$B_p = \frac{-2d\omega_i^3 x_i}{(\omega^2 - \omega_i^2)^2 + 4d^2\omega_i^2}, \textit{check}.]$$

Since $d^2 < \omega^2$, the system is sub-critically damped, and the homogenous solution is $x_h = e^{-dt}(A\sin\omega_d t + B\cos\omega_d t)$, where $\omega_d = \sqrt{\omega^2 - d^2} = 3$ rad/s. The complete solution is $x = x_h + x_p$. Apply the initial conditions: at $t = 0$, $x_0 = 0$, from which $0 = B + B_p$, and $0 = -dB + \omega_d A + \omega_i A_p$. Solve:

$$B = -B_p = 3.077,\ A = \frac{dB}{\omega_d} - \frac{\omega_i A_p}{\omega_d} = -2.051$$

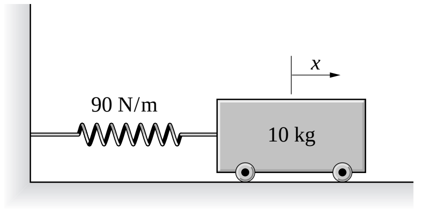
The solution is

$$x = e^{-dt}(A\sin\omega t + B\cos\omega t) + A_p \sin\omega_i t + B_p \cos\omega_i t:$$

$$\boxed{\begin{aligned} x = {} & -e^{-t}(2.051\sin 3t - 3.077\cos 3t) + 4.615\sin 2t \\ & - 3.077\cos 2t \text{ mm} \end{aligned}}$$

Problem 21.74 The coordinate x measures the displacement of the mass relative to the position in which the spring is unstretched. The mass is given the initial conditions

$$t = 0 \begin{cases} x = 0.1 \text{ m}, \\ \dfrac{dx}{dt} = 0. \end{cases}$$



(a) Determine the position of the mass as a function of time.
(b) Draw graphs of the position and velocity of the mass as functions of time for the first 5 s of motion.

Solution: The canonical equation of motion is

$$\frac{d^2x}{dt^2} + \omega^2 x = 0,$$

where $\omega = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{90}{10}} = 3$ rad/s.

(a) The position is

$$x(t) = A \sin \omega t + B \cos \omega t,$$

and the velocity is

$$\frac{dx(t)}{dt} = \omega A \cos \omega t - \omega B \sin \omega t.$$

At $t = 0$, $x(0) = 0.1 = B$, and

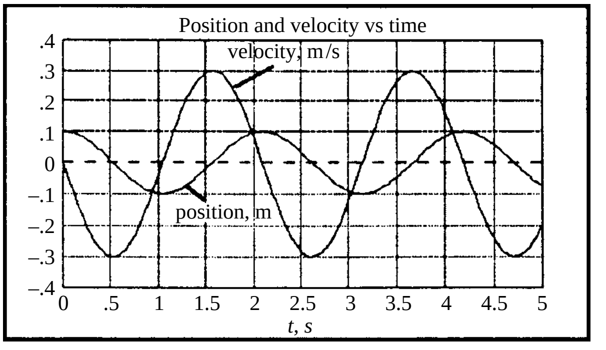
$$\frac{dx(0)}{dt} = 0 = \omega A,$$

from which $A = 0$, $B = 0.1$ m, and

$$\boxed{x(t) = 0.1 \cos 3t \text{ m}},$$

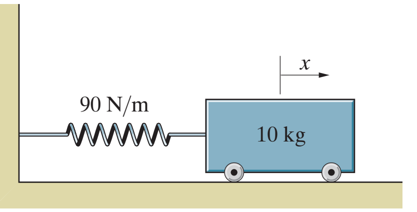
$$\frac{dx(t)}{dt} = -0.3 \sin 3t \text{ m/s}.$$

(b) The graphs are shown.



Problem 21.75 When $t = 0$, the mass in Problem 21.74 is in the position in which the spring is unstretched and has a velocity of 0.3 m/s to the right. Determine the position of the mass as functions of time and the amplitude of vibration

(a) by expressing the solution in the form given by Eq. (21.8) and
(b) by expressing the solution in the form given by Eq. (21.9)



Solution: From Eq. (21.5), the canonical form of the equation of motion is

$$\frac{d^2x}{dt^2} + \omega^2 x = 0,$$

where $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{90}{10}} = 3$ rad/s.

(a) From Eq. (21.6) the position is $x(t) = A \sin \omega t + B \cos \omega t$, and the velocity is

$$\frac{dx(t)}{dt} = \omega A \cos \omega t - \omega B \sin \omega t.$$

At $t = 0$, $x(0) = 0 = B$, and $\frac{dx(0)}{dt} = 0.3 = \omega A$ m/s, from which $B = 0$ and $A = \frac{0.3}{\omega} = 0.1$ m. The position is

$$\boxed{x(t) = 0.1 \sin 3t \text{ m}}.$$

The amplitude is

$$\boxed{|x(t)_{\text{max}} = 0.1 \text{ m}}$$

(b) From Eq. (21.7) the position is $x(t) = E \sin(\omega t - \phi)$, and the velocity is

$$\frac{dx(t)}{dt} = \omega E \cos(\omega t - \phi).$$

At $t = 0$, $x(0) = -E \sin \phi$, and the velocity is

$$\frac{dx(0)}{dt} = 0.3 = \omega E \cos \phi.$$

Solve: $\phi = 0, \quad E = \frac{0.3}{\omega} = 0.1,$

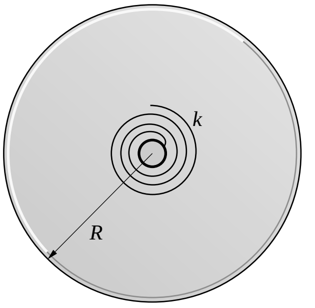
from which $\boxed{x(t) = 0.1 \sin 3t}$.

The amplitude is

$$\boxed{|x(t)| = E = 0.1 \text{ m}}$$

Problem 21.76 A homogenous disk of mass m and radius R rotates about a fixed shaft and is attached to a torsional spring with constant k. (The torsional spring exerts a restoring moment of magnitude $k\theta$, where θ is the angle of rotation of the disk relative to its position in which the spring is unstretched.) Show that the period of rotational vibrations of the disk is

$$\tau = \pi R\sqrt{\frac{2m}{k}}.$$



Solution: From the equation of angular motion, the equation of motion is $I\alpha = M$, where $M = -k\theta$, from which

$$I\frac{d^2\theta}{dt^2} + k\theta = 0,$$

and the canonical form (see Eq. (21.4)) is

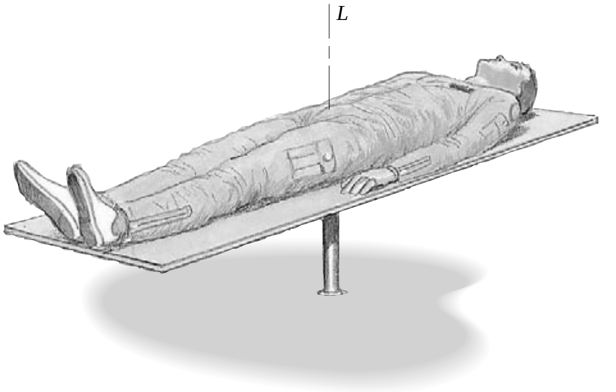
$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0, \quad \text{where} \quad \omega = \sqrt{\frac{k}{I}}.$$

For a homogenous disk the moment of inertia about the axis of rotation is

$$I = \frac{mR^2}{2}, \quad \text{from which} \quad \omega = \sqrt{\frac{2\,k}{mR^2}} = \frac{1}{R}\sqrt{\frac{2\,k}{m}}.$$

The period is $\boxed{\tau = \frac{2\pi}{\omega} = 2\pi R\sqrt{\frac{m}{2\,k}} = \pi R\sqrt{\frac{2\,m}{k}}}$

Problem 21.77 Assigned to determine the moments of inertia of astronaut candidates, an engineer attaches a horizontal platform to a vertical steel bar. The moment of inertia of the platform about L is 7.5 kg-m^2, and the frequency of torsional oscillations of the unloaded platform is 1 Hz. With an astronaut candidate in the position shown, the frequency of torsional oscillations is 0.520 Hz. What is the candidate's moment of inertia about L?



