Problem 20.1 The airplane's angular velocity relative to an earth-fixed reference frame, expressed in terms of the body-fixed coordinate system shown, is $\boldsymbol{\omega} = 0.62\mathbf{i} + 0.45\mathbf{j} - 0.23\mathbf{k}$ (rad/s). The coordinates of point A of the airplane are (3.6, 0.8, −1.2) m. What is the velocity of point A relative to the velocity of the airplane's center of mass?

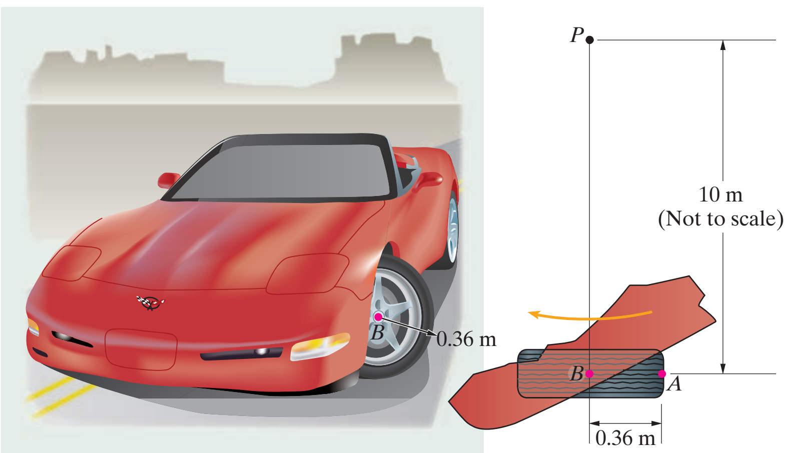
Solution:

$$\mathbf{v}_{A/G} = \boldsymbol{\omega} \times \mathbf{r}_{A/G}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.62 & 0.45 & -0.23 \\ 3.6 & 0.8 & -1.2 \end{vmatrix} \text{ (m/s)}$$

$$\boxed{\mathbf{v}_{A/G} = (-0.356\mathbf{i} - 0.084\mathbf{j} - 1.12\mathbf{k}) \text{ m/s.}}$$

Problem 20.2 In Active Example 20.1, suppose that the center of the tire moves at a constant speed of 5 m/s as the car turns. (As a result, when the angular velocity of the tire relative to an earth-fixed reference frame is expressed *in terms of components in the secondary reference frame*, $\boldsymbol{\omega} = \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}$, the components ω_x, ω_y, and ω_z are constants.) What is the angular acceleration $\boldsymbol{\alpha}$ of the tire relative to an earth-fixed reference frame?



Top view

Solution: The angular velocity of the secondary coordinate system is

$$\boldsymbol{\Omega} = -\frac{v}{R}\mathbf{k} = -\frac{(5 \text{ m/s})}{(10 \text{ m})}\mathbf{k} = -(0.5 \text{ rad/s})\mathbf{k}$$

The angular velocity of the wheel with components in the secondary coordinate system is

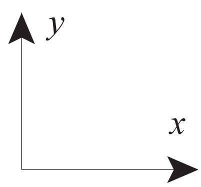
$$\boldsymbol{\omega} = \boldsymbol{\Omega} - \frac{v}{r}\mathbf{j} = -(0.5 \text{ rad/s})\mathbf{k} - \frac{(5 \text{ m/s})}{(0.36 \text{ m})}\mathbf{j}$$

$$= (-13.9\mathbf{j} - 0.5\mathbf{k}) \text{ rad/s}$$

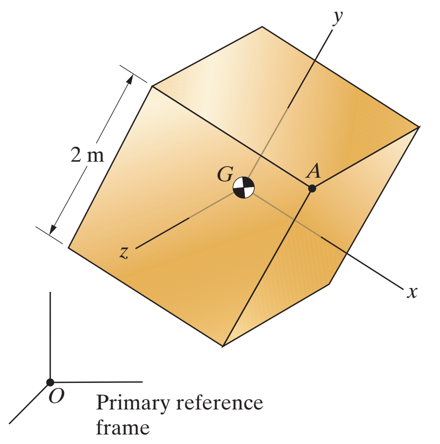
The angular acceleration is then

$$\boldsymbol{\alpha} = \boldsymbol{\Omega} \times \boldsymbol{\omega} = -(0.5 \text{ rad/s})\mathbf{k} \times (-13.9\mathbf{j} - 0.5\mathbf{k}) \text{ rad/s}$$

$$\boxed{\alpha = -6.94\mathbf{i} \text{ rad/s}^2.}$$



Problem 20.3 The angular velocity of the cube relative to the primary reference frame, expressed in terms of the body-fixed coordinate system shown is $\boldsymbol{\omega} = -6.4\mathbf{i} + 8.2\mathbf{j} + 12\mathbf{k}$ (rad/s). The velocity of the center of mass G of the cube relative to the primary reference frame at the instant shown is $\mathbf{v}_G = 26\mathbf{i} + 14\mathbf{j} + 32\mathbf{k}$ (m/s). What is the velocity of point A of the cube relative to the primary reference frame at the instant shown?



Solution: The vector from G to A is

$$\mathbf{r}_{G/A} = (\mathbf{i} + \mathbf{j} + \mathbf{k}) \text{ m.}$$

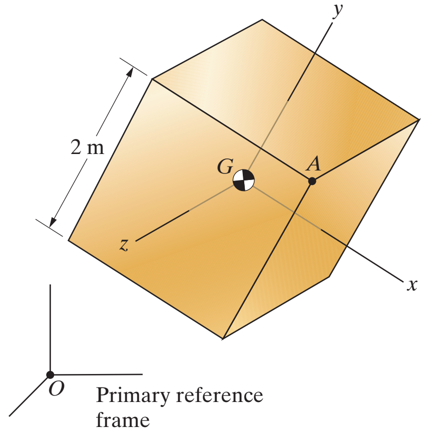
The velocity of point A is

$$\mathbf{v}_A = \mathbf{v}_G + \boldsymbol{\omega} \times \mathbf{r}_{G/A}$$

$$= (26\mathbf{i} + 14\mathbf{j} + 32\mathbf{k}) \text{ m/s} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -6.4 & 8.2 & 12 \\ 1 & 1 & 1 \end{vmatrix} \text{m/s}$$

$$\boxed{\mathbf{v}_A = (22.2\mathbf{i} + 32.4\mathbf{j} + 17.4\mathbf{k}) \text{ m/s.}}$$

Problem 20.4 The coordinate system shown is fixed with respect to the cube. The angular velocity of the cube relative to the primary reference frame, $\boldsymbol{\omega} = -6.4\mathbf{i} + 8.2\mathbf{j} + 12\mathbf{k}$ (rad/s), is constant. The acceleration of the center of mass G of the cube relative to the primary reference frame at the instant shown is $\mathbf{a}_G = 136\mathbf{i} + 76\mathbf{j} - 48\mathbf{k}$ (m/s^2). What is the acceleration of point A of the cube relative to the primary reference frame at the instant shown?



Solution: The vector from G to A is

$$\mathbf{r}_{G/A} = (\mathbf{i} + \mathbf{j} + \mathbf{k}) \text{ m.}$$

The accleration of point A is

$$\mathbf{a}_A = \mathbf{a}_G + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{G/A})$$

$$= (136\mathbf{i} + 76\mathbf{j} - 48\mathbf{k}) \text{ m/s}^2 + (-6.4\mathbf{i} + 8.2\mathbf{j} + 12\mathbf{k}) \times \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -6.4 & 8.2 & 12 \\ 1 & 1 & 1 \end{vmatrix} \text{m/s}^2$$

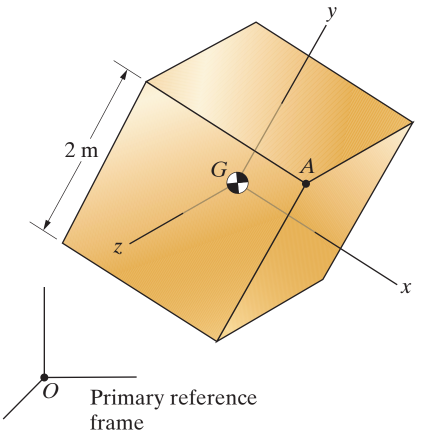
Carrying out the vector algebra, we have

$$\boxed{\mathbf{a}_A = (-205\mathbf{i} - 63.0\mathbf{j} - 135\mathbf{k}) \text{ m/s}^2.}$$

Problem 20.5 The origin of the secondary coordinate system shown is fixed to the center of mass G of the cube. The velocity of the center of mass G of the cube relative to the primary reference frame at the instant shown is $\mathbf{v}_G = 26\mathbf{i} + 14\mathbf{j} + 32\mathbf{k}$ (m/s). The cube is rotating relative to the secondary coordinate system with angular velocity $\boldsymbol{\omega}_{\text{rel}} = 6.2\mathbf{i} - 5\mathbf{j} + 8.8\mathbf{k}$ (rad/s). The secondary coordinate system is rotating relative to the primary reference frame with angular velocity $\boldsymbol{\Omega} = 2.2\mathbf{i} + 4\mathbf{j} - 3.6\mathbf{k}$ (rad/s).

(a) What is the velocity of point A of the cube relative to the primary reference frame at the instant shown?

(b) If the components of the vectors $\boldsymbol{\omega}_{\text{rel}}$ and $\boldsymbol{\Omega}$ are constant, what is the cube's angular acceleration relative to the primary reference frame?



Solution:

(a) $\mathbf{v}_A = \mathbf{v}_G + (\boldsymbol{\omega}_{\text{rel}} + \boldsymbol{\Omega}) \times \mathbf{r}_{A/G}$

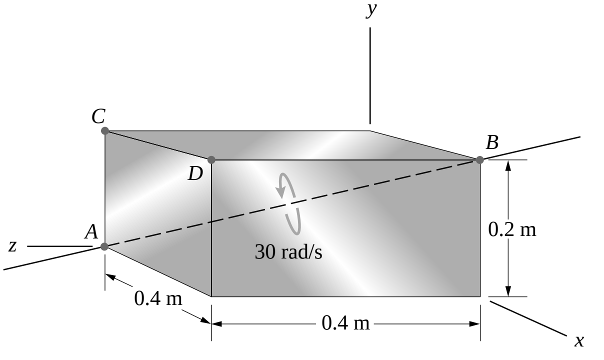
$= (26\mathbf{i} + 14\mathbf{j} + 32\mathbf{k}) + (8.4\mathbf{i} - \mathbf{j} + 5.2\mathbf{k}) \times (\mathbf{i} + \mathbf{j} + \mathbf{k})$

$\boxed{\mathbf{v}_A = (19.8\mathbf{i} + 10.8\mathbf{j} + 41.4\mathbf{k}) \text{ m/s.}}$

(b) $\boldsymbol{\alpha} = \boldsymbol{\Omega} \times \boldsymbol{\omega}_{\text{rel}} = (2.2\mathbf{i} + 4\mathbf{j} - 3.6\mathbf{k}) \times (6.2\mathbf{i} - 5\mathbf{j} + 8.8\mathbf{k})$

$\boxed{\boldsymbol{\alpha} = (17.2\mathbf{i} - 41.7\mathbf{j} - 35.8\mathbf{k}) \text{ rad/s}^2.}$

Problem 20.6 Relative to an earth-fixed reference frame, points A and B of the rigid parallelepiped are fixed and it rotates about the axis AB with an angular velocity of 30 rad/s. Determine the velocities of points C and D relative to the earth-fixed reference frame.



Solution: Given

$$\boldsymbol{\omega} = (30 \text{ rad/s})\frac{(0.4\mathbf{i} + 0.2\mathbf{j} - 0.4\mathbf{k})}{0.6} = (20\mathbf{i} + 10\mathbf{j} - 20\mathbf{k}) \text{ rad/s}$$

$\mathbf{r}_{C/A} = (0.2 \text{ m})\mathbf{j}$, $\mathbf{r}_{D/A} = (0.4\mathbf{i} + 0.2\mathbf{j})$ m

$\boxed{\mathbf{v}_C = \boldsymbol{\omega} \times \mathbf{r}_{C/A} = (4\mathbf{i} + 4\mathbf{k}) \text{ m/s}, \quad \mathbf{v}_D = \boldsymbol{\omega} \times \mathbf{r}_{D/A} = (4\mathbf{i} - 8\mathbf{j}) \text{ m/s}}$

Problem 20.7 Relative to the xyz coordinate system shown, points A and B of the rigid parallelepiped are fixed and the parallelepiped rotates about the axis AB with an angular velocity of 30 rad/s. Relative to an earth-fixed reference frame, point A is fixed and the xyz coordinate system rotates with angular velocity $\boldsymbol{\Omega} = -5\mathbf{i} + 8\mathbf{j} + 6\mathbf{k}$ (rad/s). Determine the velocities of points C and D relative to the earth-fixed reference frame.

Solution: Given

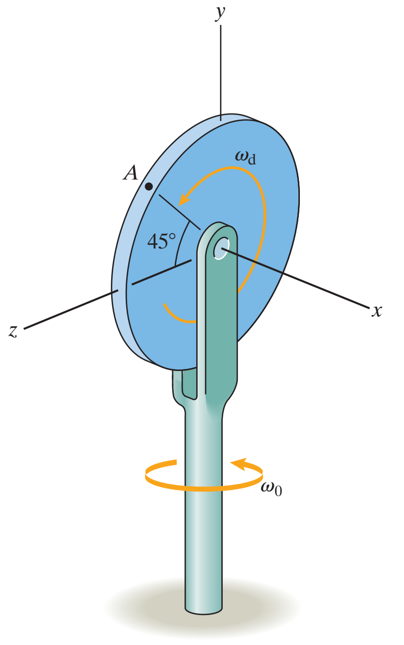
$$\boldsymbol{\omega} = (30 \text{ rad/s})\frac{(0.4\mathbf{i} + 0.2\mathbf{j} - 0.4\mathbf{k})}{0.6} = (20\mathbf{i} + 10\mathbf{j} - 20\mathbf{k}) \text{ rad/s}$$

$\boldsymbol{\Omega} = (-5\mathbf{i} + 8\mathbf{j} + 6\mathbf{k})$ rad/s, $\mathbf{r}_{C/A} = (0.2 \text{ m})\mathbf{j}$,

$\mathbf{r}_{D/A} = (0.4\mathbf{i} + 0.2\mathbf{j})$ m

$$\boxed{\begin{aligned} \mathbf{v}_C &= \mathbf{v}_A + (\boldsymbol{\Omega} + \boldsymbol{\omega}) \times \mathbf{r}_{C/A} = (2.8\mathbf{i} + 3.0\mathbf{k}) \text{ m/s} \\ \mathbf{v}_D &= \mathbf{v}_A + (\boldsymbol{\Omega} + \boldsymbol{\omega}) \times \mathbf{r}_{D/A} = (2.8\mathbf{i} - 5.6\mathbf{j} - 4.2\mathbf{k}) \text{ m/s} \end{aligned}}$$

Problem 20.8 Relative to an earth-fixed reference frame, the vertical shaft rotates about its axis with angular velocity $\omega_0 = 4$ rad/s. The secondary xyz coordinate system is *fixed with respect to the shaft* and its origin is stationary. Relative to the secondary coordinate system, the disk (radius = 8 cm) rotates with constant angular velocity $\omega_\mathrm{d} = 6$ rad/s. At the instant shown, determine the velocity of pint A (a) relative to the secondary reference frame, and (b) relative to the earth-fixed reference frame.



Solution:

(a) Relative to the secondary system

$$\mathbf{v}_A = \boldsymbol{\omega}_\mathrm{rel} \times \mathbf{r}_A = \omega_d \mathbf{i} \times r(\sin 45^\circ \mathbf{j} + \cos 45^\circ \mathbf{k})$$

$$= (6\mathbf{i}) \times (8)(\sin 45^\circ \mathbf{j} + \cos 45^\circ \mathbf{k})$$

$$= (-33.9\mathbf{j} + 33.9\mathbf{k}) \text{ cm/s.}$$

$$\boxed{\mathbf{v}_A = (-33.9\mathbf{j} + 33.9\mathbf{k}) \text{ cm/s.}}$$

(b) Relative to the earth-fixed reference frame

$$\mathbf{v}_A = (\boldsymbol{\omega}_\mathrm{rel} + \boldsymbol{\Omega}) \times \mathbf{r}_A = (\omega_d \mathbf{i} + \omega_0 \mathbf{j}) \times r(\sin 45^\circ \mathbf{j} + \cos 45^\circ \mathbf{k})$$

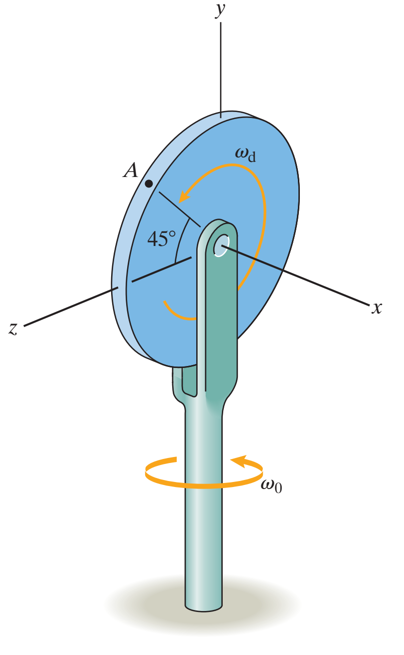
$$= (6\mathbf{i} + 4\mathbf{j}) \times (8)(\sin 45^\circ \mathbf{j} + \cos 45^\circ \mathbf{k})$$

$$= (22.6\mathbf{i} - 33.9\mathbf{j} + 33.9\mathbf{k}) \text{ cm/s.}$$

$$\boxed{\mathbf{v}_A = (22.6\mathbf{i} - 33.9\mathbf{j} + 33.9\mathbf{k}) \text{ cm/s.}}$$

Problem 20.9 Relative to an earth-fixed reference frame, the vertical shaft rotates about its axis with angular velocity $\omega_0 = 4$ rad/s. The secondary xyz coordinate system is *fixed with respect to the shaft* and its origin is stationary. Relative to the secondary coordinate system, the disk (radius = 8 cm) rotates with constant angular velocity $\omega_\mathrm{d} = 6$ rad/s.

(a) What is the angular acceleration of the disk relative to the earth-fixed reference frame?
(b) At the instant shown, determine the acceleration of point A relative to the earth-fixed reference frame.



Solution:

(a) The angular acceleration

$$\boldsymbol{\alpha} = \boldsymbol{\Omega} \times \boldsymbol{\omega}_\mathrm{rel} = \omega_0 \mathbf{j} \times \omega_d \mathbf{i}$$

$$= -\omega_0 \omega_d \mathbf{k} = -(4)(6)\mathbf{k} = -24\mathbf{k}$$

$$\boxed{\boldsymbol{\alpha} = -24\mathbf{k} \text{ rad/s}^2.}$$

(b) The acceleration of point A.

$$\mathbf{a}_A = \boldsymbol{\alpha} \times \mathbf{r}_A + (\boldsymbol{\Omega} + \boldsymbol{\omega}_\mathrm{rel}) \times [(\boldsymbol{\Omega} + \boldsymbol{\omega}_\mathrm{rel}) \times \mathbf{r}_A]$$

$$= (-24\mathbf{k}) \times (8 \sin 45^\circ \mathbf{j} + 8 \cos 45^\circ \mathbf{k})$$

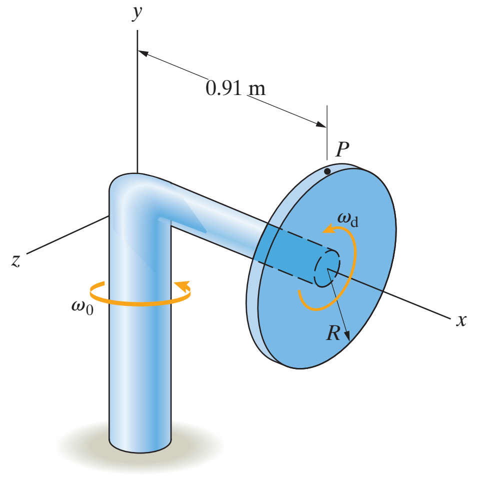
$$+ (6\mathbf{i} + 4\mathbf{j}) \times [(6\mathbf{i} + 4\mathbf{j}) \times (8 \sin 45^\circ \mathbf{j} + 8 \cos 45^\circ \mathbf{k})]$$

$$\boxed{\mathbf{a}_A = (272\mathbf{i} - 204\mathbf{j} - 294\mathbf{k}) \text{ cm/s}^2.}$$

Problem 20.10 The radius of the disk is $R = 0.61$ m. It is perpendicular to the horizontal part of the shaft and rotates relative to it with constant angular velocity $\omega_d = 36$ rad/s. Relative to an earth-fixed reference frame, the shaft rotates about the vertical axis with constant angular velocity $\omega_0 = 8$ rad/s.