Solution: The natural frequency of the unloaded platform is

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{I}} = 1 \text{ Hz},$$

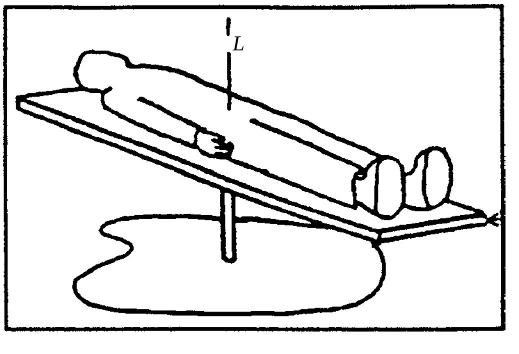
from which $k = (2\pi f)^2 I = (2\pi)^2 7.5 = 296.1$ N-m/rad.

The natural frequency of the loaded platform is

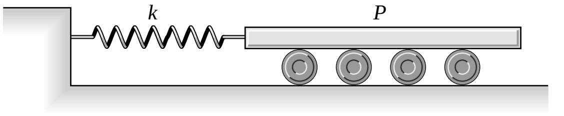
$$f_1 = \frac{\omega_1}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{I_1}} = 0.520 \text{ Hz},$$

from which $I_1 = \left(\frac{1}{2\pi f_1}\right)^2 k = 27.74$ kg-m^2,

from which $\boxed{I_A = I_1 - I = 20.24 \text{ kg-m}^2}$



Problem 21.78 The 22-kg platen P rests on four roller bearings that can be modeled as 1-kg homogenous cylinders with 30-mm radii. The spring constant is $k = 900$ N/m. What is the frequency of horizontal vibrations of the platen relative to its equilibrium position?



Solution: The kinetic energy is the sum of the kinetic energies of translation of the platen P and the roller bearings, and the kinetic energy of rotation of the roller bearings. Denote references to the platen by the subscript P and references to the ball bearings by the subscript B:

$$T = \frac{1}{2}m_P\left(\frac{dx_P}{dt}\right)^2 + \frac{1}{2}(4m_B)\left(\frac{dx_B}{dt}\right)^2 + \frac{1}{2}(4I_B)\left(\frac{d\theta}{dt}\right)^2.$$

The potential energy is the energy stored in the spring:

$$V = \frac{1}{2}kx_P^2.$$

From kinematics, $\quad -R\theta = x_B$

and $\quad x_B = \dfrac{x_P}{2}.$

Since the system is conservative, $T + V = \text{const.}$ Substitute the kinematic relations and reduce:

$$\left(\frac{1}{2}\right)\left(m_P + m_B + \frac{I_B}{R^2}\right)\left(\frac{dx_P}{dt}\right)^2 + \left(\frac{1}{2}\right)kx_P^2 = \text{const.}$$

Take the time derivative:

$$\left(\frac{dx}{dt}\right)\left[\left(m_P + m_B + \frac{I_B}{R^2}\right)\left(\frac{d^2x_P}{dt^2}\right) + kx\right] = 0.$$

This has two possible solutions,

$$\left(\frac{dx}{dt}\right) = 0$$

or $\quad \left(m_P + m_B + \dfrac{I_B}{R^2}\right)\left(\dfrac{d^2x_P}{dt^2}\right) + kx_P = 0.$

The first can be ignored, from which the canonical form of the equation of motion is

$$\frac{d^2x_P}{dt^2} + \omega^2 x_P = 0,$$

where $\quad \omega = R\sqrt{\dfrac{k}{R^2(m_P + m_B) + I_B}}.$

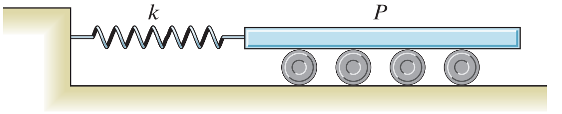
For a homogenous cylinder,

$$I_B = \frac{m_B R^2}{2},$$

from which $\quad \omega = \sqrt{\dfrac{k}{m_P + \dfrac{3}{2}m_B}} = 6.189$ rad/s.

The frequency is $\boxed{f = \dfrac{\omega}{2\pi} = 0.985\text{ Hz}}$.

Problem 21.79 At $t = 0$, the platen described in Problem 21.78 is 0.1 m to the left of its equilibrium position and is moving to the right at 2 m/s. What are the platen's position and velocity at $t = 4$ s?



Solution: The position is

$$x(t) = A \sin \omega t + B \cos \omega t,$$

and the velocity is

$$\frac{dx}{dt} = \omega A \cos \omega t - \omega B \sin \omega t,$$

where, from the solution to Problem 21.78, $\omega = 6.189$ rad/s. At $t = 0$, $x(0) = -0.1$ m, and

$$\left[\frac{dx}{dt}\right]_{t=0} = 2 \text{ m/s},$$

from which $B = -0.1$ m, $A = \frac{2}{\omega} = 0.3232$ m.

The position and velocity are

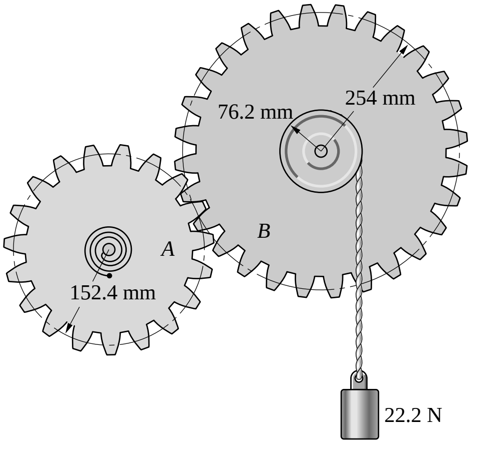
$$x(t) = 0.3232 \sin(6.189\, t) - 0.1 \cos(6.189\, t) \text{ (m)},$$

$$\frac{dx}{dt} = 2 \cos(6.189\, t) + 0.6189 \sin(6.189\, t) \text{ (m/s)}.$$

At $t = 4$ s, $\boxed{x = -0.2124 \text{ m}}$,

$$\boxed{\frac{dx}{dt} = 1.630 \text{ m/s}}.$$

Problem 21.80 The moments of inertia of gears A and B are $I_A = 0.019$ kg-m^2 and $I_B = 0.136$ kg-m^2. Gear A is attached to a torsional spring with constant $k = 2.71$ N-m/rad. What is the frequency of angular vibrations of the gears relative to their equilibrium position?



Solution: The system is conservative. The strategy is to determine the equilibrium position from the equation of motion about the unstretched spring position. Choose a coordinate system with the y axis positive upward. Denote $R_A = 0.152$ m, $R_B = 0.254$ m, and $R_M = 0.762$ m, and $W = 22.2$ N. The kinetic energy of the system is

$$T = \frac{1}{2} I_A \dot{\theta}_A^2 + \frac{1}{2} I_B \dot{\theta}_B^2 + \frac{1}{2}\frac{W}{g} v^2,$$

where $\dot{\theta}_A$, $\dot{\theta}_B$ are the angular velocities of gears A and B respectively, and v is the velocity of the 22.2 N weight. The potential energy is the sum of the energy stored in the spring plus the energy gain due to the increase in the height of the 22.2 N weight:

$$V = \frac{1}{2} k\theta_A^2 + W_y.$$

From kinematics,

$$v = R_M \dot{\theta}_B,$$

$$\dot{\theta}_B = -\left(\frac{R_A}{R_B}\right)\dot{\theta}_A,$$

$$y = R_M \theta_B = -R_M \left(\frac{R_A}{R_B}\right)\theta_A.$$

Substitute, $T + V = \text{const.}$

$$= \left(\frac{1}{2}\right)\left[I_A + \left(\frac{R_A}{R_B}\right)^2 I_B + \left(\frac{W}{g}\right)(R_M^2)\left(\frac{R_A}{R_B}\right)^2\right]\dot{\theta}_A^2 + \left(\frac{1}{2}\right) k\theta_A^2 - (R_M)\left(\frac{R_A}{R_B}\right)\theta_A W.$$

Define $M = I_A + \left(\frac{R_A}{R_B}\right)^2 I_B + \frac{W}{g}(R_M^2)\left(\frac{R_A}{R_B}\right)^2$

$$= 0.0725 \text{ kg-m}^2,$$

and take the time derivative:

$$\dot{\theta}_A\left[M\left(\frac{d^2\theta_A}{dt^2}\right) + k\theta_A - W(R_M)\left(\frac{R_A}{R_B}\right)\right] = 0.$$

Ignore the possible solution $\dot{\theta}_A = 0$, to obtain

$$\frac{d^2\theta_A}{dt^2} + \omega^2\theta_A = F,$$

where $F = \frac{W R_M}{M}\left(\frac{R_A}{R_B}\right)$,

and $\omega = \sqrt{\frac{k}{M}} = 6.114$ rad/s,

is the equation of motion *about the unstretched spring position*. Note that

$$\frac{F}{\omega^2} = \frac{W}{k} R_M \left(\frac{R_A}{R_B}\right) = 0.375 \text{ rad}$$

is the equilibrium position of θ_A, obtained by setting the acceleration to zero (since the non-homogenous term F is a constant). Make the change of variable:

$$\tilde{\theta} = \theta_A - \frac{F}{\omega^2},$$

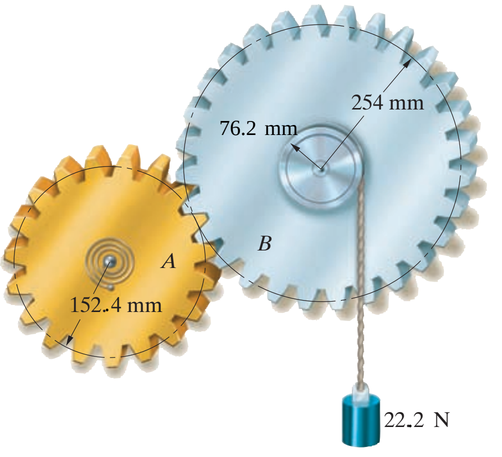
from which the canonical form (see Eq. (21.4)) of the equation of motion *about the equilibrium point* is

$$\frac{d^2\tilde{\theta}}{dt^2} + \omega^2\tilde{\theta} = 0,$$

and the natural frequency is

$$\boxed{f = \frac{\omega}{2\pi} = 0.9732 \text{ Hz.}}$$

Problem 21.81 The 22.2 N weight in Problem 21.80 is raised 12.7 mm from its equilibrium position and released from rest at $t = 0$. Determine the counterclockwise angular position of gear B relative to its equilibrium position as a function of time.



Solution: From the solution to Problem 21.80, the equation of motion for gear A is

$$\frac{d^2\theta_A}{dt^2} + \omega^2\theta_A = F,$$

where $M = I_A + \left(\frac{R_A}{R_B}\right)^2 I_B + \frac{W}{g}(R_M^2)\left(\frac{R_A}{R_B}\right)^2$

$$= 0.0725 \text{ kg-m}^2,$$

$$\omega = \sqrt{\frac{k}{M}} = 6.114 \text{ rad/s},$$

and $F = \dfrac{W R_M\left(\dfrac{R_A}{R_B}\right)}{M} = 14.02 \text{ rad/s}^2.$

As in the solution to Problem 21.80, the *equilibrium angular position* θ_A associated with the equilibrium position of the weight is

$$[\theta_A]_{\text{eq}} = \frac{F}{\omega^2} = 0.375 \text{ rad}.$$

Make the change of variable:

$$\tilde{\theta}_A = \theta_A - [\theta_A]_{\text{eq}},$$

from which the canonical form of the equation of motion *about the equilibrium point* is

$$\frac{d^2\tilde{\theta}_A}{dt^2} + \omega^2\tilde{\theta}_A = 0.$$

Assume a solution of the form

$$\tilde{\theta}_A = A\sin\omega t + B\cos\omega t.$$

The displacement from the equilibrium position is, from kinematics,

$$\tilde{\theta}_A(t=0) = \left(\frac{R_B}{R_A}\right)\theta_A$$

$$= -\frac{1}{R_M}\left(\frac{R_B}{R_A}\right)y(t=0)$$

$$= -0.2778 \text{ rad},$$

from which the initial conditions are

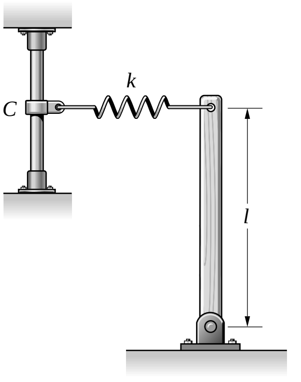
$$\tilde{\theta}_A(t=0) = -0.2778 \text{ rad} \quad \text{and} \quad \left[\frac{d\tilde{\theta}_A}{dt}\right]_{t=0} = 0,$$

from which $B = -0.2778$, $A = 0$. The angular position of gear A is $\theta_a = -0.2778\cos(6.114t)$ rad, from which the angular position of gear B is

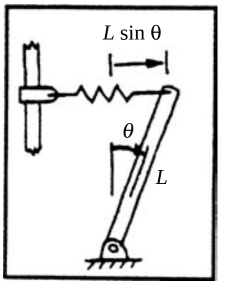
$$\boxed{\tilde{\theta}_B = -\left(\frac{R_A}{R_B}\right)\tilde{\theta}_A = 0.1667\cos(6.114t) \text{ rad}}$$

about the equilibrium position.

Problem 21.82 The mass of the slender bar is m. The spring is unstretched when the bar is vertical. The light collar C slides on the smooth vertical bar so that the spring remains horizontal. Determine the frequency of small vibrations of the bar.