(a) Determine the velocity relative to the earth-fixed reference frame of point P, which is the uppermost point of the disk.
(b) Determine the disk's angular acceleration vector $\boldsymbol{\alpha}$ relative to the earth-fixed reference frame.

(See Example 20.2.)



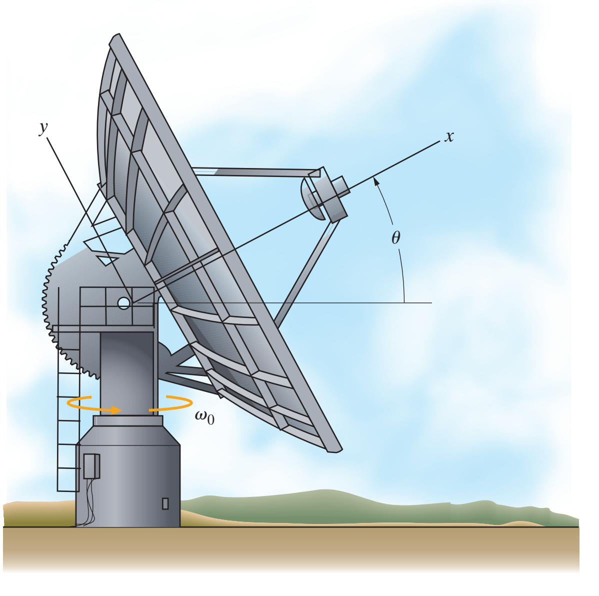
Solution:

(a) $\boxed{\begin{aligned}\mathbf{v}_P &= (8\text{ rad/s})\mathbf{j} \times (0.91\text{ m})\mathbf{i} + [(36\mathbf{i} + 8\mathbf{j})\text{ rad/s}] \times (0.61\text{ m})\,\mathbf{j} \\ &= (14.7\text{ m/s})\mathbf{k}\end{aligned}}$

(b) $\boxed{\boldsymbol{\alpha} = (8\text{ rad/s})\mathbf{j} \times [(36\mathbf{i} + 8\mathbf{j})\text{ rad/s}] = -(288\text{ rad/s}^2)\mathbf{k}}$

Problem 20.11 The vertical shaft supporting the disk antenna is rotating with a constant angular velocity $\omega_0 = 0.2$ rad/s. The angle θ from the horizontal to the antenna's axis is 30° at the instant shown and is increasing at a constant rate of 15° per second. The secondary xyz coordinate system shown is fixed with respect to the dish.

(a) What is the dish's angular velocity relative to an earth-fixed reference frame?
(b) Determine the velocity of the point of the antenna with coordinates (4,0,0) m relative to an earth-fixed reference frame.



Solution: The relative angular velocity is

$$\omega_{\text{rel}} = (15^\circ/\text{s})\left(\frac{\pi\text{ rad}}{180^\circ}\right) = \frac{\pi}{12}\text{ rad/s.}$$

(a) $\boldsymbol{\omega} = \boldsymbol{\Omega} + \boldsymbol{\omega}_{\text{rel}} = (0.2\sin 30^\circ\mathbf{i} + 0.2\cos 30^\circ\mathbf{j}) + \left(\frac{\pi}{12}\right)\mathbf{k}$

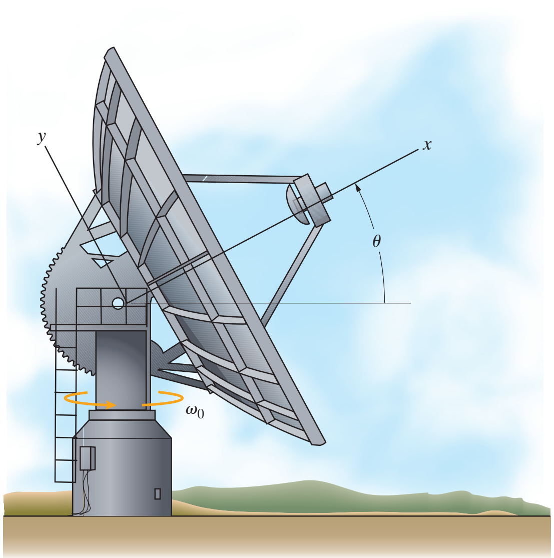
$\boxed{\boldsymbol{\omega} = (0.1\mathbf{i} + 0.173\mathbf{j} + 0.262\mathbf{k})\text{ rad/s.}}$

(b) $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} = (0.1\mathbf{i} + 0.173\mathbf{j} + 0.262\mathbf{k}) \times (4\mathbf{i})$

$\boxed{\mathbf{v} = (1.05\mathbf{j} - 0.693\mathbf{k})\text{ m/s.}}$

Problem 20.12 The vertical shaft supporting the disk antenna is rotating with a constant angular velocity $\omega_0 = 0.2$ rad/s. The angle θ from the horizontal to the antenna's axis is 30° at the instant shown and is increasing at a constant rate of 15° per second. The secondary *xyz* coordinate system shown is fixed with respect to the dish.

(a) What is the dish's angular acceleration relative to an earth-fixed reference frame?
(b) Determine the acceleration of the point of the antenna with coordinates (4, 0, 0) m relative to an earth-fixed reference frame.



Solution: The angular velocity is

$$\omega_{\text{rel}} = (15^\circ/\text{s})\left(\frac{\pi \text{ rad}}{180^\circ}\right) = \frac{\pi}{12} \text{ rad/s.}$$

$$\boldsymbol{\omega} = \boldsymbol{\Omega} + \boldsymbol{\omega}_{\text{rel}} = (0.2\sin 30^\circ\mathbf{i} + 0.2\cos 30^\circ\mathbf{j}) + \left(\frac{\pi}{12}\right)\mathbf{k}$$

$$= (0.1\mathbf{i} + 0.173\mathbf{j} + 0.262\mathbf{k}) \text{ rad/s}$$

(a) The angular acceleration is

$$\boldsymbol{\alpha} = \boldsymbol{\Omega} \times \boldsymbol{\omega}_{\text{rel}} = (0.2\sin 30^\circ\mathbf{i} + 0.2\cos 30^\circ\mathbf{j}) \times \left(\frac{\pi}{12}\mathbf{k}\right)$$

$$\boxed{\boldsymbol{\alpha} = (0.0453\mathbf{i} - 0.0262\mathbf{j}) \text{ rad/s}^2.}$$

(b) The acceleration of the point

$$\mathbf{a} = \boldsymbol{\alpha} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

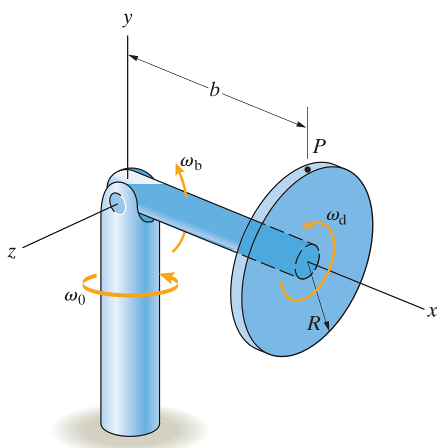
$$= (0.0453\mathbf{i} - 0.0262\mathbf{j}) \times (4\mathbf{i})$$

$$+ (0.1\mathbf{i} + 0.173\mathbf{j} + 0.262\mathbf{k}) \times [(0.1\mathbf{i} + 0.173\mathbf{j} + 0.262\mathbf{k}) \times (4\mathbf{i})]$$

$$\boxed{\mathbf{a} = (-0.394\mathbf{i} + 0.0693\mathbf{j} + 0.209\mathbf{k}) \text{ m/s}^2.}$$

Problem 20.13 The radius of the circular disk is $R = 0.2$ m, and $b = 0.3$ m. The disk rotates with angular velocity $\omega_d = 6$ rad/s relative to the horizontal bar. The horizontal bar rotates with angular velocity $\omega_b = 4$ rad/s relative to the vertical shaft, and the vertical shaft rotates with angular velocity $\omega_0 = 2$ rad/s relative to an earth-fixed reference frame. Assume that the secondary reference frame shown is fixed with respect to the horizontal bar.

(a) What is the angular velocity vector $\boldsymbol{\omega}_{\text{rel}}$ of the disk relative to the secondary reference frame?
(b) Determine the velocity relative to the earth-fixed reference frame of point P, which is the uppermost point of the disk.



Solution:

(a) The angular velocity of the disk relative to the secondary reference frame is

$$\boldsymbol{\omega}_{\text{rel}} = \omega_d\mathbf{i} = 6\mathbf{i} \text{ (rad/s).}$$

(b) The angular velocity of the reference frame is

$$\boldsymbol{\Omega} = \omega_0\mathbf{j} + \omega_b\mathbf{k} = 2\mathbf{j} + 4\mathbf{k} \text{ (rad/s),}$$

so the disk's angular velocity is

$$\boldsymbol{\omega} = \boldsymbol{\Omega} + \boldsymbol{\omega}_{\text{rel}} = 6\mathbf{i} + 2\mathbf{j} + 4\mathbf{k} \text{ (rad/s).}$$

Let O be the origin and C the center of the disk. The velocity of C is

$$\mathbf{v}_c = \mathbf{v}_0 + \boldsymbol{\Omega} \times \mathbf{r}_{C/O} = \mathbf{0} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & 4 \\ 0.3 & 0 & 0 \end{vmatrix}$$

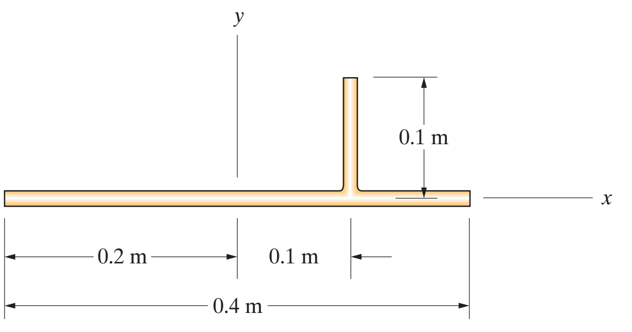
$$= 1.2\mathbf{j} - 0.6\mathbf{k} \text{ (m/s).}$$

The velocity of P is

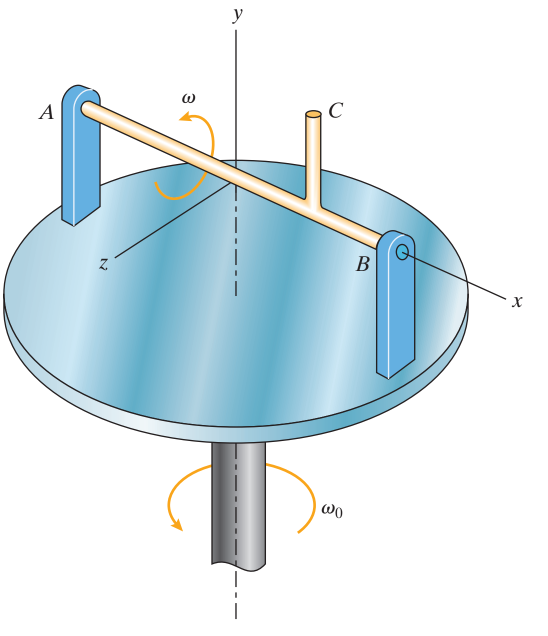
$$\mathbf{v}_P = \mathbf{v}_C + \boldsymbol{\omega} \times \mathbf{r}_{P/C} = 1.2\mathbf{j} - 0.6\mathbf{k} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 2 & 4 \\ 0 & 0.2 & 0 \end{vmatrix}$$

$$= -0.8\mathbf{i} + 1.2\mathbf{j} + 0.6\mathbf{k} \text{ (m/s).}$$

Problem 20.14 The Object in Fig. a is supported by bearings at A and B in Fig. b. The horizontal circular disk is supported by a vertical shaft that rotates with angular velocity $\omega_0 = 6$ rad/s. The horizontal bar rotates with angular velocity $\omega = 10$ rad/s. At the instant shown, what is the velocity relative to an earth-fixed reference frame of the end C of the vertical bar?



(a)



(b)

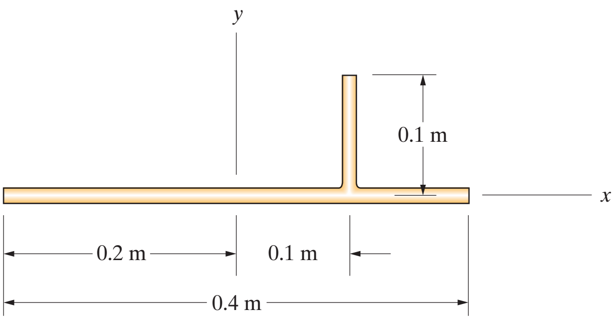
Solution:

$\mathbf{v}_c = (\mathbf{\Omega} + \boldsymbol{\omega}_{\text{rel}}) \times \mathbf{r} = (10\mathbf{i} + 6\mathbf{k}) \times (0.1\mathbf{i} + 0.1\mathbf{j})$

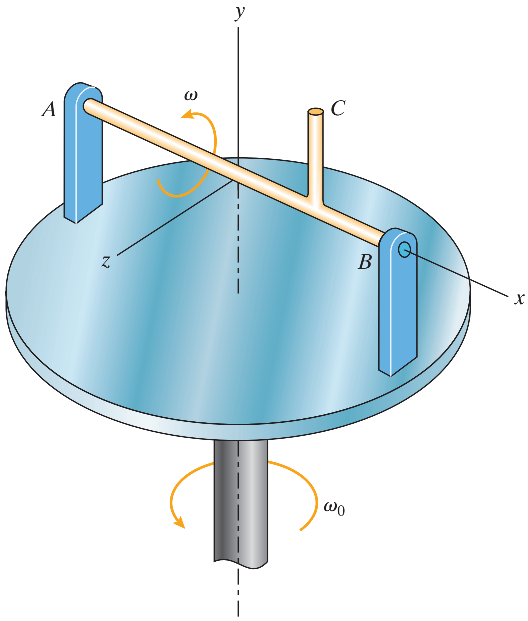
$\boxed{\mathbf{v}_c = (0.4 \text{ m/s})\mathbf{k}}$

Problem 20.15 The object in Fig. a is supported by bearings at A and B in Fig. b. The horizontal circular disk is supported by a vertical shaft that rotates with angular velocity $\omega_0 = 6$ rad/s. The horizontal bar rotates with angular velocity $\omega = 10$ rad/s.

(a) What is the angular acceleration of the object relative to an earth-fixed reference frame?
(b) At the instant shown, what is the acceleration relative to an earth-fixed reference frame of the end C of the vertical bar?



(a)



(b)

Solution:

(a) $\boldsymbol{\alpha} = \boldsymbol{\Omega} \times \boldsymbol{\omega}_{\text{rel}} = (6\mathbf{j}) \times (10\mathbf{i}) = -60\mathbf{k}$ $\boxed{\boldsymbol{\alpha} = -(60 \text{ rad/s}^2)\mathbf{k}}$

(b)

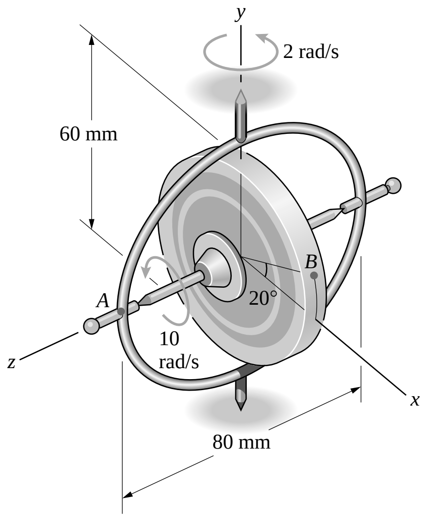
$$\mathbf{a}_c = \boldsymbol{\alpha} \times \mathbf{r} + (\boldsymbol{\Omega} + \boldsymbol{\omega}_{\text{rel}}) \times [(\Omega + \boldsymbol{\omega}_{\text{rel}}) \times \mathbf{r}]$$

$$= (-60\mathbf{k}) \times (0.1\mathbf{i} + 0.1\mathbf{j}) + (10\mathbf{i} + 6\mathbf{j}) \times [(10\mathbf{i} + 6\mathbf{j}) \times (0.1\mathbf{i} + 0.1\mathbf{j})]$$

$$= (8.4\mathbf{i} - 10\mathbf{j}) \text{ m/s}^2.$$

$$\boxed{\mathbf{a}_c = (8.4\mathbf{i} - 10\mathbf{j}) \text{ m/s}^2.}$$

Problem 20.16 Relative to a primary reference frame, the gyroscope's circular frame rotates about the vertical axis at 2 rad/s. The 60-nm diameter wheel rotates at 10 rad/s relative to the frame. Determine the velocities of points A and B relative to the primary reference frame.



Solution: Let the secondary reference frame shown be fixed with respect to the gyroscope's frame. The angular velocity of the SRF is $\mathbf{\Omega} = 2\mathbf{j}$ (rad/s). The angular velocity of the wheel relative to the SRF is $\boldsymbol{\omega}_{\text{rel}} = 10\mathbf{k}$ (rad/s), so the wheel's angular velocity is

$$\boldsymbol{\omega} = \mathbf{\Omega} + \boldsymbol{\omega}_{\text{rel}} = 2\mathbf{j} + 10\mathbf{k} \text{ (rad/s)}.$$

Let O denote the origin. The velocity of pt. A is

$$\mathbf{v}_A = \mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r}_{A/O} = 0 + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & 10 \\ 0 & 0 & 40 \end{vmatrix} = 80\mathbf{i} \text{ (mm/s)}.$$

The velocity of pt. B is

$$\mathbf{v}_B = \mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r}_{B/O}$$

$$= \mathbf{0} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & 10 \\ 30\cos 20^\circ & 30\sin 20^\circ & 0 \end{vmatrix}$$

$$= -102.6\mathbf{i} + 281.9\mathbf{j} - 56.4\mathbf{k} \text{ (mm/s)}.$$

Problem 20.17 Relative to a primary reference frame, the gyroscope's circular frame rotates about the vertical axis with a constant angular velocity of 2 rad/s. The 60-mm diameter wheel rotates with a constant angular velocity of 10 rad/s relative to the frame. Determine the accelerations of points A and B relative to the primary reference frame.

Solution: See the solution of Problem 20.16. From Eq. (20.4), the wheel's angular acceleration is

$$\boldsymbol{\alpha} = \mathbf{\Omega} \times \boldsymbol{\omega} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & 0 \\ 0 & 2 & 10 \end{vmatrix} = 20\mathbf{i} \text{ (rad/s}^2\text{)}.$$

The acceleration of pt. A is

$$\mathbf{a}_A = \mathbf{a}_0 + \boldsymbol{\alpha} \times \mathbf{r}_{A/O} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{A/O})$$

$$= 0 + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 20 & 0 & 0 \\ 0 & 0 & 40 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & 10 \\ 80 & 0 & 0 \end{vmatrix}$$

$$= -160\mathbf{k} \text{ (mm/s}^2\text{)}.$$

The acceleration of pt. B is

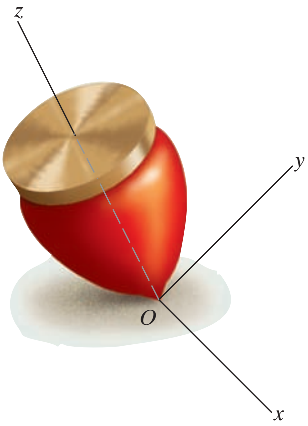
$$\mathbf{a}_B = \mathbf{a}_0 + \boldsymbol{\alpha} \times \mathbf{r}_{B/O} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{B/O})$$

$$= \mathbf{0} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 20 & 0 & 0 \\ 30\cos 20^\circ & 30\sin 20^\circ & 0 \end{vmatrix}$$

$$+ \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & 10 \\ -102.6 & 281.9 & -56.4 \end{vmatrix}$$

$$= -2932\mathbf{i} - 1026\mathbf{j} + 410\mathbf{k} \text{ (mm/s}^2\text{)}.$$

Problem 20.18 The point of the spinning top remains at a fixed point on the floor, which is the origin O of the secondary reference frame shown. The top's angular velocity relative to the secondary reference frame, $\boldsymbol{\omega}_{\text{rel}} = 50\mathbf{k}$ (rad/s), is constant. The angular velocity of the secondary reference frame relative to an earth-fixed primary reference frame is $\boldsymbol{\Omega} = 2\mathbf{j} + 5.6\mathbf{k}$ (rad/s). The components of this vector are constant. (Notice that it is expressed in terms of the secondary reference frame.) Determine the velocity relative to the earth-fixed reference frame of the point of the top with coordinates (0, 20, 30) mm.