Solution: The system is conservative. Denote the angle between the bar and the vertical by θ. The base of the bar is a fixed point. The kinetic energy of the bar is



$$T = \frac{1}{2} I \left(\frac{d\theta}{dt}\right)^2.$$

Denote the datum by $\theta = 0$. The potential energy is the result of the change in the height of the center of mass of the bar from the datum and the stretch of the spring,

$$V = -\frac{mgL}{2}(1 - \cos\theta) + \frac{1}{2}k(L\sin\theta)^2.$$

The system is conservative,

$$T + V = \text{const} = \frac{1}{2} I \left(\frac{d\theta}{dt}\right)^2 + \frac{kL^2}{2}\sin^2\theta - \frac{mgL}{2}(1 - \cos\theta).$$

Take the time derivative:

$$\left(\frac{d\theta}{dt}\right)\left[I\frac{d^2\theta}{dt^2} + kL^2\sin\theta\cos\theta - \frac{mgL}{2}\sin\theta\right] = 0.$$

From which

$$I\frac{d^2\theta}{dt^2} + kL^2\sin\theta\cos\theta - \frac{mgL}{2}\sin\theta = 0.$$

For small angles, $\sin\theta \to \theta$, $\cos\theta \to 1$. The moment of inertia about the fixed point is

$$I = \frac{mL^2}{3},$$

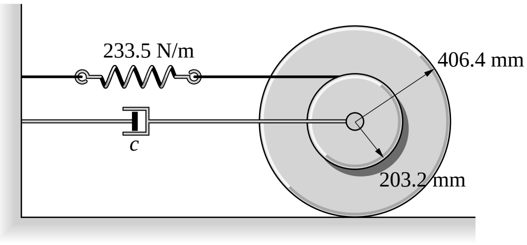
from which $\dfrac{d^2\theta}{dt^2} + \omega^2\theta = 0,$

where $\omega = \sqrt{\dfrac{3k}{m} - \dfrac{3g}{2L}}.$

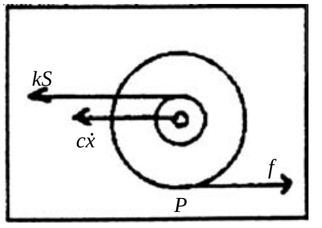
The frequency is

$$\boxed{f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{3k}{m} - \frac{3g}{2L}}}$$

Problem 21.86 The stepped disk weighs 89 N, and its moment of inertia is $I = 0.81$ kg-m^2. It rolls on the horizontal surface. If $c = 116.7$ N-s/m, what is the frequency of vibration of the disk?



Solution: The strategy is to apply the free body diagram to obtain equations for both θ and x, and then to eliminate one of these. An essential element in the strategy is the determination of the stretch of the spring. Denote $R = 0.203$ m, and the stretch of the spring by S. Choose a coordinate system with the positive x axis to the right. The sum of the moments about the center of the disk is



$$\sum M_C = RkS + 2Rf.$$

From the equation of angular motion,

$$I\frac{d^2\theta}{dt^2} = \sum M_C = RkS + 2Rf.$$

Solve for the reaction at the floor:

$$f = \frac{I}{2R}\frac{d^2\theta}{dt^2} - \frac{k}{2}S.$$

The sum of the horizontal forces:

$$\sum F_x = -kS - c\frac{dx}{dt} + f.$$

From Newton's second law:

$$m\frac{d^2x}{dt^2} = \sum F_x = -kS - c\frac{dx}{dt} + f.$$

Substitute for f and rearrange:

$$m\frac{d^2x}{dt^2} + \frac{I}{2R}\frac{d^2\theta}{dt^2} + c\frac{dx}{dt} + \frac{3}{2}kS = 0$$

From kinematics, the displacement of the center of the center of the disk is $x = -2R\theta$. The stretch of the spring is the amount wrapped around the disk plus the translation of the disk,

$$S = -R\theta - 2R\theta = -3R\theta = \frac{3}{2}x.$$

Substitute: $$\left(m + \frac{I}{(2R)^2}\right)\frac{d^2x}{dt^2} + c\frac{dx}{dt} + \left(\frac{3}{2}\right)^2 kx = 0.$$

Define $\quad M = m + \dfrac{I}{(2R)^2} = 14$ kg.

The canonical form of the equation of motion is

$$\frac{d^2x}{dt^2} + 2d\frac{dx}{dt} + \omega^2 x = 0,$$

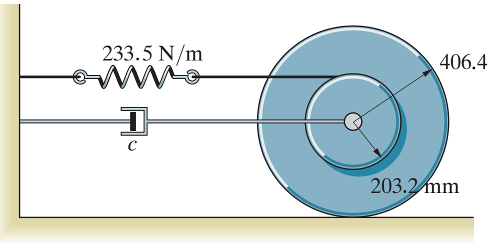
where $\quad d = \dfrac{c}{2M} = 4.170$ rad/s,

$$\omega^2 = \left(\frac{3}{2}\right)^2 \frac{k}{M} = 37.53 \text{ (rad/s)}^2.$$

Therefore $\omega_d = \sqrt{\omega^2 - d^2} = 4.488$ rad/s, and the frequency is

$$\boxed{f_d = \frac{\omega_d}{2\pi} = 0.714 \text{ Hz}}$$

Problem 21.87 The stepped disk described in Problem 21.86 is initially in equilibrium, and at $t = 0$ it is given a clockwise angular velocity of 1 rad/s. Determine the position of the center of the disk relative to its equilibrium position as a function of time.



Solution: From the solution to Problem 21.86, the equation of motion is

$$\frac{d^2x}{dt^2} + 2d\frac{dx}{dt} + \omega^2 x = 0,$$

where $d = 4.170$ rad/s, $\omega^2 = 37.53$ (rad/s)^2. Therefore

$$\omega_d = \sqrt{\omega^2 - d^2} = 4.488 \text{ rad/s}.$$

Since $d^2 < \omega^2$, the system is sub-critically damped. The solution is of the form

$$x = e^{-dt}(A \sin \omega_d t + B \cos \omega_d t).$$

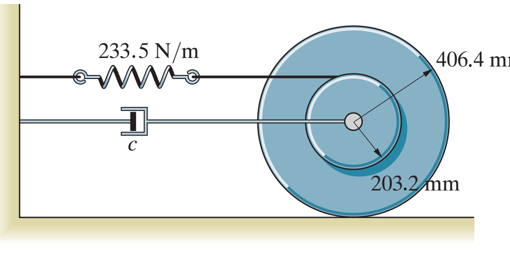
Apply the initial conditions: at $t = 0$, $\theta_0 = 0$, and $\dot{\theta}_0 = -1$ rad/s. From kinematics, $\dot{x}_0 = -2R\dot{\theta}_0 = 2R$ m. Substitute, to obtain

$$B = 0 \quad \text{and} \quad A = \frac{\dot{x}_0}{\omega_d} = 0.0906,$$

and the position of the center of the disk is

$$\boxed{x = 0.0906e^{-4.170t} \sin 4.488t}$$

Problem 21.88 The stepped disk described in Problem 21.86 is initially in equilibrium, and at $t = 0$ it is given a clockwise angular velocity of 1 rad/s. Determine the position of the center of the disk relative to its equilibrium position as a function of time if $c = 233.5$ N-s/m.