Solution:

$$\mathbf{v} = (\boldsymbol{\Omega} + \boldsymbol{\omega}_{\text{rel}}) \times \mathbf{r} = (2\mathbf{j} + 55.6\mathbf{k}) \times (0.02\mathbf{j} + 0.03\mathbf{k})$$

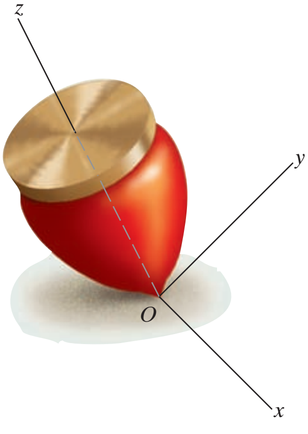
$$= (-1.05 \text{ m/s})\mathbf{i}$$

$$\boxed{\mathbf{v} = (-1.05 \text{ m/s})\mathbf{i}}$$

Problem 20.19 The point of the spinning top remains at a fixed point on the floor, which is the origin O of the secondary reference frame shown. The top's angular velocity relative to the secondary reference frame, $\boldsymbol{\omega}_{\text{rel}} = 50\mathbf{k}$ (rad/s), is constant. The angular velocity of the secondary reference frame relative to an earth-fixed primary reference frame is $\boldsymbol{\Omega} = 2\mathbf{j} + 5.6\mathbf{k}$ (rad/s). The components of this vector are constant. (Notice that it is expressed in terms of the secondary reference frame.)

(a) What is the top's angular acceleration relative to the earth-fixed reference frame?

(b) Determine the acceleration relative to the earth-fixed reference frame of the point of the top with coordinates (0, 20, 30) mm.



Solution:

(a) $\boldsymbol{\alpha} = \boldsymbol{\Omega} \times \boldsymbol{\omega}_{\text{rel}} = (2\mathbf{j} + 5.6\mathbf{k}) \times (50\mathbf{k}) = 100\mathbf{i}$ $\boxed{\boldsymbol{\alpha} = (100 \text{ rad/s}^2)\mathbf{i}}$

(b) $\mathbf{a} = \boldsymbol{\alpha} \times \mathbf{r} + (\boldsymbol{\Omega} + \boldsymbol{\omega}_{\text{rel}}) \times [(\boldsymbol{\Omega} + \omega_{\text{rel}}) \times \mathbf{r}]$

$$= (100\mathbf{i}) \times (0.02\mathbf{j} + 0.03\mathbf{k})$$

$$+ (2\mathbf{j} + 55.6\mathbf{k}) \times [(2\mathbf{j} + 55.6\mathbf{k}) \times (0.02\mathbf{j} + 0.03\mathbf{k})]$$

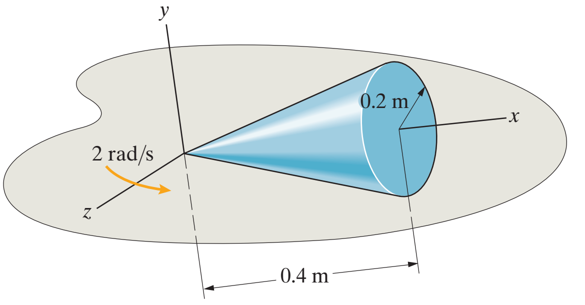
$$= (-61.5\mathbf{j} + 4.10\mathbf{k})$$

$$\boxed{\mathbf{a} = (-61.5\mathbf{j} + 4.10\mathbf{k}) \text{ m/s}^2.}$$

Problem 20.20* The cone rolls on the horizontal surface, which is fixed with respect to an earth-fixed reference frame. The x axis of the secondary reference frame remains coincident with the cone's axis, and the z axis remains horizontal. As the cone rolls, the z axis rotates in the horizontal plane with an angular velocity of 2 rad/s.

(a) What is the angular velocity vector $\mathbf{\Omega}$ of the secondary reference frame?
(b) What is the angular velocity vector $\boldsymbol{\omega}_{\text{rel}}$ of the cone relative to the secondary reference frame?

(See Example 20.3.)



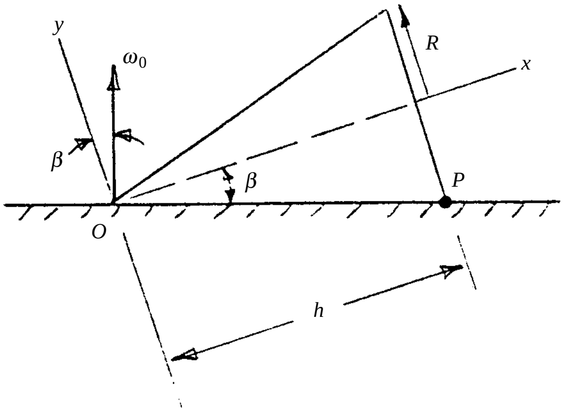
Strategy: To solve part (b), use the fact that the velocity relative to the earth-fixed reference frame of points of the cone in contact with the surface is zero.

Solution:

(a) The angle $\beta = \arctan(R/h) = \arctan(0.2/0.4) = 26.6^\circ$. The angular velocity of the secondary reference frame is

$$\mathbf{\Omega} = \omega_0 \sin\beta\,\mathbf{i} + \omega_0 \cos\beta\,\mathbf{j} = 2(\sin 26.6^\circ\,\mathbf{i} + \cos 26.6^\circ\,\mathbf{j})$$

$$= 0.894\mathbf{i} + 1.789\mathbf{j}\ \text{(rad/s)}.$$



(b) The cone's angular velocity relative to the secondary reference frame can be written $\boldsymbol{\omega}_{\text{rel}} = \omega_{\text{rel}}\mathbf{i}$, so the cone's angular velocity is

$$\boldsymbol{\omega} = \mathbf{\Omega} + \boldsymbol{\omega}_{\text{rel}}$$

$$= (0.894 + \omega_{\text{rel}})\mathbf{i} + 1.789\mathbf{j}\ \text{(rad/s)}.$$

To determine ω_{rel}, we use the fact that the point P in contact with the surface has zero velocity:

$$\mathbf{v}_P = \mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r}_{P/O} = 0 + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.894 + \omega_{\text{rel}} & 1.789 & 0 \\ 0.4 & -0.2 & 0 \end{vmatrix} = 0.$$

Solving, we obtain $\omega_{\text{rel}} = -4.47$ (rad/s), so $\boldsymbol{\omega}_{\text{rel}} = -4.47\mathbf{i}$ (rad/s).

Problem 20.21* The cone rolls on the horizontal surface, which is fixed with respect to an earth-fixed reference frame. The x axis of the secondary reference frame remains coincident with the cone's axis, and the z axis remains horizontal. As the cone rolls, the z axis rotates in the horizontal plane with an angular velocity of 2 rad/s. Determine the velocity relative to the earth-fixed reference frame of the point of the base of the cone with coordinates $x = 0.4$ m, $y = 0$, $z = 0.2$ m. (See Example 20.3.)

Solution: See the solution of Problem 20.20. The cone's angular velocity is

$$\boldsymbol{\omega} = \mathbf{\Omega} + \boldsymbol{\omega}_{\text{rel}} = (0.894\mathbf{i} + 1.789\mathbf{j}) - 4.472\mathbf{i}$$

$$= -3.578\mathbf{i} + 1.789\mathbf{j}\ \text{(rad/s)}.$$

Let A denote the pt with coordinates (0.4, 0, 0.2) m. Its velocity is

$$\mathbf{v}_A = \mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r}_{A/O} = \mathbf{0} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3.578 & 1.789 & 0 \\ 0.4 & 0 & 0.2 \end{vmatrix}$$

$$= 0.358\mathbf{i} + 0.716\mathbf{j} - 0.716\mathbf{k}\ \text{(m/s)}.$$

Problem 20.22* The cone rolls on the horizontal surface, which is fixed with respect to an earth-fixed reference frame. The x axis of the secondary reference frame remains coincident with the cone's axis, and the z axis remains horizontal. As the cone rolls, the z axis rotates in the horizontal plane with a constant angular velocity of 2 rad/s. Determine the acceleration relative to the earth-fixed reference frame of the point of the base of the cone with coordinates $x = 0.4$ m, $y = 0$, $z = 0.2$ m. (See Example 20.3.)

Solution: See the solutions of Problems 20.20 and 20.21. The cone's angular acceleration is

$$\boldsymbol{\alpha} = \boldsymbol{\Omega} \times \boldsymbol{\omega} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.894 & 1.789 & 0 \\ -3.578 & 1.789 & 0 \end{vmatrix} = 8.000\mathbf{k}\ (\text{rad/s}^2).$$

The acceleration of the point is

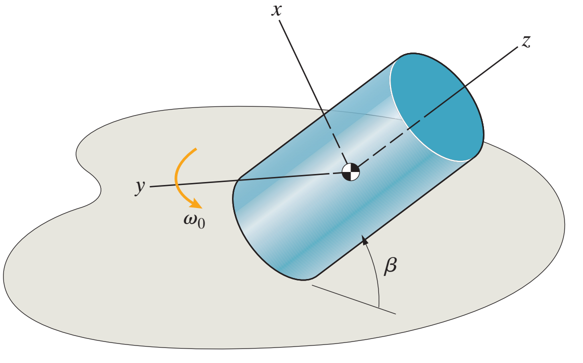
$$\mathbf{a}_A = \mathbf{a}_0 + \boldsymbol{\alpha} \times \mathbf{r}_{A/O} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{A/O})$$

$$= \mathbf{0} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 8 \\ 0.4 & 0 & 0.2 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3.578 & 1.789 & 0 \\ 0.358 & 0.716 & -0.716 \end{vmatrix}$$

$$= -1.28\mathbf{i} + 0.64\mathbf{j} - 3.20\mathbf{k}\ (\text{m/s}^2).$$

Problem 20.23* The radius and length of the cylinder are $R = 0.1$ m and $l = 0.4$ m. The horizontal surface is fixed with respect to an earth-fixed reference frame. One end of the cylinder rolls on the surface while its center, the origin of the secondary reference frame, remains stationary. The angle $\beta = 45^\circ$. The z axis of the secondary reference frame remains coincident with the cylinder's axis, and the y axis remains horizontal. As the cylinder rolls, the y axis rotates in a horizontal plane with angular velocity $\omega_0 = 2$ rad/s.

(a) What is the angular velocity vector $\boldsymbol{\Omega}$ of the secondary reference frame?
(b) What is the angular velocity vector $\boldsymbol{\omega}_{\text{rel}}$ of the cylinder relative to the secondary reference frame?



Solution:

(a) The angular velocity of the secondary reference frame is

$$\boldsymbol{\Omega} = \omega_0 \sin 45^\circ \mathbf{i} + \omega_0 \cos 45^\circ \mathbf{k}$$

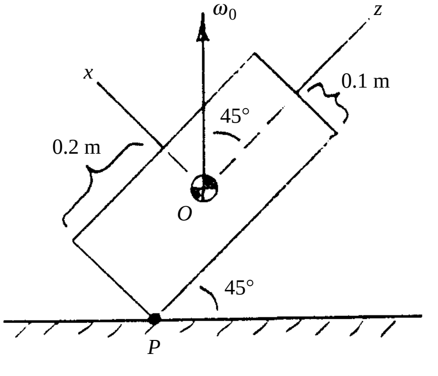
$$= (2)\sin 45^\circ \mathbf{i} + (2)\cos 45^\circ \mathbf{k}$$

$$= 1.414\mathbf{i} + 1.414\mathbf{k}\ (\text{rad/s}).$$

(b) The cylinder's angular velocity relative to the SRF can be written $\boldsymbol{\omega}_{\text{rel}} = \omega_{\text{rel}}\mathbf{k}$, so $\boldsymbol{\omega} = \boldsymbol{\Omega} + \boldsymbol{\omega}_{\text{rel}} = 1.414\mathbf{i} + (1.414 + \omega_{\text{rel}})\mathbf{k}$. We determine ω_{rel} from the condition that the velocity of pt. P is zero:

$$\mathbf{v}_P = \mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r}_{P/O} = 0 + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1.414 & 0 & 1.414 + \omega_{\text{rel}} \\ -0.1 & 0 & -0.2 \end{vmatrix} = 0.$$

Solving, we obtain $\omega_{\text{rel}} = 1.414$ rad/s, so $\boldsymbol{\omega}_{\text{rel}} = 1.414\mathbf{k}$ (rad/s).



Problem 20.24* The radius and length of the cylinder are $R = 0.1$ m and $l = 0.4$ m. The horizontal surface is fixed with respect to an earth-fixed reference frame. One end of the cylinder rolls on the surface while its center, the origin of the secondary reference frame, remains stationary. The angle $\beta = 45^\circ$. The z axis of the secondary reference frame remains coincident with the cylinder's axis, and the y axis remains horizontal. As the cylinder rolls, the y axis rotates in a horizontal plane with angular velocity $\omega_0 = 2$ rad/s. Determine the velocity relative to the earth-fixed reference frame of the point of the upper end of the cylinder with coordinates $x = 0.1$ m, $y = 0$, $z = 0.2$ m.

Solution: See the solution of Problem 20.23. The cylinder's angular velocity is

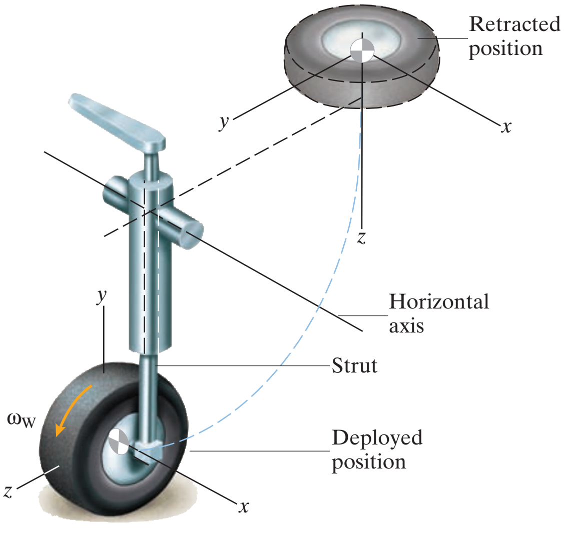
$$\boldsymbol{\omega} = \boldsymbol{\Omega} + \boldsymbol{\omega}_{\text{rel}} = (1.414\mathbf{i} + 1.414\mathbf{k}) + 1.414\mathbf{k}$$

$$= 1.414\mathbf{i} + 2.828\mathbf{k} \text{ (rad/s)}.$$

Let A devote the pt with coordinates (0.1, 0, 0.2) m. Its velocity is

$$\mathbf{v}_A = \mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r}_{A/O} = \mathbf{0} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1.414 & 0 & 2.828 \\ 0.1 & 0 & 0.2 \end{vmatrix} = \mathbf{0}.$$

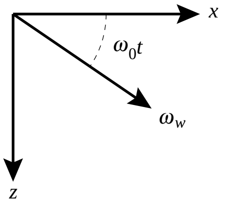
Problem 20.25* The landing gear of the P-40 airplane used in World War II retracts by rotating 90° about the horizontal axis toward the rear of the airplane. As the wheel retracts, a linkage rotates the strut supporting the wheel 90° about the strut's longitudinal axis so that the wheel is horizontal in the retracted position. (Viewed from the horizontal axis toward the wheel, the strut rotates in the clockwise direction.) The x axis of the coordinate system shown remains parallel to the horizontal axis and the y axis remains parallel to the strut as the wheel retracts. Let ω_W be the magnitude of the wheel's angular velocity when the airplane lifts off, and assume that it remains constant. Let ω_0 be the magnitude of the constant angular velocity of the strut about the horizontal axis as the landing gear is retracted. The magnitude of the angular velocity of the strut about its longitudinal axis also equals ω_0. The landing gear begins retracting at $t = 0$. Determine the wheel's angular velocity relative to the airplane as a function of time.



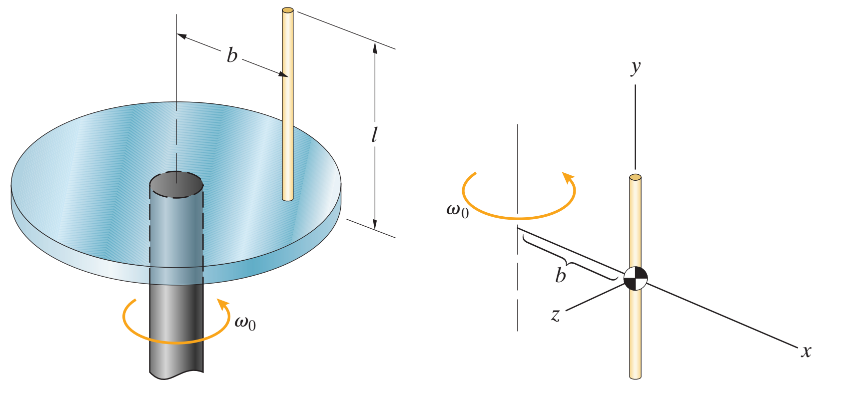
Solution: The angular velocity is given by

$$\boldsymbol{\omega} = \omega_0\mathbf{i} + \omega_0\mathbf{j} + \omega_W[(\cos\omega_0 t)\mathbf{i} + (\sin\omega_0 t)\mathbf{k}]$$

$$\boxed{\boldsymbol{\omega} = (\omega_0 + \omega_W\cos\omega_0 t)\mathbf{i} + \omega_0\mathbf{j} + (\omega_0 + \omega_W\cos\omega_0 t)\mathbf{k}}$$



Problem 20.26 In Active Example 20.4, suppose that the shaft supporting the disk is initially stationary, and at $t = 0$ it is subjected to a constant angular acceleration α_0 in the counterclockwise direction viewed from above the disk. Determine the force and couple exerted on the bar by the disk at that instant.



Solution: We have Use Newton's Second Law for rigid bodies to find the force $\mathbf{F}$ at the base of the rod

$$\mathbf{F} = mg\mathbf{j} = m\mathbf{a} = m(-b\alpha_0\mathbf{k}) \quad \Rightarrow \quad \boxed{\mathbf{F} = mg\mathbf{j} - mb\alpha_0\mathbf{k}}$$

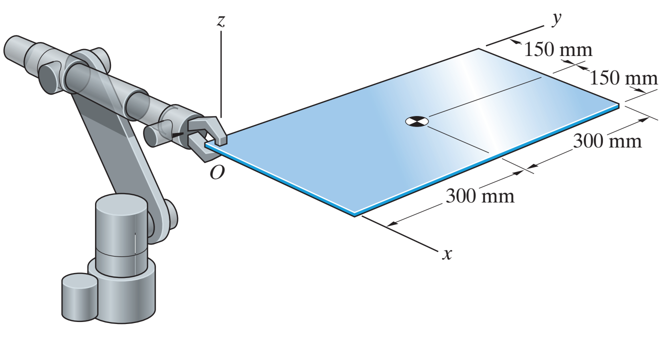
Now use Euler's equations to find the moment $\mathbf{C}$ exerted by the disk on the base of the rod. Note that the angular velocity is zero, and the only nonzero inertias are $I_{xx} = I_{zz} = ml^2/12$.

$$\mathbf{C} + \left(-\frac{1}{2}\mathbf{j}\right) \times \mathbf{F} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{Bmatrix} 0 \\ \alpha_0 \\ 0 \end{Bmatrix} = 0$$

$$\mathbf{C} = \frac{1}{2}\mathbf{j}(mg\mathbf{j} - mb\alpha_0\mathbf{k})$$

$$\boxed{\mathbf{C} = -\frac{1}{2}mbl\alpha_0\mathbf{i}}$$

Problem 20.27 In Example 20.5, suppose that the horizontal plate is initially stationary, and at $t = 0$ the robotic manipulator exerts a couple $\mathbf{C}$ on the plate at the fixed point O such that the plate's angular acceleration at this instant is $\boldsymbol{\alpha} = 150\mathbf{i} + 320\mathbf{j} + 25\mathbf{k}$ (rad/s^2). Determine $\mathbf{C}$.