Solution: From the solution to Problem 21.86, the equation of motion is

$$\frac{d^2x}{dt^2} + 2d\frac{dx}{dt} + \omega^2 x = 0,$$

where $d = \dfrac{c}{2M} = 8.340$ rad/s,

$$\omega^2 = \left(\frac{3}{2}\right)^2 \frac{k}{M} = 37.53 \text{ (rad/s)}^2.$$

Since $d^2 > \omega^2$,

the system is supercritically damped. The solution is of the form (see Eq. (21.24)) $x = e^{-dt}(Ce^{ht} + De^{-ht})$, where $h = \sqrt{d^2 - \omega^2} = 5.659$ rad/s. Apply the initial conditions: at $t = 0$, $\theta_0 = 0$, and $\dot{\theta}_0 = -1$ rad/s. From kinematics, $\dot{x}_0 = -2R\dot{\theta}_0 = 2R$ m. Substitute, to obtain $0 = C + D$ and $\dot{x}_0 = -(d - h)C - (d + h)D$. Solve:

$$C = \frac{x_0}{2h} = 0.036,$$

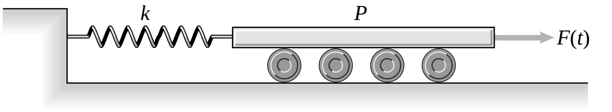
$$D = -\frac{x_0}{2h} = -0.036,$$

from which the position of the center of the disk is

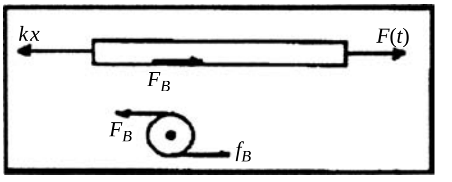
$$x = 0.036e^{-8.340t}(e^{5.659t} - e^{-5.659t})$$

$$\boxed{= 0.036(e^{-2.680t} - e^{-14.00t}) \text{ m}}$$

Problem 21.89 The 22-kg platen P rests on four roller bearings that can be modeled as 1-kg homogeneous cylinders with 30-mm radii. The spring constant is $k = 900$ N/m. The platen is subjected to a force $F(t) = 100 \sin 3t$ N. What is the magnitude of the platen's steady-state horizontal vibration?



Solution: Choose a coordinate system with the origin at the wall and the x axis parallel to the plane surface. Denote the roller bearings by the subscript B and the platen by the subscript P.



The roller bearings: The sum of the moments about the mass center of a roller bearing is

$$\sum M_{B\text{-cm}} = +RF_B + Rf_B.$$

From Newton's second law:

$$I_B \frac{d^2\theta}{dt^2} = RF_B + Rf_B.$$

Solve for the reaction at the floor:

$$f_B = \frac{I_B}{R}\frac{d^2\theta}{dt^2} - F_B.$$

The sum of the *horizontal* forces on each roller bearing:

$$\sum F_x = -F_B + f_P.$$

From Newton's second law

$$m_B \frac{d^2 x_B}{dt^2} = -F_B + f_B,$$

where x_B is the translation of the center of mass of the roller bearing. Substitute

$$f_p, \quad m_B \frac{d^2 x_B}{dt^2} = \frac{I_B}{R}\frac{d^2\theta}{dt^2} - 2F_B.$$

From kinematics, $\theta_B = -\dfrac{x_B}{R}$,

from which $\left(m_B + \dfrac{I_B}{R^2}\right)\dfrac{d^2 x_B}{dt^2} = -2F_B.$

The platen: The sum of the forces on the platen are

$$\sum F_P = -kx + 4F_B + F(t).$$

From Newton's second law,

$$m_P \frac{d^2 x_P}{dt^2} = -kx_p + 4F_B + F(t).$$

Substitute for F_B and rearrange:

$$m_p \frac{d^2 x_p}{dt^2} + kx_P + 2\left(m_B + \frac{I_B}{R^2}\right)\frac{d^2 x_B}{dt^2} = F(t).$$

From kinematics, $x_B = \dfrac{x_P}{2}$,

from which $\left(m_P + m_B + \dfrac{I_B}{R^2}\right)\dfrac{d^2 x_P}{dt^2} + kx_P = F(t).$

For a homogenous cylinder,

$$I_B = \frac{m_B R^2}{2},$$

from which we define

$$M = m_p + \frac{3}{2}m_B = 23.5 \text{ kg}.$$

For $d = 0$, the canonical form of the equation of motion (see Eq. (21.26)) is

$$\frac{d^2 x_p}{dt^2} + \omega^2 x_p = a(t),$$

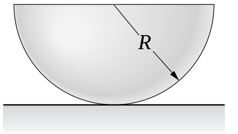
where $\omega^2 = 38.30 \text{ (rad/s)}^2$,

and $a(t) = \dfrac{F(t)}{M} = 4.255 \sin 3t \text{ (m/s}^2\text{)}.$

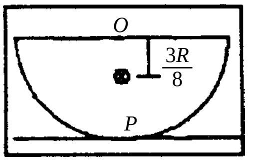
The amplitude of the steady state motion is given by Eq. (21.31):

$$\boxed{E_p = \frac{4.255}{(\omega^2 - 3^2)} = 0.1452 \text{ m}}$$

Problem 21.83 A homogeneous hemisphere of radius R and mass m rests on a level surface. If you rotate the hemisphere slightly from its equilibrium position and release it, what is the frequency of its vibrations?



Solution: The system is conservative. The distance from the center of mass to point O is $h = 3R/8$. Denote the angle of rotation about P by θ. Rotation about P causes the center of mass to rotate relative to the radius center OP, suggesting the analogy with a pendulum suspended from O. The kinetic energy is $T = (1/2)I_p\dot{\theta}^2$. The potential energy is $V = mgh(1-\cos\theta)$, where $h(1-\cos\theta)$ is the increase in height of the center of mass. $T + V = \text{const} = I_p\dot{\theta}^2 + mgh(1-\cos\theta)$. Take the time derivative:



$$\dot{\theta}\left[I_p\frac{d^2\theta}{dt^2} + mgh\sin\theta\right] = 0,$$

from which $\dfrac{d^2\theta}{dt^2} + \dfrac{mgh}{I_p}\sin\theta = 0.$

For small angles $\sin\theta \to \theta$, and the moment of inertia about P is

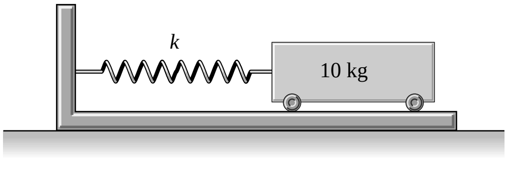
$$I_p = I_{CM} + m(R-h)^2 = \frac{83}{320}mR^2 + \left(\frac{5}{8}\right)^2 mR^2 = \frac{13}{20}mR^2,$$

from which $\dfrac{d^2\theta}{dt^2} + \omega^2\theta = 0,$

where $\omega = \sqrt{\dfrac{20(3)g}{13(8)R}} = \sqrt{\dfrac{15g}{26R}}.$

The frequency is $\boxed{f = \dfrac{1}{2\pi}\sqrt{\dfrac{15g}{26R}}}$

Problem 21.84 The frequency of the spring-mass oscillator is measured and determined to be 4.00 Hz. The oscillator is then placed in a barrel of oil, and its frequency is determined to be 3.80 Hz. What is the logarithmic decrement of vibrations of the mass when the oscillator is immersed in oil?



Solution: The undamped and damped frequencies are $f = 4$ Hz and $f_d = 3.8$ Hz, so

$$\tau_d = \frac{1}{f_d} = 0.263 \text{ s},$$

$$\omega = 2\pi f = 25.13 \text{ rad/s},$$

$$\omega_d = 2\pi f_d = 23.88 \text{ rad/s}.$$

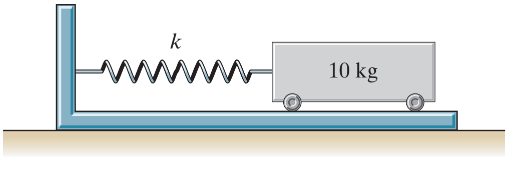
From the relation

$$\omega_d = \sqrt{\omega^2 - d^2},$$

we obtain $d = 7.85$ rad/s, so the logarithmic decrement is $\delta = d\tau_d = (7.85)(0.263) = 2.07$.