Solution: The mass of the plate is 4 kg. Point O is a fixed point. The nonzero inertias are

$$I_{xx} = \frac{1}{3}(4\text{ kg})(0.6\text{ m})^2 = 0.48\text{ kg-m}^2,$$

$$I_{yy} = \frac{1}{3}(4\text{ kg})(0.3\text{ m})^2 = 0.12\text{ kg-m}^2$$

$$I_{zz} = I_{xx} + I_{yy} = 0.60\text{ kg-m}^2,$$

$$I_{xy} = (4\text{ kg})(0.15\text{ m})(0.3\text{ m}) = 0.18\text{ kg-m}^2.$$

Euler's equations can be written as (note that the angular velocity is zero)

$$\mathbf{C} + (0.15\mathbf{i} + 0.3\mathbf{j}) \times (-[4][9.81]\mathbf{k}) = [I]\boldsymbol{\alpha}$$

$$\mathbf{C} = -(0.15\mathbf{i} + 0.3\mathbf{j}) \times (-[4][9.81]\mathbf{k}) + [I]\boldsymbol{\alpha}$$

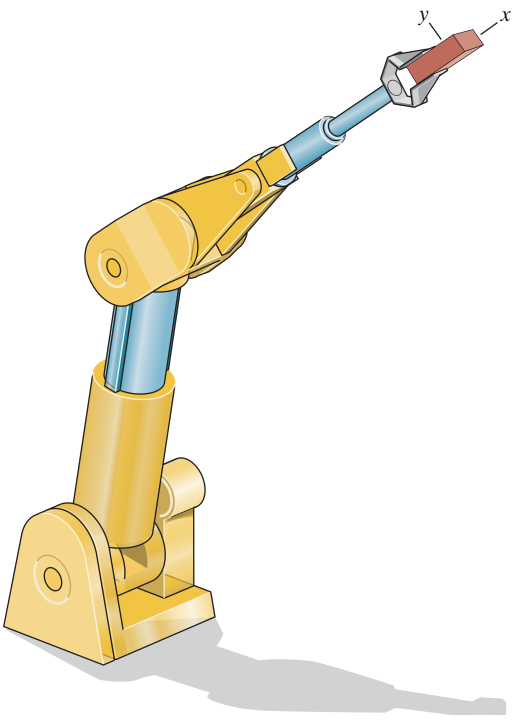
In Matrix Form this is

$$\begin{Bmatrix} C_x \\ C_y \\ C_z \end{Bmatrix} = \begin{Bmatrix} 11.8 \\ -5.89 \\ 0 \end{Bmatrix} + \begin{bmatrix} 0.48 & -0.18 & 0 \\ -0.18 & 0.12 & 0 \\ 0 & 0 & 0.60 \end{bmatrix} \begin{Bmatrix} 150 \\ 320 \\ 25 \end{Bmatrix}$$

Solving we find $C_x = 26.2, \quad C_y = 5.51, \quad C_z = 15.$

Thus $\boxed{\mathbf{C} = (26.2\mathbf{i} + 5.51\mathbf{j} + 15\mathbf{k})\text{ N-m.}}$

Problem 20.28 A robotic manipulator moves a casting. The inertia matrix of the casting in terms of a body-fixed coordinate system with its origin at the center of mass is shown. At the present instant, the angular velocity and angular acceleration of the casting are $w = 1.2\mathbf{i} + 0.8\mathbf{j} - 0.4\mathbf{k}$ (rad/s) and $\boldsymbol{\alpha} = 0.26\mathbf{i} - 0.07\mathbf{j} + 0.13\mathbf{k}$ (rad/s^2). What moment is exerted about the center of mass of the casting by the manipulator?



$$\begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} = \begin{bmatrix} 0.05 & -0.03 & 0 \\ -0.03 & 0.08 & 0 \\ 0 & 0 & 0.04 \end{bmatrix} \text{ kg-m}^2.$$

Solution:

$$\begin{Bmatrix} M_x \\ M_y \\ M_z \end{Bmatrix} = \begin{bmatrix} 0.05 & -0.03 & 0 \\ -0.03 & 0.08 & 0 \\ 0 & 0 & 0.04 \end{bmatrix} \begin{Bmatrix} 0.26 \\ -0.07 \\ 0.13 \end{Bmatrix} \text{ N-m}$$

$$+ \begin{bmatrix} 0 & 0.4 & 0.8 \\ -0.4 & 0 & -1.2 \\ -0.8 & 1.2 & 0 \end{bmatrix} \begin{bmatrix} 0.05 & -0.03 & 0 \\ -0.03 & 0.08 & 0 \\ 0 & 0 & 0.04 \end{bmatrix}$$

$$\times \begin{Bmatrix} 1.2 \\ 0.8 \\ -0.4 \end{Bmatrix} \text{ N-m}$$

$$\boxed{\mathbf{M} = (M_x\mathbf{i} + M_y\mathbf{j} + M_z\mathbf{k}) = (0.0135\mathbf{i} - 0.0086\mathbf{j} + 0.01\mathbf{k}) \text{ N-m}}$$

Problem 20.29 A robotic manipulator holds a casting. The inertia matrix of the casting in terms of a body-fixed coordinate system with its origin at the center of mass is shown. At the present instant, the casting is stationary. If the manipulator exerts a moment $\Sigma\mathbf{M} = 0.042\mathbf{i} + 0.036\mathbf{j} + 0.066\mathbf{k}$ (N-m) about the center of mass, what is the angular acceleration of the casting at that instant?

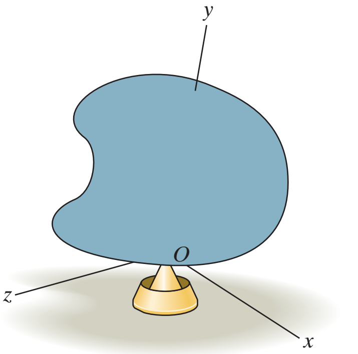
Solution:

$$\begin{Bmatrix} 0.042 \\ 0.036 \\ 0.066 \end{Bmatrix} \text{ N-m} = \begin{bmatrix} 0.05 & -0.03 & 0 \\ -0.03 & 0.08 & 0 \\ 0 & 0 & 0.04 \end{bmatrix} \text{ kg-m}^2 \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{Bmatrix}$$

Solving we find

$$\boxed{\boldsymbol{\alpha} = (\alpha_x\mathbf{i} + \alpha_y\mathbf{j} + \alpha_z\mathbf{k}) = (1.43\mathbf{i} + 0.987\mathbf{j} + 1.65\mathbf{k}) \text{ rad/s}^2}$$

Problem 20.30 The rigid body rotates about the fixed point O. Its inertia matrix in terms of the body-fixed coordinate system is shown. At the present instant, the rigid body's angular velocity is $\boldsymbol{\omega} = 6\mathbf{i} + 6\mathbf{j} - 4\mathbf{k}$ (rad/s) and its angular acceleration is zero. What total moment about O is being exerted on the rigid body?



$$\begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} = \begin{bmatrix} 4 & -2 & 0 \\ -2 & 3 & 1 \\ 0 & 1 & 5 \end{bmatrix} \text{ kg-m}^2.$$

Solution:

$$\begin{Bmatrix} M_x \\ M_y \\ M_z \end{Bmatrix} = \begin{bmatrix} 0 & 4 & 6 \\ -4 & 0 & -6 \\ -6 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & -2 & 0 \\ -2 & 3 & 1 \\ 0 & 1 & 5 \end{bmatrix} \begin{Bmatrix} 6 \\ 6 \\ -4 \end{Bmatrix} \text{ N-m}$$

$$\boxed{\mathbf{M} = (M_x\mathbf{i} + M_y\mathbf{j} + M_z\mathbf{k}) = (-76\mathbf{i} + 36\mathbf{j} - 60\mathbf{k}) \text{ N-m}}$$

Problem 20.31 The rigid body rotates about the fixed point O. Its inertia matrix in terms of the body-fixed coordinate system is shown. At the present instant, the rigid body's angular velocity is $\boldsymbol{\omega} = 6\mathbf{i} + 6\mathbf{j} - 4\mathbf{k}$ (rad/s). The total moment about O due to the forces and couples acting on the rigid body is zero. What is its angular acceleration?

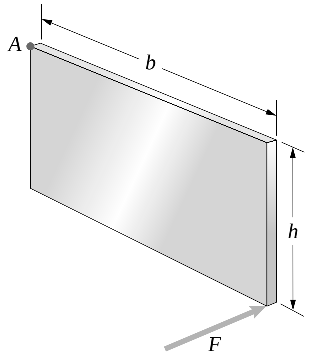
Solution:

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} 4 & -2 & 0 \\ -2 & 3 & 1 \\ 0 & 1 & 5 \end{bmatrix} \text{ kg-m}^2 \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{Bmatrix}$$

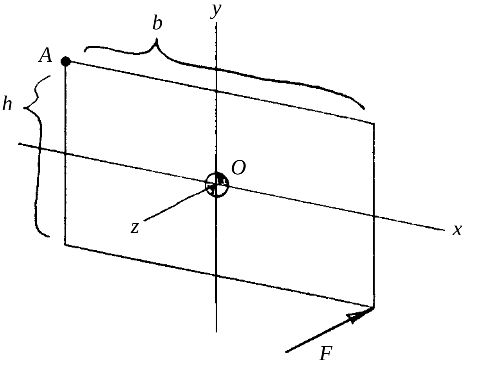
$$+ \begin{bmatrix} 0 & 4 & 6 \\ -4 & 0 & -6 \\ -6 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & -2 & 0 \\ -2 & 3 & 1 \\ 0 & 1 & 5 \end{bmatrix} \begin{Bmatrix} 6 \\ 6 \\ -4 \end{Bmatrix} \text{ N-m}$$

Solving we find $\boxed{\boldsymbol{\alpha} = (16.2\mathbf{i} - 5.56\mathbf{j} + 13.1\mathbf{k}) \text{ rad/s}^2}$

Problem 20.32 The dimensions of the 20-kg thin plate are $h = 0.4$ m and $b = 0.6$ m. The plate is stationary relative to an inertial reference frame when the force $F = 10$ N is applied in the direction perpendicular to the plate. No other forces or couples act on the plate. At the instant F is applied, what is the magnitude of the acceleration of point A relative to the inertial reference frame?



Solution: From Appendix C, the inertia matrix in terms of the body-fixed reference frame shown is



$$[I] = \begin{bmatrix} \frac{1}{12}mh^2 & 0 & 0 \\ 0 & \frac{1}{12}mb^2 & 0 \\ 0 & 0 & \frac{1}{12}m(b^2+h^2) \end{bmatrix}$$

$$= \begin{bmatrix} 0.267 & 0 & 0 \\ 0 & 0.6 & 0 \\ 0 & 0 & 0.867 \end{bmatrix} \text{kg-m}^2.$$

The moment of the force about the center of mass is

$$\mathbf{M} = \left(\frac{b}{2}\mathbf{i} - \frac{h}{2}\mathbf{j}\right) \times (-F\mathbf{k}) = 2\mathbf{i} + 3\mathbf{j} \text{ (N-m)}.$$

From Eq. (20.19) with $\boldsymbol{\omega} = \boldsymbol{\Omega} = \mathbf{0}$,

$$\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.267 & 0 & 0 \\ 0 & 0.6 & 0 \\ 0 & 0 & 0.367 \end{bmatrix} \begin{bmatrix} d\omega_x/dt \\ d\omega_y/dt \\ d\omega_z/dt \end{bmatrix}$$

Solving, we obtain $\boldsymbol{\alpha} = 7.5\mathbf{i} + 5\mathbf{j}$ (m/s^2). From Newton's second law, $\sum \mathbf{F} = -F\mathbf{k} = m\mathbf{a}_0$, the acceleration of the center of mass is $\mathbf{a}_0 = -\dfrac{F}{m}\mathbf{k} = -0.5\mathbf{k}$ (m/s^2). The acceleration of pt A is

$$\mathbf{a}_A = \mathbf{a}_0 + \boldsymbol{\alpha} \times \mathbf{r}_{A/O} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{A/O})$$

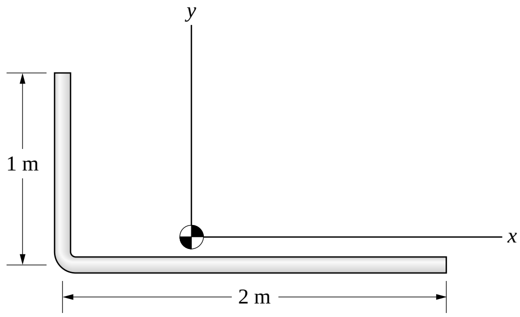
$$= -0.5\mathbf{k} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7.5 & 5 & 0 \\ -0.3 & 0.2 & 0 \end{vmatrix} + \mathbf{0}$$

$$= 2.5\mathbf{k} \text{ (m/s}^2).$$

We see that $|\mathbf{a}_A| = 2.5$ m/s^2.

Problem 20.33 In terms of the coordinate system shown, the inertia matrix of the 6-kg slender bar is

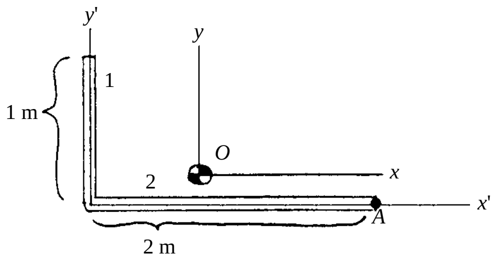
$$\begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} = \begin{bmatrix} 0.500 & 0.667 & 0 \\ 0.667 & 2.667 & 0 \\ 0 & 0 & 3.167 \end{bmatrix} \text{ kg-m}^2.$$



The bar is stationary relative to an inertial reference frame when the force $\mathbf{F} = 12\mathbf{k}$ (N) is applied at the right end of the bar. No other forces or couples act on the bar. Determine

(a) the bar's angular acceleration relative to the inertial reference frame and
(b) the acceleration of the right end of the bar relative to the inertial reference frame at the instant the force is applied.

Solution:



(a) In terms of the primed reference frame shown, the coordinates of the center of mass are

$$\mathbf{x}' = \frac{\mathbf{x}'_1 m_1 + \mathbf{x}'_2 m_2}{m_1 + m_2} = \frac{(0)\frac{1}{3}(6) + (1)\frac{2}{3}(6)}{\frac{1}{3}(6) + \frac{2}{3}(6)}$$

$$= 0.667 \text{ m},$$

$$\mathbf{y}' = \frac{\mathbf{y}'_1 m_1 + \mathbf{y}'_2 m_2}{m_1 + m_2} = \frac{(0.5)\frac{1}{3}(6) + (0)\frac{2}{3}(6)}{\frac{1}{3}(6) + \frac{2}{3}(6)}$$

$$= 0.167 \text{ m}.$$

The moment of $\mathbf{F}$ about the center of mass is

$$\mathbf{M} = (1.333\mathbf{i} - 0.167\mathbf{j}) \times 12\mathbf{k}$$

$$= -2\mathbf{i} - 16\mathbf{j} \text{ (N-m)}.$$

From Eq. (20.19) with $\boldsymbol{\omega} = \boldsymbol{\Omega} = \mathbf{0}$,

$$\begin{bmatrix} -2 \\ -16 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.667 & 0 \\ 0.667 & 2.667 & 0 \\ 0 & 0 & 3.167 \end{bmatrix} \begin{bmatrix} d\omega_x/dt \\ d\omega_y/dt \\ d\omega_z/dt \end{bmatrix}.$$

Solving, we obtain $\boldsymbol{\alpha} = 6.01\mathbf{i} - 7.50\mathbf{j}$ (rad/s^2).

(b) From Newton's second law, $\sum \mathbf{F} = 12\mathbf{k} = (6)\mathbf{a}_0$, the acceleration of the center of mass is $\mathbf{a}_0 = 2\mathbf{k}$ (m/s^2). The acceleration of pt A is

$$\mathbf{a}_A = \mathbf{a}_0 + \boldsymbol{\alpha} \times \mathbf{r}_{A/O} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{A/O})$$

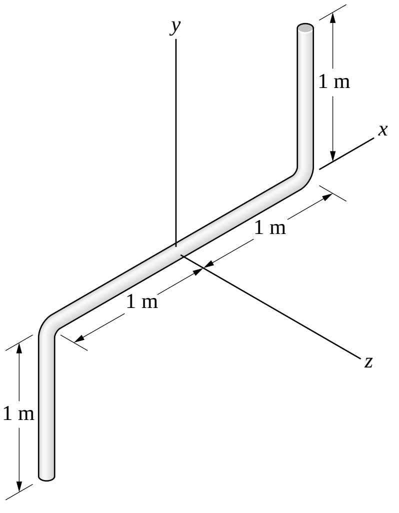
$$= 2\mathbf{k} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6.01 & -7.50 & 0 \\ 1.333 & -0.167 & 0 \end{vmatrix} + \mathbf{0}$$

$$= 11.0\mathbf{k} \text{ (m/s}^2).$$

Problem 20.34 In terms of the coordinate system shown, the inertia matrix of the 12-kg slender bar is

$$\begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} = \begin{bmatrix} 2 & -3 & 0 \\ -3 & 8 & 0 \\ 0 & 0 & 10 \end{bmatrix} \text{ kg-m}^2.$$

The bar is stationary relative to an inertial reference frame when a force $\mathbf{F} = 20\mathbf{i} + 40\mathbf{k}$ (N) is applied at the point $x = 1$ m, $y = 1$ m. No other forces or couples act on the bar. Determine (a) the bar's angular acceleration and (b) the acceleration of the point $x = -1$ m, $y = -1$ m, relative to the inertial reference frame at the instant the force is applied.



Solution:

(a) The moment of the force about the center of mass is

$$\mathbf{M} = (\mathbf{i} + \mathbf{j}) \times (20\mathbf{i} + 40\mathbf{k})$$

$$= 40\mathbf{i} - 40\mathbf{j} - 20\mathbf{k} \text{ (N-m)}.$$

From Eq. (20.19) with $\boldsymbol{\omega} = \boldsymbol{\Omega} = \mathbf{0}$,

$$\begin{bmatrix} 40 \\ -40 \\ -20 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 0 \\ -3 & 8 & 0 \\ 0 & 0 & 10 \end{bmatrix} \begin{bmatrix} d\omega_x/dt \\ d\omega_y/dt \\ d\omega_z/dt \end{bmatrix}.$$

Solving, we obtain

$$\boldsymbol{\alpha} = 28.57\mathbf{i} + 5.71\mathbf{j} - 2\mathbf{k} \text{ (rad/s}^2).$$

From Newton's second law,

$$\sum \mathbf{F} = 20\mathbf{i} + 40\mathbf{k} = (12)\mathbf{a}_0,$$

the acceleration of the center of mass is

$$\mathbf{a}_0 = 1.67\mathbf{i} + 3.33\mathbf{k} \text{ (N/s}^2).$$

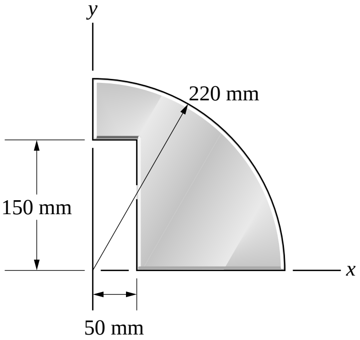
The acceleration of the pt with coordinates $(-1, -1, 0)$ is

$$\mathbf{a}_A = \mathbf{a}_0 + \boldsymbol{\alpha} \times \mathbf{r}_{A/O} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times r_{A/O})$$

$$= 1.67\mathbf{i} + 3.33\mathbf{k} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 28.57 & 5.71 & -2 \\ -1 & -1 & 0 \end{vmatrix} + \mathbf{0}$$

$$= -0.333\mathbf{i} + 2.000\mathbf{j} - 19.524\mathbf{k} \text{ (m/s}^2)$$

Problem 20.35 The inertia matrix of the 2.4-kg plate in terms of the given coordinate system is shown. The angular velocity of the plate is $\boldsymbol{\omega} = 6.4\mathbf{i} + 8.2\mathbf{j} + 14\mathbf{k}$ (rad/s), and its angular acceleration is $\boldsymbol{\alpha} = 60\mathbf{i} + 40\mathbf{j} - 120\mathbf{k}$ (rad/s^2). What are the components of the total moment exerted on the plate about its center of mass?



Solution: In the solution of Problem 20.87, the location of the center of mass, $\mathbf{x} = 0.1102$ (m), $\mathbf{y} = 0.0979$ (m) and the moments of inertia in terms of a parallel coordinate system with its origin at the center of mass are determined:

$$I_{x'x'} = 0.00876 \text{ (kg-m}^2\text{)}, \; I_{y'y'} = I_{yy} = 0.00655 \text{ (kg-m}^2\text{)},$$

$$I_{z'z'} = 0.01531 \text{ (kg-m}^2\text{)}, \; I_{x'y'} = -0.00396 \text{ (kg-m}^2\text{)}, \; I_{y'z'} = I_{z'x'} = 0.$$

The components of the total moment are given by Equation (20.19) with $\boldsymbol{\Omega} = \boldsymbol{\omega}$:

$$\begin{bmatrix} \sum M_x \\ \sum M_y \\ \sum M_z \end{bmatrix} = \begin{bmatrix} 0.00876 & 0.00396 & 0 \\ 0.00396 & 0.00655 & 0 \\ 0 & 0 & 0.01531 \end{bmatrix} \begin{bmatrix} 60 \\ 40 \\ -120 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & -14.0 & 8.2 \\ 14.0 & 0 & -6.4 \\ -8.2 & 6.4 & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} 0.00876 & 0.00396 & 0 \\ 0.00396 & 0.00655 & 0 \\ 0 & 0 & 0.01531 \end{bmatrix} \begin{bmatrix} 6.4 \\ 8.2 \\ 14.0 \end{bmatrix}$$

$$= \begin{bmatrix} 1.335 \\ 0.367 \\ -2.057 \end{bmatrix} \text{ (N-m).}$$

Problem 20.36 The inertia matrix of the 2.4-kg plate in terms of the given coordinate system is shown. At $t = 0$, the plate is stationary and is subjected to a force $\mathbf{F} = -10\mathbf{k}$ (N) at the point with coordinates (220,0,0) mm. No other forces or couples act on the plate. Determine (a) the acceleration of the plate's center of mass and (b) the plate's angular acceleration at the instant the force is applied.

Solution:

(a) From Newton's second law, $\sum \mathbf{F} = m\mathbf{a}$: $-10\mathbf{k} = 2.4\mathbf{a}$, and the acceleration of the center of mass is $\mathbf{a} = -4.17\mathbf{k}$ (m/s^2).

(b) From the solution of Problem 20.87, the center of mass is at $\mathbf{x} = 0.1102$ (m), $\mathbf{y} = 0.0979$ (m). Therefore, the moment of the force about the center of mass is

$$\sum \mathbf{M} = [(0.22 - 0.1102)\mathbf{i} - 0.0979\mathbf{j}] \times (-10\mathbf{k})$$

$$= 0.979\mathbf{i} + 1.098\mathbf{j} \text{ (N-m).}$$

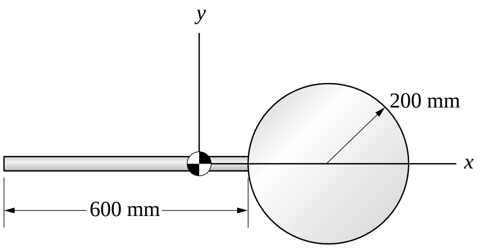
Equation (20.19) is

$$\begin{bmatrix} 0.979 \\ 1.098 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.00876 & 0.00396 & 0 \\ 0.00396 & 0.00655 & 0 \\ 0 & 0 & 0.01531 \end{bmatrix} \begin{bmatrix} d\omega_x/dt \\ d\omega_y/dt \\ d\omega_z/dt \end{bmatrix}.$$

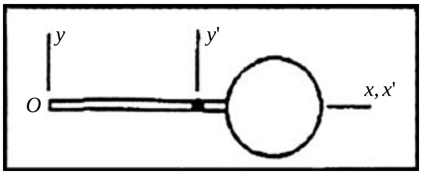
Solving these equations, we obtain

$$\boldsymbol{\alpha} = d\boldsymbol{\omega}/dt = 49.5\mathbf{i} + 137.7\mathbf{j} \text{ (rad/s}^2\text{).}$$

Problem 20.37 A 3-kg slender bar is rigidly attached to a 2-kg thin circular disk. In terms of the body-fixed coordinate system shown, the angular velocity of the composite object is $\boldsymbol{\omega} = 100\mathbf{i} - 4\mathbf{j} + 6\mathbf{k}$ (rad/s) and its angular acceleration is zero. What are the components of the total moment exerted on the object about its center of mass?



Solution: Choose an x, y, z coordinate system with the origin at O and the x axis parallel to the slender rod, as shown. From the solution to Problem 20.92, the coordinates of the center of mass in the x, y, z system are (0.5, 0, 0), and the inertia matrix about a parallel coordinate system with origin at the center of mass is:



$$[I]_G = \begin{bmatrix} 0.02 & 0 & 0 \\ 0 & 0.41 & 0 \\ 0 & 0 & 0.43 \end{bmatrix} \text{ kg-m}^2.$$

Since the coordinate system is body fixed, $\boldsymbol{\Omega} = \boldsymbol{\omega}$, and Eq. (20.19) reduces to

$$\begin{bmatrix} \sum M_{Ox} \\ \sum M_{Oy} \\ \sum M_{Oz} \end{bmatrix} = \begin{bmatrix} 0 & -6 & -4 \\ 6 & 0 & -100 \\ 4 & 100 & 0 \end{bmatrix} \begin{bmatrix} 0.02 & 0 & 0 \\ 0 & 0.41 & 0 \\ 0 & 0 & 0.43 \end{bmatrix} \begin{bmatrix} 100 \\ -4 \\ 6 \end{bmatrix},$$

$$\begin{bmatrix} \sum M_{Ox} \\ \sum M_{Oy} \\ \sum M_{Oz} \end{bmatrix} = \begin{bmatrix} 0 & -2.46 & -1.72 \\ 0.12 & 0 & -43 \\ 0.08 & 41 & 0 \end{bmatrix} \begin{bmatrix} 100 \\ -4 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} -0.48 \\ -246 \\ -156 \end{bmatrix} \text{ N-m},$$

$$\mathbf{M}_0 = -0.48\mathbf{i} - 246\mathbf{j} - 156\mathbf{k} \text{ N-m}.$$

Problem 20.38 A 3-kg slender bar is rigidly attached to a 2-kg thin circular disk. At $t = 0$, the composite object is stationary and is subjected to the moment $\Sigma\mathbf{M} = -10\mathbf{i} + 10\mathbf{j}$ (N-m) about its center of mass. No other forces or couples act on the object. Determine the object's angular acceleration at $t = 0$.

Solution: From the solution to Problem 20.92, the inertia matrix in terms of the parallel coordinate system with origin at the center of mass is

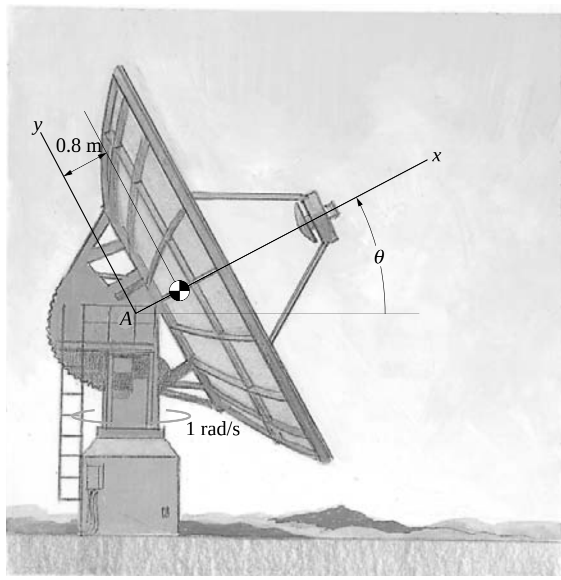
$$[I]_G = \begin{bmatrix} 0.02 & 0 & 0 \\ 0 & 0.41 & 0 \\ 0 & 0 & 0.43 \end{bmatrix} \text{ kg-m}^2.$$

Since the coordinates are body-fixed and the object is stationary at $t = 0$, $\boldsymbol{\Omega} = \boldsymbol{\omega} = 0$, and Eq. (20.19) reduces to:

$$\begin{bmatrix} -10 \\ 10 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.02 & 0 & 0 \\ 0 & 0.41 & 0 \\ 0 & 0 & 0.43 \end{bmatrix} \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{bmatrix} = \begin{bmatrix} 0.02\alpha_x \\ 0.41\alpha_y \\ 0.43\alpha_z \end{bmatrix}.$$

Solve: $\boldsymbol{\alpha} = -500\mathbf{i} + 24.4\mathbf{j}$ (rad/s^2).

Problem 20.39 The vertical shaft supporting the dish antenna is rotating with a constant angular velocity of 1 rad/s. The angle $\theta = 30^\circ$, $d\theta/dt = 20^\circ/s^2$, and $d^2\theta/dt^2 = -40^\circ/s^2$. The mass of the antenna is 280 kg, and its moments and products of inertia, in kg-m^2, are $I_{xx} = 140$, $I_{yy} = I_{zz} = 220$, $I_{xy} = I_{yz} = I_{zx} = 0$. Determine the couple exerted on the antenna by its support at A at the instant shown.



Solution: The reactions at the support arise from (a) the Euler moments about the point A, and (b) the weight unbalance due to the offset center of mass. *The Euler Equations*: Express the reactions in the x, y, z system. The angular velocity in the x, y, z system

$$\boldsymbol{\omega} = \mathbf{i}\sin\theta + \mathbf{j}\cos\theta + (d\theta/dt)\mathbf{k}$$

$$= 0.5\mathbf{i} + 0.866\mathbf{j} + 0.3491\mathbf{k} \text{ (rad/s)}.$$

The angular acceleration is

$$\boldsymbol{\alpha} = \frac{d\boldsymbol{\omega}}{dt} = (\mathbf{i}\cos\theta - \mathbf{j}\sin\theta)\frac{d\theta}{\mathrm{dt}} + \frac{d^2\theta}{\mathrm{dt}^2}\mathbf{k}$$

$$= 0.3023\mathbf{i} - 0.1745\mathbf{j} - 0.6981\mathbf{k} \text{ (rad/s}^2).$$

Since the coordinates are body fixed $\boldsymbol{\Omega} = \boldsymbol{\omega}$, and Eq. (20.13) is

$$\begin{bmatrix} \sum M_{Ox} \\ \sum M_{Oy} \\ \sum M_{Oz} \end{bmatrix} = \begin{bmatrix} 140 & 0 & 0 \\ 0 & 220 & 0 \\ 0 & 0 & 220 \end{bmatrix} \begin{bmatrix} 0.302 \\ -0.1745 \\ -0.6981 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & -0.3491 & 0.866 \\ 0.3491 & 0 & -0.5 \\ -0.866 & 0.5 & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} 140 & 0 & 0 \\ 0 & 220 & 0 \\ 0 & 0 & 220 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.866 \\ 0.3491 \end{bmatrix}.$$

$$\begin{bmatrix} \sum M_{Ox} \\ \sum M_{Oy} \\ \sum M_{Oz} \end{bmatrix} = \begin{bmatrix} 42.32 \\ -38.39 \\ -153.6 \end{bmatrix} + \begin{bmatrix} 0 & -76.79 & 190.52 \\ 48.87 & 0 & -110 \\ -121.2 & 110 & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} 0.5 \\ 0.866 \\ 0.3491 \end{bmatrix}$$

$$= \begin{bmatrix} 42.32 \\ -38.39 \\ -153.6 \end{bmatrix} + \begin{bmatrix} 0 \\ -13.96 \\ 34.64 \end{bmatrix} = \begin{bmatrix} 42.32 \\ -52.36 \\ -118.9 \end{bmatrix},$$

$$\mathbf{M}_0 = 42.32\mathbf{i} - 52.36\mathbf{j} - 118.9\mathbf{k} \text{ N-m}$$

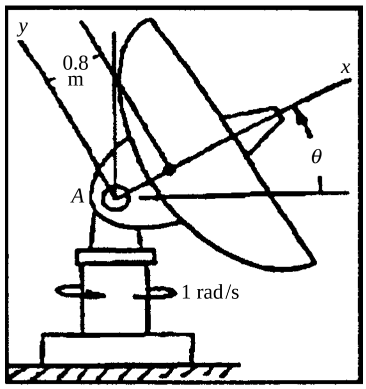
The unbalance exerted by the offset center of mass: The weight of the antenna acting through the center of mass in the x, y, z system is

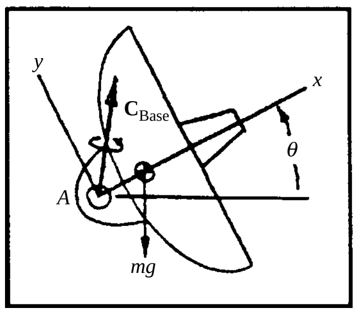
$$\mathbf{W} = \mathrm{mg}(-\mathbf{i}\sin\theta - \mathbf{j}\cos\theta) = -1373.4\mathbf{i} - 2378.8\mathbf{j} \text{ (N)}.$$

The vector distance to the center of mass is in the x, y, z system is $\mathbf{r}_{G/O} = 0.8\mathbf{i}$ (m). The moment exerted by the weight is

$$\mathbf{M}_W = \mathbf{r}_{G/O} \times \mathbf{W} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.8 & 0 & 0 \\ -1373.4 & -2378.8 & 0 \end{bmatrix}$$

$$= 1903.0\mathbf{k}.$$





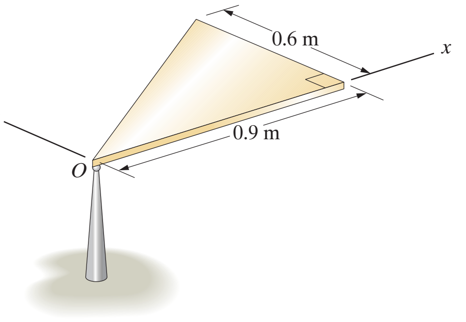
The couple exerted by the base:

$$\begin{bmatrix} C_x \\ C_y \\ C_z \end{bmatrix} = \begin{bmatrix} \sum M_x \\ \sum M_y \\ \sum M_z \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ -1903.0 \end{bmatrix}$$

$$= \begin{bmatrix} 42.32 \\ -52.36 \\ -118.9 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ +1903.0 \end{bmatrix} = \begin{bmatrix} 42.32 \\ -52.36 \\ 1784.1 \end{bmatrix} \text{ (N-m)},$$

$$\mathbf{C}_{\text{Base}} = 42.35\mathbf{i} - 52.36\mathbf{j} + 1784.1\mathbf{k} \text{ (N-m)}$$

Problem 20.40 The 5-kg triangular plate is connected to a ball-and-socket support at O. If the plate is released from rest in the horizontal position, what are the components of its angular acceleration at that instant?



Solution: From the appendix to Chapter 20 on moments of inertia, the inertia matrix in terms of the reference frame shown is

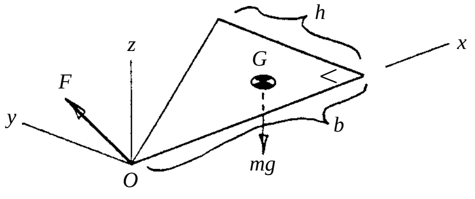
$$[I] = \begin{bmatrix} \frac{m}{A}\left(\frac{1}{12}bh^3\right) & -\frac{m}{A}\left(\frac{1}{8}b^2h^2\right) & 0 \\ -\frac{m}{A}\left(\frac{1}{8}b^2h^2\right) & \frac{m}{A}\left(\frac{1}{4}hb^3\right) & 0 \\ 0 & 0 & \frac{m}{A}\left(\frac{1}{12}bh^3 + \frac{1}{4}hb^3\right) \end{bmatrix}$$

$$= \begin{bmatrix} 0.300 & -0.675 & 0 \\ -0.675 & 2.025 & 0 \\ 0 & 0 & 2.325 \end{bmatrix} \text{kg-m}^2.$$

The moment exerted by the weight about the fixed pt. 0 is

$$\sum \mathbf{m}_0 = \left(\frac{2}{3}b\mathbf{i} + \frac{1}{3}h\mathbf{j}\right) \times (-mg\mathbf{k})$$

$$= -9.81\mathbf{i} + 29.43\mathbf{j} \text{ (N-m).}$$



From Eq. (20.13) with $\boldsymbol{\omega} = \boldsymbol{\Omega} = \mathbf{0}$,

$$\begin{bmatrix} -9.81 \\ 29.43 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.300 & -0.675 & 0 \\ -0.675 & 2.025 & 0 \\ 0 & 0 & 2.325 \end{bmatrix} \begin{bmatrix} d\omega_x/dt \\ d\omega_y/dt \\ d\omega_z/dt \end{bmatrix}.$$

Solving, we obtain $\boldsymbol{\alpha} = 14.5\ \mathbf{j}$ (rad/s^2).

Problem 20.41 If the 5-kg plate is released from rest in the horizontal position, what force is exerted on it by the ball-and-socket support at that instant?

Solution: See the solution of Problem 20.40. Let G denote the center of mass and Let $\mathbf{F}$ be the force exerted by the support.

The acceleration of the center of mass is

$$\mathbf{a}_G = \mathbf{a}_O + \boldsymbol{\alpha} \times \mathbf{r}_{G/O} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{G/O})$$

$$= \mathbf{0} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 14.5 & 0 \\ \frac{2}{3}b & \frac{1}{3}h & 0 \end{vmatrix} + \mathbf{0}$$

$$= -8.72\ \mathbf{k} \text{ (m/s}^2)$$

From Newton's second law,

$$\Sigma\mathbf{F} = m\mathbf{a}_G: \quad \mathbf{F} - (5)(9.81)\mathbf{k} = (5)(-8.72\mathbf{k}),$$

we obtain $\mathbf{F} = 5.45\mathbf{k}$ (N).

Problem 20.42 The 5-kg triangular plate is connected to a ball-and-socket support at O. If the plate is released in the horizontal position with angular velocity $\boldsymbol{\omega} = 4\mathbf{i}$ (rad/s), what are the components of its angular acceleration at that instant?

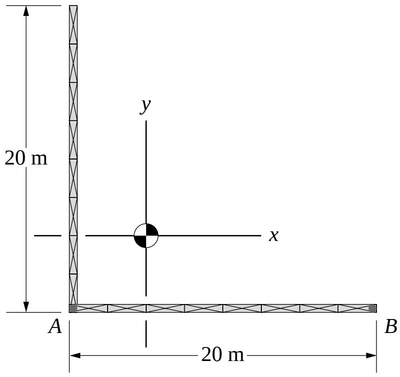
Solution: See the solution of Problem 20.40. From Eq. (20.13) with $\boldsymbol{\omega} = \boldsymbol{\Omega} = 4\mathbf{i}$ (rad/s);

$$\begin{bmatrix} -9.81 \\ 29.43 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.300 & -0.675 & 0 \\ -0.675 & 2.025 & 0 \\ 0 & 0 & 2.325 \end{bmatrix} \begin{bmatrix} \mathrm{d}\omega_x/\mathrm{d}t \\ \mathrm{d}\omega_y/\mathrm{d}t \\ \mathrm{d}\omega_z/\mathrm{d}t \end{bmatrix}$$

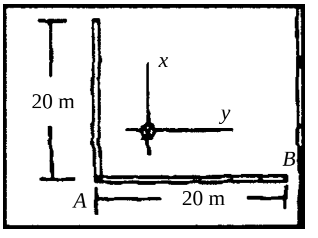
$$+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -4 \\ 0 & 4 & 0 \end{bmatrix} \begin{bmatrix} 0.300 & -0.675 & 0 \\ -0.675 & 2.025 & 0 \\ 0 & 0 & 2.325 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}.$$

Solving, we obtain $\boldsymbol{\alpha} = 14.53\mathbf{j} + 4.65\mathbf{k}$ (rad/s^2).

Problem 20.43 A subassembly of a space station can be modeled as two rigidly connected slender bars, each with a mass of 5000 kg. The subassembly is not rotating at $t = 0$, when a reaction control motor exerts a force $\mathbf{F} = 400\mathbf{k}$ (N) at B. What is the acceleration of point A relative to the center of mass of the subassembly at $t = 0$?



Solution: Choose a x', y', z' coordinate system with the origin at A and the x' axis parallel to the horizontal bar, and a parallel x, y, z system with origin at the center of mass.



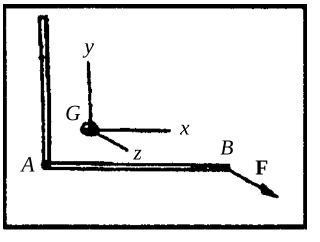
The Euler Equations: The center of mass in the x', y', z' system has the coordinates

$$x_G = \frac{10(5000) + 0(5000)}{10000} = 5 \text{ m},$$

$$y_G = \frac{10(5000) + 0(5000)}{10000} = 5 \text{ m},$$

$$z_G = 0,$$

from which $(d_x, d_y, d_z) = (5, 5, 0)$ m.