$\underline{\delta = 2.07}$.

Problem 21.85 Consider the oscillator immersed in oil described in Problem 21.84. If the mass is displaced 0.1 m to the right of its equilibrium position and released from rest, what is its position relative to the equilibrium position as a function of time?



Solution: The mass and spring constant are unknown. The canonical form of the equation of motion (see Eq. (21.16)) is

$$\frac{d^2x}{dt^2} + 2d\frac{dx}{dt} + \omega^2 x = 0,$$

where, from the solution to Problem 21.84, $d = 7.848$ rad/s, and $\omega = 2\pi(4) = 25.13$ rad/s. The solution is of the form (see Eq. (21.19)) $x = e^{-dt}(A\sin\omega_d t + B\cos\omega_d t)$, where $\omega_d = 2\pi(3.8) = 23.88$ rad/s. Apply the initial conditions: at $t = 0$, $x_0 = 0.1$ m, and $\dot{x}_0 = 0$, from which $B = x_0$, and $0 = -dB + \omega_d A$,

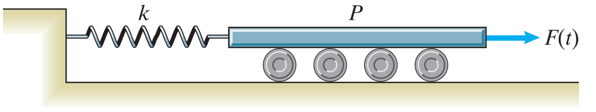
from which $A = \dfrac{dx_0}{\omega_d},$

The solution is

$$x = x_0 e^{-dt}\left(\frac{d}{\omega_d}\sin\omega_d + \cos\omega_d\right)$$

$$\boxed{= 0.1e^{-7.848t}(0.3287\sin 23.88t + \cos 23.88t)}$$

Problem 21.90 At $t = 0$, the platen described in Problem 21.89 is 0.1 m to the right of its equilibrium position and is moving to the right at 2 m/s. Determine the platen's position relative to its equilibrium position as a function of time.



Solution: From the solution to Problem 21.89, the equation of motion is

$$\frac{d^2 x_p}{dt^2} + \omega^2 x_p = a(t),$$

where $\omega^2 = 38.30$ (rad/s)^2, and

$$a(t) = \frac{F(t)}{M} = 4.255 \sin 3t \text{ m/s}^2.$$

The solution is in the form $x = x_h + x_p$, where the homogenous solution is of the form $x_h = A \sin \omega t + B \cos \omega t$ and the particular solution x_p is given by Eq. (21.30), with $d = 0$ and $b_0 = 0$. The result:

$$x = A \sin \omega t + B \cos \omega t + \frac{a_0}{(\omega^2 - \omega_0^2)} \sin \omega_0 t,$$

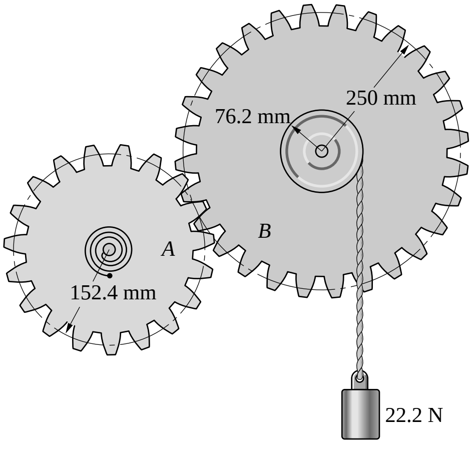
where $a_0 = 4.255$ m, $\omega = 6.189$ rad/s, and $\omega_0 = 3$ rad/s. Apply the initial conditions: at $t = 0$, $x_0 = 0.1$ m, and $\dot{x}_0 = 2$ m/s, from which $B = 0.1$, and

$$A = \frac{2}{\omega} - \left(\frac{\omega_0}{\omega}\right) \frac{a_0}{(\omega^2 - \omega_0^2)} = 0.2528,$$

from which

$$\boxed{x = 0.253 \sin 6.19t + 0.1 \cos 6.19t + 0.145 \sin 3t \text{ m}}$$

Problem 21.91 The moments of inertia of gears A and B are $I_A = 0.019$ kg-m^2 and $I_B = 0.136$ kg-m^2. Gear A is connected to a torsional spring with constant $k = 2.71$ N-m/rad. The bearing supporting gear B incorporates a damping element that exerts a resisting moment on gear B of magnitude $2.03(d\theta_B/dt)$ N-m, where $d\theta_B/dt$ is the angular velocity of gear B in rad/s. What is the frequency of angular vibration of the gears?



Solution: Choose a coordinate system with the x axis positive downward. The sum of the moments on gear A is $\sum M = -k\theta_A + R_A F$, where the moment exerted by the spring opposes the angular displacement θ_A. From the equation of angular motion,

$$I_A \frac{d^2\theta_A}{dt^2} = \sum M = -k\theta_A + R_A F,$$

from which $F = \left(\frac{I_A}{R_A}\right)\frac{d^2\theta_A}{dt^2} + \left(\frac{k}{R_A}\right)\theta_A.$

The sum of the moments acting on gear B is

$$\sum M = -2.03\frac{d\theta_B}{dt} + R_B F - R_W F_W,$$

where $W = 22.2$ N, and the moment exerted by the damping element opposes the angular velocity of B. From the equation of angular motion applied to B,

$$I_B \frac{d^2\theta_B}{dt^2} = \sum M = -2.03\frac{d\theta_B}{dt} + R_B F - R_W F_W.$$

The sum of the forces on the weight are $\sum F = +F_W - W$. From Newton's second law applied to the weight,

$$\left(\frac{W}{g}\right)\frac{d^2x}{dt^2} = F_W - W,$$

from which $F_W = \left(\frac{W}{g}\right)\frac{d^2x}{dt^2} + W.$

Substitute for F and F_W to obtain the equation of motion for gear B:

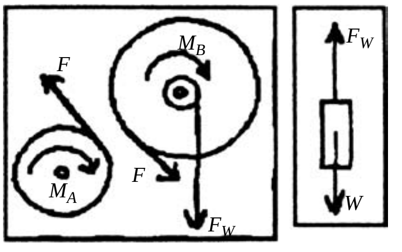
$$I_B\frac{d^2\theta_B}{dt^2} + 2.03\frac{d\theta_B}{dt} - \left(\frac{R_B}{R_A}\right)\left(I_A\frac{d^2\theta_A}{dt^2} + k\theta_A\right)$$

$$- R_W\left(\left(\frac{W}{g}\right)\frac{d^2x}{dt^2} + W\right) = 0.$$

From kinematics, $\theta_A = -\left(\frac{R_B}{RA}\right)\theta_B$, and $x = -R_W\theta_B$, from which

$$M\frac{d^2\theta_B}{dt^2} + 2.03\frac{d\theta_B}{dt} + \left(\frac{R_B}{R_A}\right)^2 k\theta_B = R_W W,$$

where $M = I_A + \left(\frac{R_B}{R_A}\right)^2 I_A + R_W^2\left(\frac{W}{g}\right) = 0.201$ kg-m^2



The canonical form of the equation of motion is

$$\frac{d^2\theta_B}{dt^2} + 2d\frac{d\theta_B}{dt} + \omega^2\theta_B = P,$$

where $d = \frac{2.03}{2M} = 5.047$ rad/s,

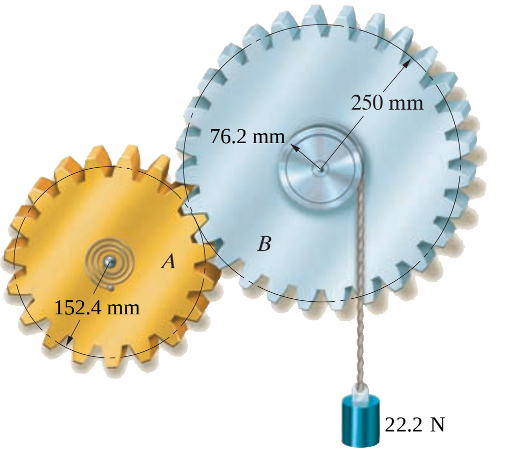
$$\omega^2 = \frac{\left(\frac{R_B}{R_A}\right)^2 k}{M} = 37.39 \text{ (rad/s)}^2,$$

and $P = \frac{R_W W}{M} = 8.412$ (rad/s)^2.