From Appendix C, the moments and products of inertia of each bar about A are

$$I_{xx}^A = I_{yy}^A = \frac{mL^2}{3},$$

$$I_{zz}^A = I_{xx}^A + I_{yy}^A = \frac{2mL^2}{3}$$

$$I_{xy}^A = I_{xz}^A = I_{yz}^A = 0,$$

where $m = 5000$ kg, and $L = 20$ m. The moment of inertia matrix is

$$[I^A] = \begin{bmatrix} 0.6667 & 0 & 0 \\ 0 & 0.6667 & 0 \\ 0 & 0 & 1.333 \end{bmatrix} \text{ Mg-m}^2.$$

From the parallel axis theorem, Eq. (20.42), the moments and products of inertia about the center of mass are:

$$I_{xx} = I_{xx}^A - (d_z^2 + d_y^2)(2 \text{ m}) = 0.4167 \text{ Mg-m}^2,$$

$$I_{yy} = I_{yy}^A - (d_x^2 + d_z^2)(2 \text{ m}) = 0.4167 \text{ Mg-m}^2,$$

$$I_{zz} = I_{zz}^A - (d_x^2 + d_y^2)(2 \text{ m}) = 0.8333 \text{ Mg-m}^2.$$

$$I_{xy} = I_{xy}^A - d_x d_y (2 \text{ m}) = -0.2500 \text{ Mg-m}^2,$$

from which the inertia matrix is

$$[I] = \begin{bmatrix} 0.4167 & 0.2500 & 0 \\ 0.2500 & 0.4167 & 0 \\ 0 & 0 & 0.8333 \end{bmatrix} \text{ Mg-m}^2.$$

The vector distance from the center of mass to the point B is

$$\mathbf{r}_{B/G} = (20 - 5)\mathbf{i} + (0 - 5)\mathbf{j} = 15\mathbf{i} - 5\mathbf{j} \text{ (m)}.$$

The moment about the center of mass is

$$\mathbf{M}_G = \mathbf{r}_{B/G} \times \mathbf{F} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 15 & -5 & 0 \\ 0 & 0 & 400 \end{bmatrix}$$

$$= -2000\mathbf{i} - 6000\mathbf{j} \text{ (N-m)}.$$

The coordinates are body-fixed, and the object is initially stationary, from which $\boldsymbol{\Omega} = \boldsymbol{\omega} = 0$, and Eq. (20.19) reduces to

$$\begin{bmatrix} -2000 \\ -6000 \\ 0 \end{bmatrix} = [I]$$

$$= \begin{bmatrix} 4.167 \times 10^5 & 2.5 \times 10^5 & 0 \\ 2.5 \times 10^5 & 4.167 \times 10^5 & 0 \\ 0 & 0 & 8.333 \times 10^5 \end{bmatrix} \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{bmatrix}.$$

Carry out the matrix multiplication to obtain:

$$4.167 \times 10^5 \alpha_x + 2.5 \times 10^5 \alpha_y = -2000,$$

$$2.5 \times 10^5 \alpha_x + 4.167 \times 10^5 \alpha_y = -6000,$$

and $\alpha_z = 0$. Solve: $\boldsymbol{\alpha} = 0.006\mathbf{i} - 0.018\mathbf{j}$ (rad/s^2).

Newton's second law: The acceleration of the center of mass of the object from Newton's second law is

$$\mathbf{a}_G = \left(\frac{1}{2 \text{ m}}\right)\mathbf{F} = 0.04\mathbf{k} \text{ (m/s}^2\text{)}.$$

The acceleration of point A: The vector distance from the center of mass to the point A is $\mathbf{r}_{A/G} = -5\mathbf{i} - 5\mathbf{j}$ (m). The acceleration of point A is

$$\mathbf{a}_A = \mathbf{a}_G + \boldsymbol{\alpha} \times \mathbf{r}_{A/G} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{A/G}).$$

Since the object is initially stationary, $\boldsymbol{\omega} = 0$.

$$\mathbf{a}_A = \mathbf{a}_G + \boldsymbol{\alpha} \times \mathbf{r}_{A/G} = 0.04\mathbf{k} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.006 & -0.018 & 0 \\ -5 & -5 & 0 \end{bmatrix}$$

$$= -0.08\mathbf{k} \text{ (m/s}^2\text{)}, \quad \mathbf{a}_A = -0.08\mathbf{k} \text{ (m/s}^2\text{)}$$

Problem 20.44 A subassembly of a space station can be modeled as two rigidly connected slender bars, each with a mass of 5000 kg. If the subassembly is rotating about the x axis at a constant rate of 1 revolution every 10 minutes, what is the magnitude of the couple its reaction control system is exerting on it?

Solution: (See Figure in solution to Problem 20.100.) The angular acceleration of the disk is given by

$$\frac{d\boldsymbol{\omega}}{dt} = \frac{d}{dt}(\omega_d \mathbf{i} + \omega_O \mathbf{j}) + \boldsymbol{\omega}_O \times \boldsymbol{\omega}_d = 0 + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \omega_O & 0 \\ \omega_d & 0 & 0 \end{bmatrix}$$

$$= -\omega_O \omega_d \mathbf{k}.$$

The velocity of point A relative to O is

$$\mathbf{a}_{A/O} = \boldsymbol{\alpha} \times \mathbf{r}_{A/O} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{A/O})$$

$$= (-\omega_O \omega_d)(\mathbf{k} \times \mathbf{r}_{A/O}) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{A/O}).$$

Term by term:

$$-\omega_O \omega_d (\mathbf{k} \times \mathbf{r}_{A/O}) = -\omega_O \omega_d \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ b & R\sin\theta & -R\cos\theta \end{bmatrix}$$

$$= \omega_O \omega_d R \sin\theta \mathbf{i} - \omega_O \omega_d b \mathbf{j},$$

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{A/O}) = \boldsymbol{\omega} \times \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_d & \omega_O & 0 \\ b & R\sin\theta & -R\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_d & \omega_O & 0 \\ -R\omega_O\cos\theta & R\omega_d\cos\theta & R\omega_d\sin\theta - b\omega_O \end{bmatrix}$$

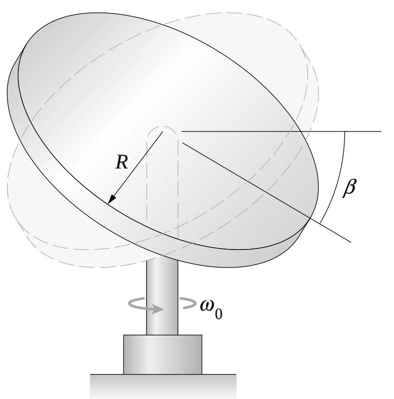
$$= (R\omega_d \sin\theta - b\omega_O)(\omega_o \mathbf{i} - \omega_d \mathbf{j}) + (R\cos\theta)(\omega_d^2 + \omega_O^2)\mathbf{k}.$$

Collecting terms:

$$\mathbf{a}_{A/O} = (2R\omega_O\omega_d \sin\theta - b\omega_O^2)\mathbf{i} - (R\omega_d^2 \sin\theta)\mathbf{j}$$

$$+ (R\omega_d^2 \cos\theta + R\omega_O^2 \cos\theta)\mathbf{k}.$$

Problem 20.45 The thin circular disk of radius $R = 0.2$ m and mass $m = 4$ kg is rigidly attached to the vertical shaft. The plane of the disk is slanted at an angle $\beta = 30^\circ$ relative to the horizontal. The shaft rotates with constant angular velocity $\omega_0 = 25$ rad/s. Determine the magnitude of the couple exerted on the disk by the shaft.



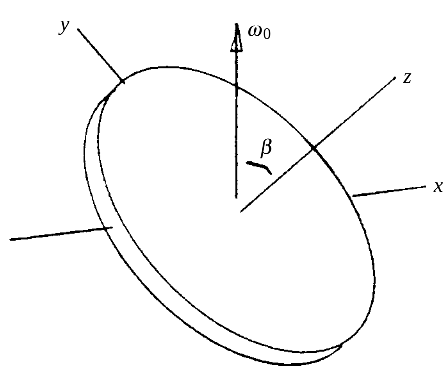
Solution: In terms of the body-fixed reference frame shown, the disk's inertia matrix is

$$[I] = \begin{bmatrix} \frac{1}{4}mR^2 & 0 & 0 \\ 0 & \frac{1}{4}mR^2 & 0 \\ 0 & 0 & \frac{1}{2}mR^2 \end{bmatrix} = \begin{bmatrix} 0.04 & 0 & 0 \\ 0 & 0.04 & 0 \\ 0 & 0 & 0.08 \end{bmatrix} \text{ kg-m}^2.$$

The disk's angular velocity is

$$\boldsymbol{\omega} = \boldsymbol{\Omega} = \omega_0 \sin\beta \mathbf{j} + \omega_0 \cos\beta \mathbf{k}$$

$$= 12.50\mathbf{j} + 21.65\mathbf{k} \text{ (rad/s)}.$$

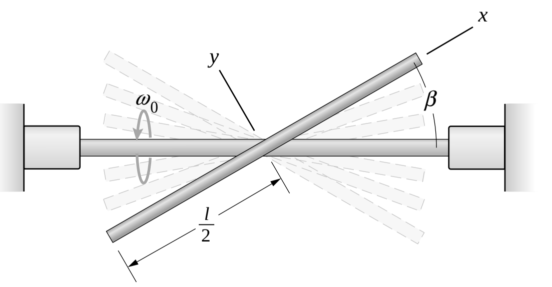


From Eq. (20.19) with $d\omega_x/dt = d\omega_y/dt = d\omega_z/dt = 0$,

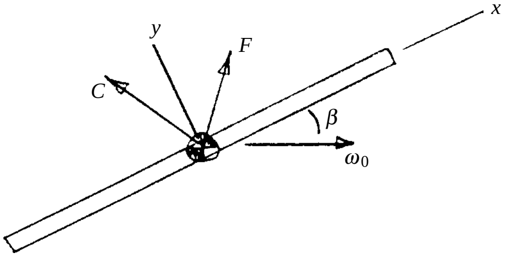
$$\begin{bmatrix} \sum M_x \\ \sum M_y \\ \sum M_z \end{bmatrix} = \begin{bmatrix} 0 & -21.65 & 12.5 \\ 21.65 & 0 & 0 \\ -12.5 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.04 & 0 & 0 \\ 0 & 0.04 & 0 \\ 0 & 0 & 0.08 \end{bmatrix}$$

$$\times \begin{bmatrix} 0 \\ 12.5 \\ 21.65 \end{bmatrix} = \begin{bmatrix} 10.8 \\ 0 \\ 0 \end{bmatrix} \text{ N-m.}$$

Problem 20.46 The slender bar of mass $m = 8$ kg and length $l = 1.2$ m is welded to a horizontal shaft that rotates with constant angular velocity $\omega_0 = 25$ rad/s. The angle $\beta = 30^\circ$. Determine the magnitudes of the force **F** and couple **C** exerted on the bar by the shaft. (Write the equations of angular motion in terms of the body-fixed coordinate system shown.)



Solution: In terms of the body-fixed reference frame shown, the inertia matrix is



$$[I] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{12}ml^2 & 0 \\ 0 & 0 & \frac{1}{12}ml^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.96 & 0 \\ 0 & 0 & 0.96 \end{bmatrix} \text{ kg-m}^2.$$

The bar's angular velocity is

$$\boldsymbol{\omega} = \omega_0 \cos\beta\mathbf{i} - \omega_0 \sin\beta\mathbf{j}$$

$$= 21.65\mathbf{i} - 12.50\mathbf{j} \text{ (rad/s)}.$$

The acceleration of the center of mass is zero, so the force **F** must be equal and opposite to the force exerted by the bar's weight. Therefore $|\mathbf{F}| = mg = 78.5$ N. From Eq. (20.19) with $d\omega_x/dt = d\omega_y/dt = d\omega_z/dt = 0$,

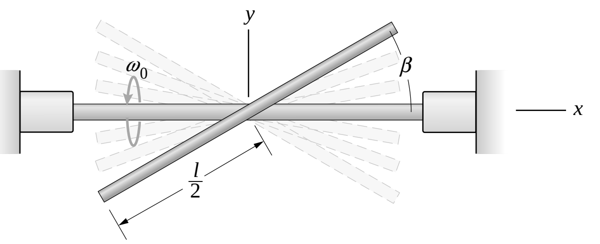
$$\begin{bmatrix} C_x \\ C_y \\ C_z \end{bmatrix} = \begin{bmatrix} 0 & 0 & -12.5 \\ 0 & 0 & -21.65 \\ 12.5 & 21.65 & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.96 & 0 \\ 0 & 0 & 0.96 \end{bmatrix} \begin{bmatrix} 21.65 \\ -12.5 \\ 0 \end{bmatrix}$$

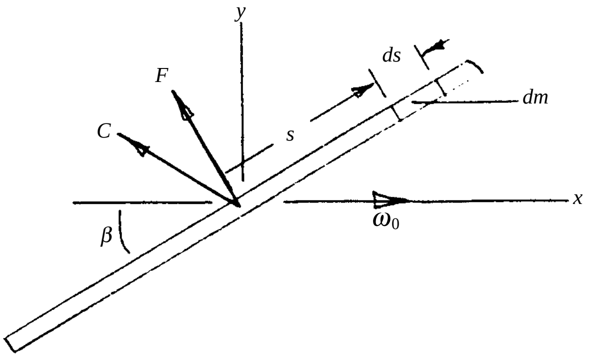
$$= \begin{bmatrix} 0 \\ 0 \\ -260 \end{bmatrix} \text{ N-m}$$

We see that $|\mathbf{C}| = 260$ N-m.

Problem 20.47 The slender bar of mass $m = 8$ kg and length $l = 1.2$ m is welded to a horizontal shaft that rotates with constant angular velocity $\omega_0 = 25$ rad/s. The angle $\beta = 30^\circ$. Determine the magnitudes of the force **F** and couple **C** exerted on the bar by the shaft. (Write the equations of angular motion in terms of the body-fixed coordinate system shown. See Problem 20.98.)



Solution: Let ρ be the bar's density and A its cross-sectional area. The mass dm is $dm = \rho A\, ds$. The bar's moment of inertia about the x axis is



$$I_{xx} = \int_m y^2 dm = \int_{-l/2}^{l/2} (s \sin\beta)^2 \rho A\, ds$$

$$= \rho A \sin^2\beta \left[\frac{5^3}{3}\right]_{-\frac{l}{2}}^{\frac{l}{2}}$$

$$= \frac{1}{12}\rho A\ l^3 \sin^2\beta$$

$$= \frac{1}{12} ml^2 \sin^2\beta.$$

The moment of inertia about the y axis is

$$I_{yy} = \int_m x^2\, dm = \int_{-l/2}^{l/2} (s \cos\beta)^2 \rho A\, ds$$

$$= \frac{1}{12} ml^2 \cos^2\beta,$$

and the product of inertia I_{xy} is

$$I_{xy} = \int_m xy\, dm = \int_{-l/2}^{l/2} (s^2 \sin\beta\cos\beta)\rho A\, ds$$

$$= \frac{1}{12} ml^2 \sin\beta\cos\beta.$$

The inertia matrix is

$$[I] = \frac{1}{12} ml^2 \begin{bmatrix} \sin^2\beta & -\sin\beta\cos\beta & 0 \\ -\sin\beta\cos\beta & \cos^2\beta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.240 & -0.416 & 0 \\ -0.416 & 0.720 & 0 \\ 0 & 0 & 0.960 \end{bmatrix} \text{kg-m}^2.$$

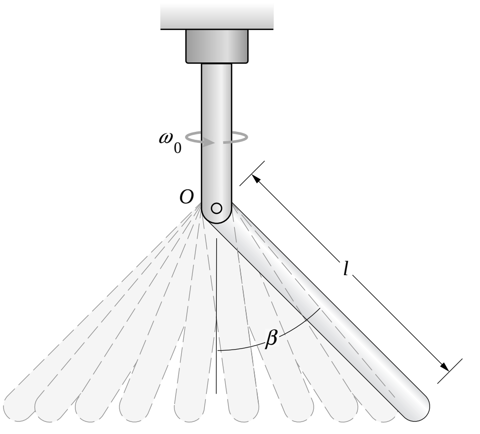
The bar's angular velocity is $\boldsymbol{\omega} = \omega_0 \mathbf{i}$. The acceleration of the center of mass is zero, so the force **F** must be equal and opposite to the force exerted by the bar's weight. Therefore $|\mathbf{F}| = mg = 78.5$ N. From Eq. (20.19) with $d\omega_x/dt = d\omega_y/dt = d\omega_z/dt = 0$,

$$\begin{bmatrix} C_x \\ C_y \\ C_z \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -25 \\ 0 & 25 & 0 \end{bmatrix} \begin{bmatrix} 0.240 & -0.416 & 0 \\ -0.416 & 0.720 & 0 \\ 0 & 0 & 0.960 \end{bmatrix} \begin{bmatrix} 25 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ -260 \end{bmatrix} \text{N-m}.$$

We see that $|\mathbf{C}| = 260$ N-m.

Problem 20.48 The slender bar of length l and mass m is pinned to the vertical shaft at O. The vertical shaft rotates with a constant angular velocity ω_0. Show that the value of ω_0 necessary for the bar to remain at a constant angle β relative to the vertical is $\omega_0 = \sqrt{3g/2l\cos\beta}$.



Solution: This is motion about a fixed point so Eq. (20.13) is applicable. Choose a body-fixed x, y, z coordinate system with the origin at O, the positive x axis parallel to the slender bar, and z axis out of the page. The angular velocity of the vertical shaft is

$$\mathbf{\Omega} = \omega_0(-\mathbf{i}\cos\beta + \mathbf{j}\sin\beta).$$

The vector from O to the center of mass of the bar is $\mathbf{r}_{G/O} = (L/2)\mathbf{i}$. The weight is

$$\mathbf{W} = mg(\mathbf{i}\cos\beta - \mathbf{j}\sin\beta).$$

The moment about the point O is

$$\mathbf{M}_G = \mathbf{r}_{G/O} \times \mathbf{W} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{L}{2} & 0 & 0 \\ mg\cos\beta & -mg\sin\beta & 0 \end{bmatrix}$$

$$= \left(-\frac{mgL}{2}\sin\beta\right)\mathbf{k}$$

The moments and products of inertia about O in the x, y, z system are

$$I_{xx} = 0,$$

$$I_{yy} = I_{zz} = mL^2/3,$$

$$I_{xy} = I_{xz} = I_{yz} = 0.$$

The body-fixed coordinate system rotates with angular velocity

$$\mathbf{\Omega} = \boldsymbol{\omega} = \omega_0(-\mathbf{i}\cos\beta + \mathbf{j}\sin\beta).$$

Eq. (20.13) reduces to

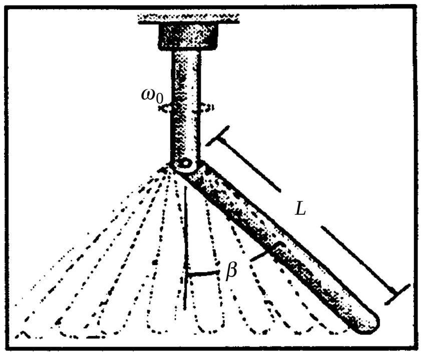
$$\begin{bmatrix} M_{Ox} \\ M_{Oy} \\ M_{Oz} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \omega_0\sin\beta \\ 0 & 0 & \omega_0\cos\beta \\ -\omega_0\sin\beta & -\omega_0\cos\beta & 0 \end{bmatrix}$$

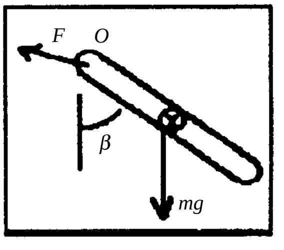
$$\times \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{mL^2}{3} & 0 \\ 0 & 0 & \frac{mL^2}{3} \end{bmatrix} \begin{bmatrix} -\omega_0\cos\beta \\ \omega_0\sin\beta \\ 0 \end{bmatrix}.$$

Carry out the matrix multiplication,

$$\begin{bmatrix} M_{Ox} \\ M_{Oy} \\ M_{Oz} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{\omega_0 mL^2\sin\beta}{3} \\ 0 & 0 & \frac{\omega_0 mL^2\cos\beta}{3} \\ 0 & -\frac{\omega_0 mL^2\cos\beta}{3} & 0 \end{bmatrix} \begin{bmatrix} -\omega_0\cos\beta \\ \omega_0\sin\beta \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ -\frac{\omega_0^2 mL^2\cos\beta\sin\beta}{3} \end{bmatrix}.$$





Equate the z components:

$$M_{Oz} = -\frac{\omega_0^2 mL^2\cos\beta\sin\beta}{3}.$$

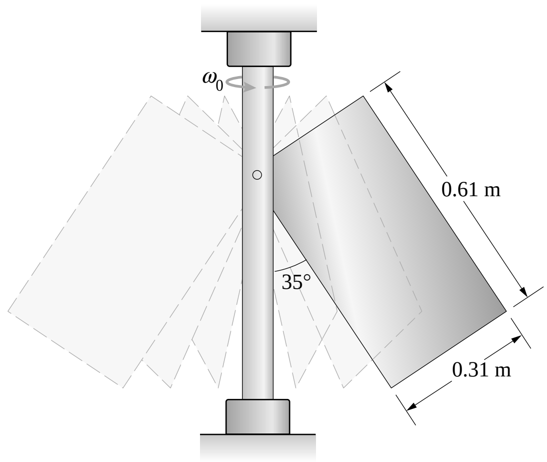
The pin-supported joint at O cannot support a couple, $\mathbf{C} = 0$, from which

$$\mathbf{M}_O = \mathbf{M}_G. - \frac{mgL}{2}\sin\beta = -\frac{\omega_0^2 mL^2\cos\beta\sin\beta}{3}$$

Assume that $\beta \neq 0$, from which $\sin\beta \neq 0$, and the equation can be solved for

$$\boxed{\omega_0 = \sqrt{\frac{3g}{2l\cos\beta}}}$$

Problem 20.49 The vertical shaft rotates with constant angular velocity ω_0. The 35° angle between the edge of the 44.5 N thin rectangular plate pinned to the shaft and the shaft remains constant. Determine ω_0.



Solution: This is motion about a fixed point, and Eq. (20.13) is applicable. Choose an x, y, z coordinate system with the origin at the pinned joint O and the x axis parallel to the lower edge of the plate, and the y axis parallel to the upper narrow edge of the plate. Denote $\beta = 35^\circ$. The plate rotates with angular velocity

$$\boldsymbol{\omega} = \omega_0(-\mathbf{i}\cos\beta + \mathbf{j}\sin\beta) = \omega_0(-0.8192\mathbf{i} + 0.5736\mathbf{j}) \text{ (rad/s).}$$

The vector from the pin joint to the center of mass of the plate is

$$\mathbf{r}_{G/O} = 0.31\mathbf{i} + (0.152)\mathbf{j} \text{ (m).}$$

The weight of the plate is

$$\mathbf{W} = 44.5\,(\mathbf{i}\cos\beta - \mathbf{j}\sin\beta)$$

$$= 36.4\mathbf{i} - 25.5\mathbf{j} \text{ (N).}$$

The moment about the center of mass is

$$\mathbf{M}_G = \mathbf{r}_{G/O} \times \mathbf{W} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.31 & 0.152 & 0 \\ 36.4 & -25.5 & 0 \end{bmatrix}$$

$$= -13.33 \text{ N-m.}$$

From Appendix C, the moments and products of inertia of a thin plate about O are

$$I_{xx} = \frac{mh^2}{3} = 0.14 \text{ kg-m}^2,$$

$$I_{yy} = \frac{mb^2}{3} = 0.56 \text{ kg-m}^2,$$

$$I_{zz} = \frac{m}{3}(h^2 + b^2) = 0.7 \text{ kg-m}^2,$$

$$I_{xy} = \frac{mbh}{4} = 0.21 \text{ kg-m}^2,$$

$$I_{xz} = I_{yz} = 0.$$

At a constant rate of rotation, the angle $\beta = 35^\circ = \text{const}$, $\boldsymbol{\alpha} = 0$. The body-fixed coordinate system rotates with angular velocity

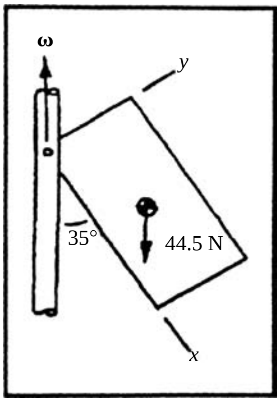
$$\boldsymbol{\Omega} = \boldsymbol{\omega} = \omega_0(-\mathbf{i}\cos\beta + \mathbf{j}\sin\beta)$$

$$= -3.64\omega_0\mathbf{i} + 2.55\omega_0\mathbf{j} \text{ (rad/s),}$$

and Eq. (20.13) reduces to:

$$\begin{bmatrix} M_{Ox} \\ M_{Oy} \\ M_{Oz} \end{bmatrix} = \omega_0^2 \begin{bmatrix} 0 & 0 & 2.55 \\ 0 & 0 & 3.64 \\ -2.55 & -3.64 & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} 0.14 & -0.21 & 0 \\ -0.21 & 0.56 & 0 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} -3.64 \\ 2.55 \\ 0 \end{bmatrix}.$$



Carry out the matrix operations to obtain:

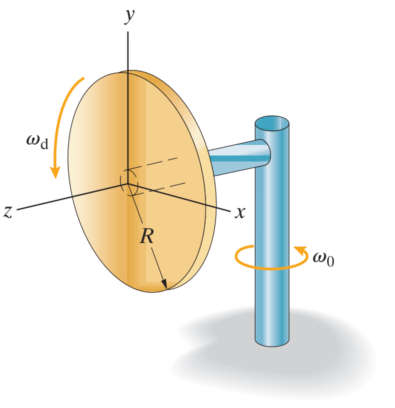
$$\begin{bmatrix} M_{Ox} \\ M_{Oy} \\ M_{Oz} \end{bmatrix} = \omega_0^2 \begin{bmatrix} 0 \\ 0 \\ -0.27 \end{bmatrix}$$

The pin support cannot support a couple, from which

$$M_{Oz} = M_{Gz}, \quad -13.33 = -\omega_0^2(0.27),$$

from which $\boxed{\omega_0 = 7.025 \text{ rad/s}}$

Problem 20.50 The radius of the 100 N thin circular disk is $R = 0.5$ m. The disk is mounted on the horizontal shaft and rotates with constant angular velocity $\omega_d = 10$ rad/s relative to the shaft. The horizontal shaft is 1 m in length. The vertical shaft rotates with constant angular velocity $\omega_0 = 4$ rad/s. Determine the force and couple exerted at the center of the disk by the horizontal shaft.



Solution: Using Newton's Second Law

$$\mathbf{F} - mg\mathbf{j} = m\mathbf{a} = -mr\omega_0^2\mathbf{k}$$

$$\mathbf{F} = (100 \text{ N})\mathbf{j} - \left(\frac{100 \text{ N}}{9.81 \text{ m/s}^2}\right)(1 \text{ m})(4 \text{ rad/s})^2\mathbf{k}$$

$$\boxed{\mathbf{F} = (100\mathbf{j} - 163\mathbf{k}) \text{ N.}}$$

In preparation to use Euler's Equations we have

$$\boldsymbol{\omega} = \omega_0\mathbf{j} + \omega_d\mathbf{k} = (4\mathbf{j} + 10\mathbf{k}) \text{ rad/s}$$

$$\boldsymbol{\alpha} = \omega_0\mathbf{j} \times \omega_d\mathbf{k} = (40 \text{ rad/s}^2)\mathbf{i}$$

$$[I] = \begin{bmatrix} \frac{1}{4}mR^2 & 0 & 0 \\ 0 & \frac{1}{4}mR^2 & 0 \\ 0 & 0 & \frac{1}{2}mR^2 \end{bmatrix} = \begin{bmatrix} 0.637 & 0 & 0 \\ 0 & 0.637 & 0 \\ 0 & 0 & 1.274 \end{bmatrix} \text{ kg-m}^2$$

Euler's Equations are now

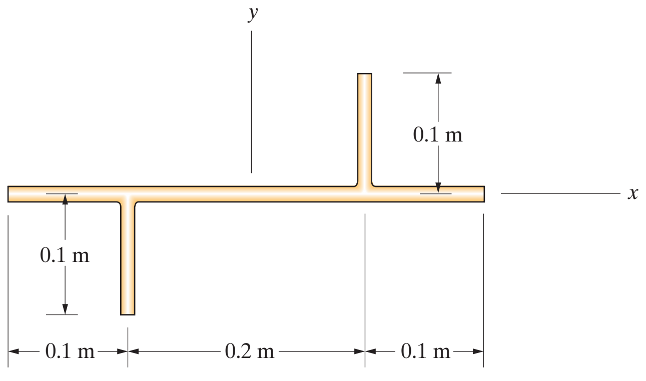
$$\mathbf{M} = [I]\boldsymbol{\alpha} + \boldsymbol{\omega} \times [I]\boldsymbol{\omega}$$

$$\mathbf{M} = \begin{bmatrix} 0.637 & 0 & 0 \\ 0 & 0.637 & 0 \\ 0 & 0 & 1.274 \end{bmatrix} \begin{Bmatrix} 40 \\ 0 \\ 0 \end{Bmatrix}$$

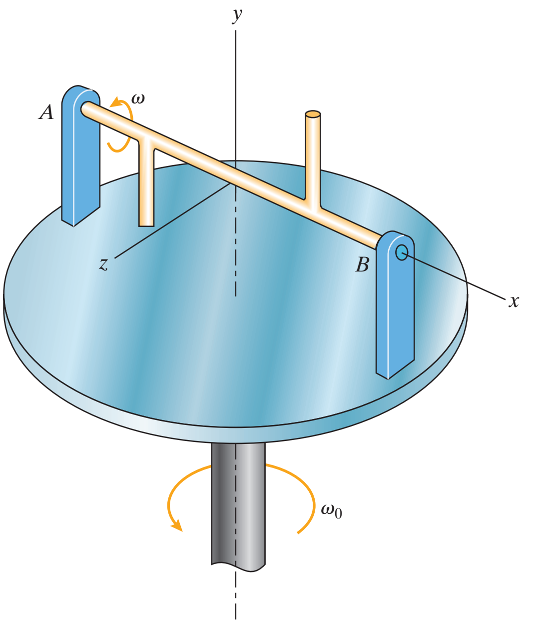
$$+ \begin{bmatrix} 0 & -10 & 4 \\ 10 & 0 & 0 \\ -4 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.637 & 0 & 0 \\ 0 & 0.637 & 0 \\ 0 & 0 & 1.274 \end{bmatrix} \begin{Bmatrix} 0 \\ 4 \\ 10 \end{Bmatrix}$$

$$\boxed{\mathbf{M} = 25.5\mathbf{i} \text{ N-m.}}$$

Problem 20.51 The object shown in Fig. a consists of two 1-kg vertical slender bars welded to the 4-kg horizontal slender bar. In Fig. b, the object is supported by bearings at A and B. The horizontal circular disk is supported by a vertical shaft that rotates with constant angular velocity $\omega_0 = 6$ rad/s. The horizontal bar rotates with constant angular velocity $\omega = 10$ rad/s. At the instant shown, determine the y and z components of the forces exerted on the object at A and B.



(a)



(b)

Solution: The nonzero inertias are

$$I_{xx} = 2\frac{1}{3}(1\text{ kg})(0.1\text{ m})^2 = 0.00667\text{ kg-m}^2,$$

$$I_{yy} = \frac{1}{12}(4\text{ kg})(0.4\text{ m})^2 + 2(1\text{ kg})(0.1\text{ m})^2 = 0.0733\text{ kg-m}^2,$$

$$I_{zz} = I_{xx} + I_{yy} = 0.08\text{ kg-m}^2,$$

$$I_{xy} = 2(1\text{ kg})(0.1\text{ m})(0.05\text{ m}) = 0.01\text{ kg-m}^2.$$

Newton's Second Law gives (the acceleration of the center of mass is zero).

$$\Sigma F_y : A_y + B_y - (6\text{ kg})(9.81\text{ m/s}^2) = 0,$$

$$\Sigma F_z : A_z + B_z = 0.$$

The angular velocity and angular acceleration are

$$\boldsymbol{\omega} = (10\mathbf{i} + 6\mathbf{j})\text{ rad/s}$$

$$\boldsymbol{\alpha} = 6\mathbf{j} \times 10\mathbf{i} = -60\mathbf{k}\text{ rad/s}^2.$$

The moment about the center of mass is

$$M_x = 0,$$

$$M_y = B_z(0.2\text{ m}) - A_z(0.2\text{ m}),$$

$$M_z = B_y(0.2\text{ m}) - A_y(0.2\text{ m}).$$

Euler's equations are now

$$\begin{Bmatrix} 0 \\ B_z - A_z \\ B_y - A_y \end{Bmatrix}(0.2)$$

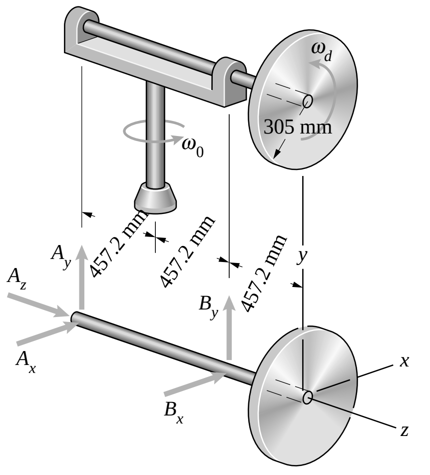
$$= \begin{bmatrix} 0.00667 & -0.01 & 0 \\ -0.01 & 0.0733 & 0 \\ 0 & 0 & 0.08 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ -60 \end{Bmatrix}$$

$$+ \begin{bmatrix} 0 & 6 & 0 \\ -6 & 0 & 10 \\ 0 & -10 & 0 \end{bmatrix}\begin{bmatrix} 0.00667 & -0.01 & 0 \\ -0.01 & 0.0733 & 0 \\ 0 & 0 & 0.08 \end{bmatrix}\begin{Bmatrix} 10 \\ 6 \\ 0 \end{Bmatrix}$$

Solving Euler's equations along with Newton's Second Law we find

$$\boxed{A_y = 33.0\text{ N}, \quad A_z = 0, \quad B_y = 25.8\text{ N}, \quad B_z = 0.}$$

Problem 20.52 The 44.5 N thin circular disk is rigidly attached to the 53.4 N slender horizontal shaft. The disk and horizontal shaft rotate about the axis of the shaft with constant angular velocity $\omega_d = 20$ rad/s. The entire assembly rotates about the vertical axis with constant angular velocity $\omega_0 = 4$ rad/s. Determine the components of the force and couple exerted on the horizontal shaft by the disk.



Solution: The shaft is $L = 3(0.457) = 1.37$ m long. The mass of the disk is

$$m_D = \frac{44.5}{9.81} = 4.54 \text{ kg}.$$

The reaction of the shaft to the disk: The moments and products of inertia of the disk are:

$$I_{xx} = I_{yy} = \frac{m_D R^2}{4} = 0.105 \text{ kg-m}^2,$$

$$I_{zz} = \frac{m_D R^2}{2} = 0.211 \text{ kg-m}^2$$

$$I_{xy} = I_{xz} = I_{yz} = 0.$$

The rotation rate is constant,

$$\boldsymbol{\alpha} = \frac{d\boldsymbol{\omega}}{dt} = 0.$$

The body-fixed coordinate system rotates with angular velocity $\boldsymbol{\Omega} = \omega_0 \mathbf{j}$ (rad/s), and $\boldsymbol{\omega} = \omega_0 \mathbf{j} + \omega_d \mathbf{k}$ (rad/s). Eq. (20.19) reduces to:

$$\begin{bmatrix} M_{dx} \\ M_{dy} \\ M_{dz} \end{bmatrix} = \omega_d \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} 0 \\ \omega_0 \\ \omega_d \end{bmatrix}$$

$$= \omega_0 \omega_d \begin{bmatrix} I_{zz} \\ 0 \\ 0 \end{bmatrix}.$$

The total moment exerted by the disk is

$$\mathbf{M}_d = \omega_0 \omega_d \frac{m_d R^2}{2} \mathbf{i} = 16.85 \mathbf{i} \text{ (N-m)}.$$

The reaction on the shaft by the disk is

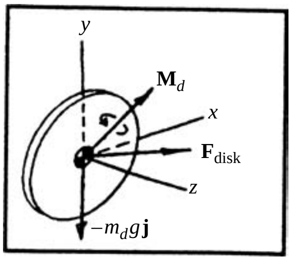
$$\boxed{\mathbf{M}_s = -\mathbf{M}_d = -16.85 \mathbf{i} \text{ N-m}}$$

The reaction of the shaft to the acceleration of the disk: The attachment point of the column to the shaft has the coordinates $(0, 0, -0.91)$ m, from which the vector distance from the attachment point to the disk is $\mathbf{r}_{D/P} = 0.91\mathbf{k}$ m. The acceleration of the disk is

$$\mathbf{a}_D = \mathbf{a}_P + \boldsymbol{\alpha} \times \mathbf{r}_{D/P} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{P/D}) = \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{P/D})$$

$$\mathbf{a}_D = \boldsymbol{\Omega} \times \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \omega_0 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \omega_0 & \\ 0.91\omega_0 & 0 & 0 \end{bmatrix}$$

$$= -0.91 \omega_0^2 \mathbf{k} = -14.6 \mathbf{k} \text{ (m/s}^2).$$



From Newton's second law,

$$m_d \mathbf{a}_D = \mathbf{F}_{\text{disk}} + \mathbf{W} = \mathbf{F}_{\text{disk}} - W_d \mathbf{j},$$

from which the external force on the disk is:

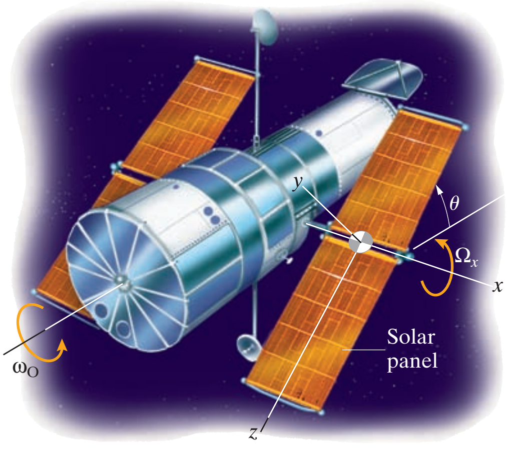
$$\mathbf{F}_{\text{disk}} = m_d g \mathbf{j} - 0.91 m_d \omega_0^2 \mathbf{k} = 44.5 \mathbf{j} - 66.4 \mathbf{k} \text{ (N)}.$$

The external force on the shaft is

$$\boxed{\mathbf{F}_{\text{shaft}} = -\mathbf{F}_{\text{disk}} = -44.5 \mathbf{j} + 66.4 \mathbf{k} \text{ (N)}}$$

Problem 20.53 The Hubble telescope is rotating about its longitudinal axis with constant angular velocity ω_0. The coordinate system is fixed with respect to the solar panel. Relative to the telescope, the solar panel rotates about the x axis with constant angular velocity ω_x. Assume that the moments of inertia I_{xx}, I_{yy}, and I_{zz} are known, and $I_{xy} = I_{yz} = I_{zx} = 0$. Show that the moment about the x axis the servomechanisms must exert on the solar panel is

$$\Sigma M_x = (I_{zz} - I_{yy})\omega_0^2 \sin\theta \cos\theta.$$



Solution: We have

$$\boldsymbol{\omega} = \omega_x \mathbf{i} + \omega_0(\sin\theta\mathbf{j} + \cos\theta\mathbf{k})$$

$$\boldsymbol{\alpha} = \frac{d\boldsymbol{\omega}}{dt} = 0\mathbf{i} + \left(\omega_0 \cos\theta \frac{d\theta}{dt}\right)\mathbf{j} - \left(\omega_0 \sin\theta \frac{d\theta}{dt}\right)\mathbf{k}$$

$$\boldsymbol{\alpha} = \omega_0\omega_x \cos\theta\mathbf{j} - \omega_0\omega_x \sin\theta\mathbf{k}$$

Thus

$$\begin{Bmatrix} M_x \\ M_y \\ M_z \end{Bmatrix} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{Bmatrix} 0 \\ \omega_0\omega_x \cos\theta \\ -\omega_0\omega_x \sin\theta \end{Bmatrix}$$

$$+ \begin{bmatrix} 0 & -\omega_0 \cos\theta & \omega_0 \sin\theta \\ \omega_0 \cos\theta & 0 & \omega_x \\ -\omega_0 \sin\theta & -\omega_x & 0 \end{bmatrix}$$

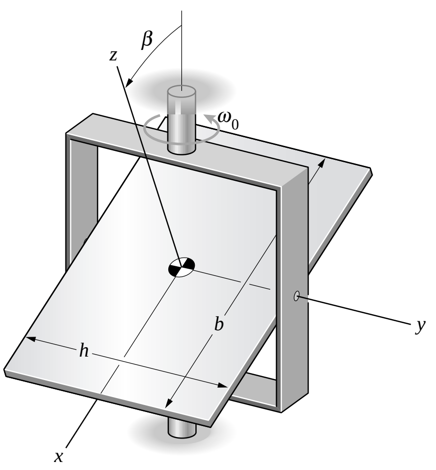
$$\times \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{Bmatrix} \omega_x \\ \omega_0 \sin\theta \\ \omega_0 \cos\theta \end{Bmatrix}$$