The system is sub critically damped, since $d^2 < \omega^2$, from which $\omega_d = \sqrt{\omega^2 - d^2} = 3.452$ rad/s, and the frequency is

$$\boxed{f_d = \frac{\omega_d}{2\pi} = 0.5493 \text{ Hz}}$$

Problem 21.92 The 22.2 N weight in Problem 21.91 is raised 12.7 mm. from its equilibrium position and released from rest at $t = 0$. Determine the counterclockwise angular position of gear B relative to its equilibrium position as a function of time.



Solution: From the solution to Problem 81.91,

$$\frac{d^2\theta_B}{dt^2} + 2d\frac{d\theta_B}{dt} + \omega^2\theta_B = P,$$

where $d = 5.047$ rad/s, $\omega^2 = 37.39$ (rad/s)^2, and $P = 8.412$ (rad/s)^2. Since the non homogenous term P is independent of time and angle, the equilibrium position is found by setting the acceleration and velocity to zero in the equation of motion and solving: $\theta_{eq} = \frac{P}{\omega^2}$. Make the transformation $\tilde{\theta}_B = \theta_B - \theta_{eq}$, from which, by substitution,

$$\frac{d^2\tilde{\theta}_B}{dt^2} + 2d\frac{d\tilde{\theta}_B}{dt} + \omega^2\tilde{\theta} = 0$$

is the equation of motion about the equilibrium point. Since $d^2 < \omega^2$, the system is sub critically damped, from which the solution is $\tilde{\theta}_B = e^{-dt}(A\sin\omega_d t + B\cos\omega_d t)$. Apply the initial conditions: $x_0 = -0.0127$ m, from which

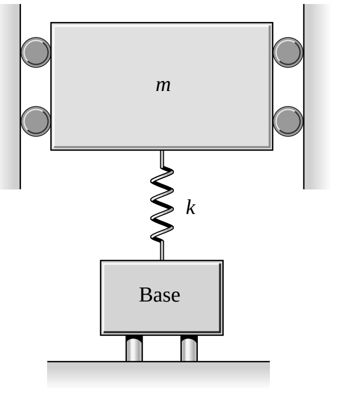
$$[\tilde{\theta}_B]_{t=0} = -\frac{x_0}{R_w} = 0.1667 \text{ rad}, \quad \frac{d\tilde{\theta}_B}{dt} = 0,$$

from which $B = 0.1667$, $A = \frac{d(0.1667)}{\omega_d} = 0.2437$.

The solution is

$$\boxed{\tilde{\theta}(t) = e^{-5.047t}(0.244\sin(3.45t) + 0.167\cos(3.45t))}$$

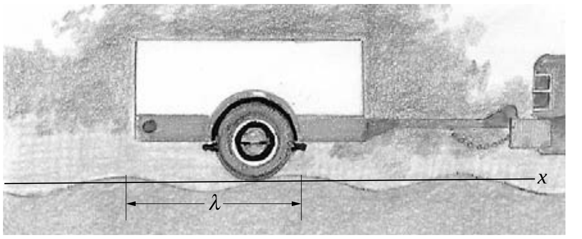
Problem 21.93 The base and mass m are initially stationary. The base is then subjected to a vertical displacement $h\sin\omega_i t$ relative to its original position. What is the magnitude of the resulting steady-state vibration of the mass m *relative to the base*?



Solution: From Eq. (21.26), for $d = 0$, $\frac{d^2x}{dt^2} + \omega^2 x = a(t)$, where $\omega^2 = \frac{k}{m}$, and $a(t) = \omega_i^2 h\sin\omega_i t$. From Eq. (21.31), the steady state amplitude is

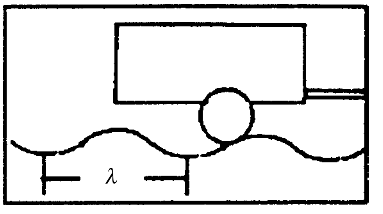
$$\boxed{E_p = \frac{\omega_i^2 h}{(\omega^2 - \omega_i^2)} = \frac{\omega_i^2 h}{\left(\frac{k}{m} - \omega_i^2\right)}}$$

Problem 21.94* The mass of the trailer, not including its wheels and axle, is m, and the spring constant of its suspension is k. To analyze the suspension's behavior, an engineer assumes that the height of the road surface relative to its mean height is $h \sin(2\pi/\lambda)$. Assume that the trailer's wheels remain on the road and its horizontal component of velocity is v. Neglect the damping due to the suspension's shock absorbers.



(a) Determine the magnitude of the trailer's vertical steady-state vibration *relative to the road surface*.
(b) At what velocity v does resonance occur?

Solution: Since the wheels and axle act as a base that moves with the disturbance, this is analogous to the transducer problem (Example 21.7). For a *constant velocity* the distance $x = \int_0^t v\,dt = vt$, from which the movement of the axle-wheel assembly as a function of time is $h_f(t) = h\sin(\omega_0 t)$, where $\omega_0 = \frac{2\pi v}{\lambda}$ rad/s. [*Check*: The velocity of the disturbing "waves" in the road relative to the trailer is v. Use the physical relationship between frequency, wavelength and velocity of propagation $\lambda f = v$. The wavelength of the road disturbance is λ, from which the forcing function frequency is $f_0 = \frac{v}{\lambda}$, and the circular frequency is $\omega_0 = 2\pi f_0 = \frac{2\pi v}{\lambda}$. *check.*] The forcing function on the spring-mass oscillator (that is, the trailer body and spring) is (see Example 21.7)



$$F(t) = -m\frac{d^2 h_f(t)}{dt^2} = m\left(\frac{2\pi v}{\lambda}\right)^2 h \sin\left(\frac{2\pi v}{\lambda}t\right).$$

For $d = 0$, the canonical form of the equation of motion is $\frac{d^2y}{dt^2} + \omega^2 y = a(t)$, where $\omega = \sqrt{\frac{k}{m}} = 8.787$ rad/s, and

$$a(t) = \frac{F(t)}{m} = \left(\frac{2\pi v}{\lambda}\right)^2 h \sin\left(\frac{2\pi v}{\lambda}t\right)$$

$$= a_0\left(\frac{2\pi v}{\lambda}\right)^2 \sin\left(\frac{2\pi v}{\lambda}t\right),$$

where $a_0 = h$.

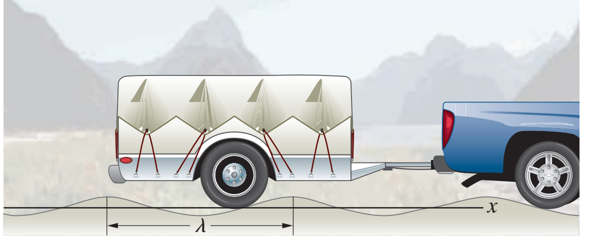
(a) The magnitude of the steady state amplitude of the motion relative to the wheel-axle assembly is given by Eq. (21.31) and the equations in Example 21.7 for $d = 0$ and $b_0 = 0$,

$$\boxed{E_p = \frac{\left(\frac{2\pi v}{\lambda}\right)^2 h}{\left(\frac{k}{m} - \left(\frac{2\pi v}{\lambda}\right)^2\right)}}$$

(b) Resonance, by definition, occurs when the denominator vanishes, from which

$$\boxed{v = \frac{\lambda}{2\pi}\sqrt{\frac{k}{m}}}$$

Problem 21.95* The trailer in Problem 21.94, not including its wheels and axle, weighs 4448 N. The spring constant of its suspension is $k = 35024$ N/m, and the damping coefficient due to its shock absorbers is $c = 2919$ N-s/m. The road surface parameters are $h = 5.1$ mm and $\lambda = 2.44$ m. The trailer's horizontal velocity is $v = 9.65$ km/h. Determine the magnitude of the trailer's vertical steady-state vibration relative to the road surface,



(a) neglecting the damping due to the shock absorbers and
(b) not neglecting the damping.

Solution:

(a) From the solution to Problem 21.94, for zero damping, the steady state amplitude relative to the road surface is

$$E_p = \frac{\left(\dfrac{2\pi v}{\lambda}\right)^2 h}{\left(\dfrac{k}{m} - \left(\dfrac{2\pi v}{\lambda}\right)^2\right)}.$$

Substitute numerical values:

$v = 9.65$ km/h $= 2.68$ m/s, $\lambda = 2.44$ m, $k = 35024$ N/m,

$$m = \frac{W}{g} = 453.5 \text{ kg},$$

to obtain $\boxed{E_p = 0.082 \text{ m} = 82 \text{ mm.}}$

(b) From Example 21.7, and the solution to Problem 21.94, the canonical form of the equation of motion is

$$\frac{d^2y}{dt^2} + 2d\frac{dy}{dt} + \omega^2 y = a(t),$$

where $d = \dfrac{c}{2m} = \dfrac{200}{2(31.08)} = 3.217$ rad/s, $\omega = 8.787$ rad/s,

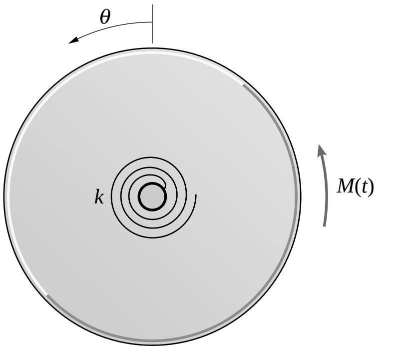
and $a(t) = a_0\left(\dfrac{2\pi v}{\lambda}\right)^2 \sin\left(\dfrac{2\pi v}{\lambda}t\right)$,

where $a_0 = h$. The system is sub-critically damped, from which $\omega_d = \sqrt{\omega^2 - d^2} = 8.177$ rad/s. From Example 21.7, the magnitude of the steady state response is

$$\boxed{E_p = \frac{\left(\dfrac{2\pi v}{\lambda}\right)^2 h}{\sqrt{\left(\omega^2 - \left(\dfrac{2\pi v}{\lambda}\right)^2\right)^2 + 4d^2\left(\dfrac{2\pi v}{\lambda}\right)^2}} = 0.045 \text{ m} = 45 \text{ mm.}}$$

Problem 21.96* A disk with moment of inertia I rotates about a fixed shaft and is attached to a torsional spring with constant k. The angle θ measures the angular position of the disk relative to its position when the spring is unstretched. The disk is initially stationary with the spring unstretched. At $t = 0$, a time-dependent moment $M(t) = M_0(1 - e^{-t})$ is applied to the disk, where M_0 is a constant. Show that the angular position of the disk as a function of time is

$$\theta = \frac{M_0}{I}\left[-\frac{1}{\omega(1+\omega^2)}\sin\omega t - \frac{1}{\omega^2(1+\omega^2)}\cos\omega t + \frac{1}{\omega^2} - \frac{1}{(1+\omega^2)}e^{-t}\right].$$



Strategy: To determine the particular solution, seek a solution of the form

$$\theta_p = A_p + B_p e^{-t},$$

where A_p and B_p are constants that you must determine.

Solution: The sum of the moments on the disk are $\sum M = -k\theta + M(t)$. From the equation of angular motion, $I\frac{d^2\theta}{dt^2} = -k\theta + M(t)$. For $d = 0$ the canonical form is $\frac{d^2\theta}{dt^2} + \omega^2\theta = a(t)$, where $\omega = \sqrt{\frac{k}{I}}$, and $a(t) = \frac{M_0}{I}(1 - e^{-t})$. (a) For the particular solution, assume a solution of the form $\theta_p = A_p + B_p e^{-t}$. Substitute into the equation of motion,

$$\frac{d^2\theta_p}{dt^2} + \omega^2\theta_p = B_p e^{-t} + \omega^2(A_p + B_p e^{-t}) = \frac{M_0}{I}(1 - e^{-t}).$$

Rearrange:

$$\omega^2 A_p + B_p(1+\omega^2)e^{-t} = \frac{M_0}{I} - \frac{M_0}{I}e^{-t}.$$

Equate like coefficients:

$$\omega^2 A_p = \frac{M_0}{I},\ B_p(1+\omega^2) = -\frac{M_0}{I},$$

from which $A_p = \frac{M_0}{\omega^2 I}$, and $B_p = -\frac{M_0}{(1+\omega^2)I}$.

The particular solution is

$$\theta_p = \frac{M_0}{I}\left[\frac{1}{\omega^2} - \frac{e^{-t}}{(1+\omega^2)}\right].$$

The system is undamped. The solution to the homogenous equation has the form $\theta_h = A\sin\omega t + B\cos\omega t$. The trial solution is:

$$\theta = \theta_h + \theta_p = A\sin\omega t + B\cos\omega t + \frac{M_0}{I}\left(\frac{1}{\omega^2} - \frac{e^{-t}}{(1+\omega^2)}\right).$$

Apply the initial conditions: at $t = 0$, $\theta_0 = 0$ and $\dot\theta_0 = 0$: $\theta_0 = 0 = B + \frac{M_0}{I}\left(\frac{1}{\omega^2} - \frac{1}{(1+\omega^2)}\right)$,

$$\dot\theta_0 = 0 = \omega A + \frac{M_0}{I}\left(\frac{1}{(1+\omega^2)}\right),$$

from which

$$A = -\frac{M_0}{I}\left(\frac{1}{\omega(1+\omega^2)}\right),$$

$$B = -\frac{M_0}{I}\left(\frac{1}{\omega^2} - \frac{1}{(1+\omega^2)}\right) = -\frac{M_0}{I}\left(\frac{1}{\omega^2(1+\omega^2)}\right),$$

and the complete solution is

$$\boxed{\theta = \frac{M_0}{I}\left[-\frac{1}{\omega(1+\omega^2)}\sin\omega t - \frac{1}{\omega^2(1+\omega^2)}\cos\omega t + \frac{1}{\omega^2} - \frac{1}{(1+\omega^2)}e^{-t}\right].}$$