Solving we find

$$\boxed{M_x = (I_{zz} - I_{yy})\omega_0{}^2 \sin\theta \cos\theta}$$

Problem 20.54 The thin rectangular plate is attached to the rectangular frame by pins. The frame rotates with constant angular velocity ω_0. Show that

$$\frac{d^2\beta}{dt^2} = -\omega_0^2 \sin\beta\cos\beta.$$



Solution: Assume that the only external moment applied to the object is the moment required to maintain a constant rotation ω_0 about the axis of rotation. Denote this moment by $\mathbf{M}_0$. In the x, y, z system $\mathbf{M}_0 = M_0(-\mathbf{i}\sin\beta + \mathbf{k}\cos\beta)$, from which

$$M_x = -M_0 \sin\beta,$$

$$M_y = 0,$$

$$M_z = M_0\cos\beta.$$

From Appendix C, in the x, y, z system the moments and products of inertia of the plate are

$$I_{xx} = \frac{mh^2}{12},$$

$$I_{yy} = \frac{mb^2}{12},$$

$$I_{zz} = \frac{m}{12}(h^2 + b^2),$$

$$I_{xy} = I_{xz} = I_{yz} = 0.$$

The plate is attached to the frame by pins, so the assumption is that the plate is free to rotate about the y-axis. The body-fixed coordinate system rotates with angular velocity

$$\boldsymbol{\Omega} = \boldsymbol{\omega} = -\mathbf{i}\omega_0\sin\beta + \mathbf{j}\left(\frac{d\beta}{dt}\right) + \mathbf{k}\omega_0\cos\beta \text{ (rad/s)},$$

where $\frac{d\beta}{dt}$ is the angular velocity about the y-axis. For $\omega_0 = \text{const.}$ *for all time*, the derivative

$$\frac{d\omega_0}{dt} = 0,$$

and the acceleration is

$$\boldsymbol{\alpha} = -\mathbf{i}\omega_0\cos\beta\left(\frac{d\beta}{dt}\right) + \mathbf{j}\left(\frac{d^2\beta}{dt^2}\right) - \mathbf{k}\omega_0\sin\beta\left(\frac{d\beta}{dt}\right).$$

Eq. (20.19) becomes

$$\begin{bmatrix} -M_0\sin\beta \\ 0 \\ M_0\cos\beta \end{bmatrix} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}\begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & -\omega_0\cos\beta & \dfrac{d\beta}{dt} \\ \omega_0\cos\beta & 0 & \omega_0\sin\beta \\ -\dfrac{d\beta}{dt} & -\omega_0\sin\beta & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}\begin{bmatrix} -\omega_0\sin\beta \\ \dfrac{d\beta}{dt} \\ \omega_0\cos\beta \end{bmatrix}$$

$$\begin{bmatrix} -M_0\sin\beta \\ 0 \\ M_0\cos\beta \end{bmatrix} = \begin{bmatrix} I_{xx}\alpha_x \\ I_{yy}\alpha_y \\ I_{zz}\alpha_z \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & -\omega_0 I_{yy}\cos\beta & I_{zz}\dfrac{d\beta}{dt} \\ \omega_0 I_{xx}\cos\beta & 0 & \omega_0 I_{zz}\sin\beta \\ -I_{xx}\dfrac{d\beta}{dt} & -\omega_0 I_{yy}\sin\beta & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} -\omega_0\sin\beta \\ \dfrac{d\beta}{dt} \\ \omega_0\cos\beta \end{bmatrix},$$

$$\begin{bmatrix} -M_0\sin\beta \\ 0 \\ M_0\cos\beta \end{bmatrix} = \begin{bmatrix} -\omega_0 I_{xx}\cos\beta\left(\dfrac{d\beta}{dt}\right) \\ I_{yy}\left(\dfrac{d\beta}{dt}\right) \\ -\omega_0 I_{zz}\sin\beta\left(\dfrac{d\beta}{dt}\right) \end{bmatrix}$$

$$+ \begin{bmatrix} \omega_0 I_{xx}\cos\beta\left(\dfrac{d\beta}{dt}\right) \\ \omega_0^2 I_{yy}\cos\beta\sin\beta \\ \omega_0(I_{xx} - I_{yy})\sin\beta\left(\dfrac{d\beta}{dt}\right) \end{bmatrix}$$

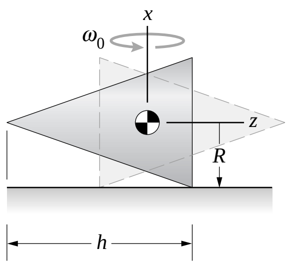
$$= \begin{bmatrix} 0 \\ I_{yy}\alpha_y + \omega_0^2 I_{yy}\cos\beta\sin\beta \\ -2\omega_0 I_{yy}\sin\beta\left(\dfrac{d\beta}{dt}\right) \end{bmatrix}$$

where $I_{zz} = I_{xx} + I_{yy}$ has been used. The y-component is

$$I_{yy}\alpha_y + \omega_0^2 I_{yy}\cos\beta\sin\beta = 0,$$

from which $\boxed{\dfrac{d^2\beta}{dt^2} = -\omega_0^2\cos\beta\sin\beta}$.

Problem 20.55* The axis of the right circular cone of mass m, height h, and radius R spins about the vertical axis with constant angular velocity ω_0. The center of mass of the cone is stationary, and its base rolls on the floor. Show that the angular velocity necessary for this motion is $\omega_0 = \sqrt{10g/3R}$. (See Example 20.6.)



Strategy: Let the z axis remain aligned with the axis of the cone and the x remain vertical.

Solution: This a problem of general motion, and Eq. (20.19) applies. The vector distance from the center of mass to the base of the cone is

$$\mathbf{r}_{B/G} = \frac{h}{4}\mathbf{k}$$

(see Appendix C). The angular velocity of rotation of the body fixed coordinate system is $\boldsymbol{\Omega} = \omega_0 \mathbf{i}$. The velocity of the center of the base is

$$\mathbf{v}_B = \boldsymbol{\Omega} \times \mathbf{r}_{B/G} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_0 & 0 & 0 \\ 0 & 0 & \dfrac{h}{4} \end{bmatrix} = -\frac{\omega_0 h}{4}\mathbf{j}.$$

Let the spin rate about the z axis be $\dot{\phi}$, so that the angular velocity, from which $\boldsymbol{\omega} = \boldsymbol{\Omega} + \dot{\phi}\mathbf{k}$. The point of contact with the surface is stationary, and the velocity of the center of the base of the cone is

$$\mathbf{v} = -\frac{\omega_0 h}{4}\mathbf{j},$$

from which $0 = \mathbf{v} + \boldsymbol{\omega} \times (-R\mathbf{i}) = \left(-\dfrac{\omega_0 h}{4} - R\dot{\phi}\right)\mathbf{j} = 0,$

from which $\dot{\phi} = -\dfrac{\omega_0 h}{4R}.$

The center of mass of the cone is at a zero distance from the axis of rotation, from which the acceleration of the center of mass is zero. The angular velocity about the z-axis,

$$\boldsymbol{\omega} = \omega_0 \mathbf{i} - \frac{\omega_0 h}{4R}\mathbf{k} \text{ (rad/s)}.$$

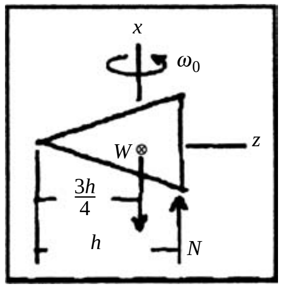
The weight of the cone is $\mathbf{W} = -mg\mathbf{i}$. The reaction of the floor on the cone is $\mathbf{N} = -\mathbf{W}$. The moment about the center of mass exerted by the weight is

$$\mathbf{M}_G = \mathbf{r}_{B/G} \times \mathbf{N} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \dfrac{h}{4} \\ mg & 0 & 0 \end{bmatrix} = +\left(\frac{mgh}{4}\right)\mathbf{j}.$$

The moments and products of inertia of a cone about its center of mass in the x, y, z system are, from Appendix C,

$$I_{xx} = I_{yy} = m\left(\frac{3}{80}h^2 + \frac{3}{20}R^2\right),$$

$$I_{zz} = \frac{3mR^2}{10}, \; I_{xy} = I_{xz} = I_{yz} = 0.$$



Since the rotation rate is constant, and the z axis remains horizontal, the angular acceleration is zero,

$$\frac{d\boldsymbol{\omega}_0}{dt} = 0.$$

The body-fixed coordinate system rotates with angular velocity $\boldsymbol{\Omega} = \omega_0 \mathbf{i}$, and

$$\boldsymbol{\omega} = \omega_0 \mathbf{i} - \frac{h\omega_0}{4R}\mathbf{k}.$$

Eq. (9.26) becomes:

$$\begin{bmatrix} M_{Gx} \\ M_{Gy} \\ M_{Gz} \end{bmatrix} = \omega_0^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -\dfrac{h}{4R} \end{bmatrix}$$

$$= \omega_0^2 \begin{bmatrix} 0 \\ \dfrac{hI_{zz}}{4R} \\ 0 \end{bmatrix}.$$

For equilibrium, $M_{Oy} = M_{Gy}$, from which

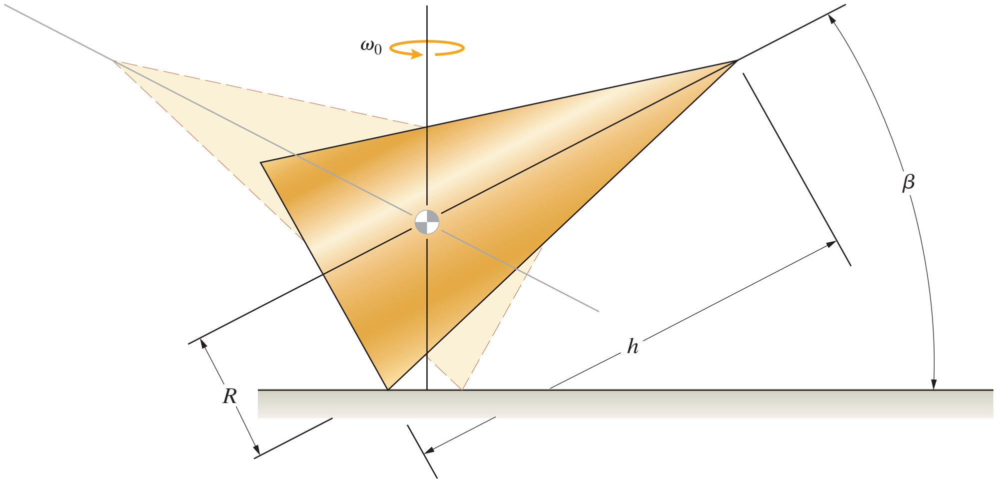
$$\frac{mgh}{4} = \frac{3mhR}{40}\omega_0^2.$$

Solve $\boxed{\omega_0 = \sqrt{\dfrac{10\text{ g}}{3R}}}$

Problem 20.56 The titled homogeneous cone undergoes a steady motion in which its flat end rolls on the floor while the center of mass remains stationary. The angle β between the axis and the horizontal remains constant, and the axis rotates about the vertical axis with constant angular velocity ω_0. The cone has mass m, radius R, and height h. Show that the angular velocity ω_0 necessary for this motion satisfies (see Example 20.6)

$$\omega_0{}^2 = \frac{g(R\sin\beta - \frac{1}{4}h\cos\beta)}{\frac{3}{20}(R^2+\frac{1}{4}h^2)\sin\beta\cos\beta - \frac{3}{40}hR\cos^2\beta}$$

(See Example 20.6.)



Solution: Following Example 20.6, we use a coordinate system with the z axis pointing along the cone axis, the y axis remains horizontal (out of the paper) and the x axis completes the set

$$\mathbf{\Omega} = \omega_0\cos\beta\mathbf{i} + \omega_0\sin\beta\mathbf{k}$$

$$\boldsymbol{\omega} = \mathbf{\Omega} + \omega_{\text{rel}}\mathbf{k} = \omega_0\cos\beta\mathbf{i} + (\omega_0\sin\beta + \omega_{\text{rel}})\mathbf{k}$$

The point P of the cone that is in contact with the ground does not move, therefore

$$\mathbf{v}_P = \mathbf{v}_C + \boldsymbol{\omega}\times\mathbf{r}_{P/C}$$

$$0 = 0 + [\omega_0\cos\beta\mathbf{i} + (\omega_0\sin\beta+\omega_{\text{rel}})\mathbf{k}]\times[-R\mathbf{i} - \tfrac{1}{4}h\mathbf{k}]$$

$$= [\tfrac{1}{4}h\omega_0\cos\beta - R(\omega_0\sin\beta + \omega_{\text{rel}})]\mathbf{j}.$$

Solving yields

$$\omega_{\text{rel}} = \left[\frac{h}{4R}\cos\beta - \sin\beta\right]\omega_0, \quad \boldsymbol{\omega} = \omega_0\cos\beta\mathbf{i} + \frac{h}{4R}\omega_0\cos\beta\mathbf{k}.$$

Since the center of mass is stationary, the floor exerts no horizontal force, and the vertical force is equal to the weight ($N = mg$). The moment about the center of mass due to the normal force is

$$\mathbf{M} = mg(R\sin\beta - \tfrac{1}{4}h\cos\beta)\mathbf{j}$$

The moments and products of inertia for the cone are

$$[I] = \begin{bmatrix} \frac{3}{20}mR^2 + \frac{3}{80}mh^2 & 0 & 0 \\ 0 & \frac{3}{20}mR^2 + \frac{3}{80}mh^2 & 0 \\ 0 & 0 & \frac{3}{10}mR^2 \end{bmatrix}$$

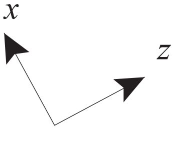
Substituting these expressions into Eq. (20.19), and evaluating the matrix products, we obtain

$$mg(R\sin\beta - \tfrac{1}{4}h\cos\beta) = [(\tfrac{3}{80}h^2 + \tfrac{3}{20}R^2)\cos\beta\sin\beta$$

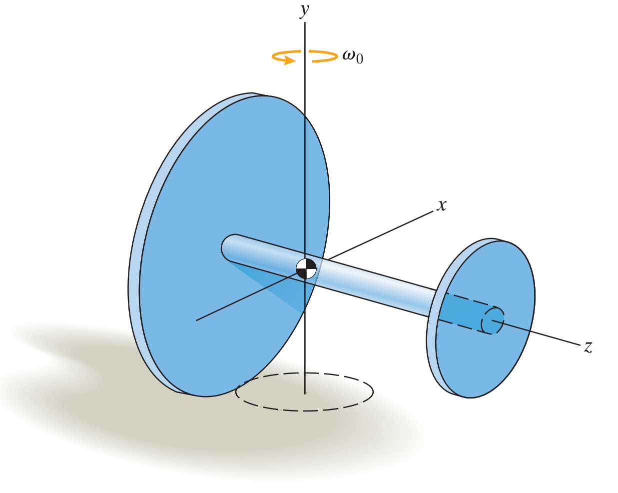
$$- \tfrac{3}{40}hR\cos^2\beta]m\omega_0{}^2$$

Solving, we find that

$$\boxed{\omega_0{}^2 = \frac{g(R\sin\beta - \frac{1}{4}h\cos\beta)}{\frac{3}{20}(R^2+\frac{1}{4}h^2)\sin\beta\cos\beta - \frac{3}{40}hR\cos^2\beta}.}$$



Problem 20.57 The two thin disks are rigidly connected by a slender bar. The radius of the large disk is 200 mm and its mass is 4 kg. The radius of the small disk is 100 mm and its mass is 1 kg. The bar is 400 mm in length and its mass is negligible. The composite object undergoes a steady motion in which it spins about the vertical y axis through its center of mass with angular velocity ω_0. The bar is horizontal during this motion and the large disk rolls on the floor. What is ω_0?



Solution: The z axis remains aligned with the bar and the y axis remains vertical.

The center of mass (measured from the large disk) is located a distance

$$d = \frac{(1\text{ kg})(0.4\text{ m})}{(5\text{ kg})} = 0.08\text{ m}$$

The inertias are

$$I_{zz} = \frac{1}{2}(4\text{ kg})(0.2\text{ m})^2 + \frac{1}{2}(1\text{ kg})(0.1\text{ m})^2 = 0.085\text{ kg-m}^2$$

$$I_{xx} = I_{yy} = \frac{1}{4}(4\text{ kg})(0.2\text{ m})^2 + (4\text{ kg})(0.08\text{ m}^2)$$

$$+\frac{1}{4}(1\text{ kg})(0.1\text{ m})^2 + (1\text{ kg})(0.32\text{ m})^2 = 0.1705\text{ kg-m}^2$$

$$I_{xy} = I_{xz} = I_{yz} = 0$$

The angular velocity of the coordinate system is $\mathbf{\Omega} = \omega_0\mathbf{j}$

Define ω_z to be the rate of rotation of the object about the z axis. Thus $\boldsymbol{\omega} = \omega_0\mathbf{j} + \omega_z\mathbf{k}$

To find ω_z, require that the velocity of the point in contact with the floor be zero

$$\mathbf{v}_P = \boldsymbol{\omega}\times\mathbf{r} = (\omega_0\mathbf{j} + \omega_z\mathbf{k})\times(-0.08\mathbf{k} - 0.2\mathbf{j})\text{ m}$$

$$= [(0.2\text{ m})\omega_z - (0.08\text{ m})\omega_0]\mathbf{i} = 0 \;\Rightarrow\; \omega_z = 0.4\omega_0$$

Since the center of mass does not move, the normal force on the contact point is equal to the weight. Therefore the moment about the center o mass is given by

$$\mathbf{M} = [(-0.2\text{ m})\mathbf{j} - (0.08\text{ m})\mathbf{k}]\times[(5\text{ kg})(9.81\text{ m/s}^2)\mathbf{j}] = (3.924\text{ N})\mathbf{i}$$

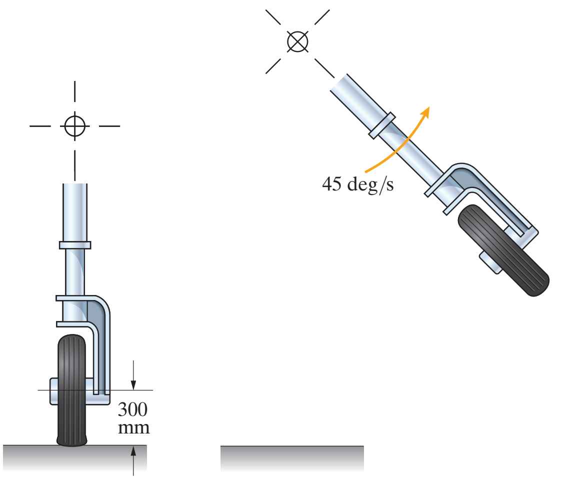
Equation 20.19 now gives $(d\omega_x/dt = d\omega_y/dt = d\omega_z/dt = 0)$

$$\begin{Bmatrix} 3.924\text{ N} \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & \omega_0 \\ 0 & 0 & 0 \\ -\omega_0 & 0 & 0 \end{bmatrix}\begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}\begin{Bmatrix} 0 \\ \omega_0 \\ 0.4\omega_0 \end{Bmatrix}$$

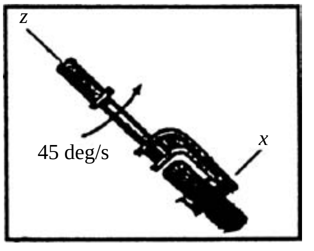
Putting in the values and solving we find

$$\boxed{\omega_0 = 10.7\text{ rad/s}}$$

Problem 20.58 The view of an airplane's landing gear as seen looking from behind the airplane is shown in Fig. (a). The radius of the wheel is 300 mm, and its moment of inertia is 2 kg-m^2. The airplane takes off at 30 m/s. After takeoff, the landing gear retracts by rotating toward the right side of the airplane, as shown in Fig. (b). Determine the magnitude of the couple exerted by the wheel on its support. (Neglect the airplane's angular motion.)



Solution: Choose a coordinate system with the origin at the center of mass of the wheel and the z axis aligned with the carriage, as shown. Assume that the angular velocities are constant, so that the angular accelerations are zero. The moments and products of inertia of the wheel are $I_{xx} = mR^2/2 = 2$ kg-m^2, from which $m = 44.44$ kg.



$I_{yy} = I_{zz} = mR^2/4 = 1$ kg-m^2.

The angular velocities are

$\mathbf{\Omega} = -(45(\pi/180))\mathbf{j} = -0.7853\mathbf{j}$ rad/s.

$$\boldsymbol{\omega} = -\left(\frac{v}{R}\right)\mathbf{i} + \mathbf{\Omega} = -\left(\frac{30}{0.3}\right)\mathbf{i} - 0.7853\mathbf{j} = -100\mathbf{i} - 0.7853\mathbf{j}.$$

Eq. (20.19) becomes

$$\begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} 0 & 0 & \Omega_y \\ 0 & 0 & 0 \\ -\Omega_y & 0 & 0 \end{bmatrix} \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ 0 \end{bmatrix}$$

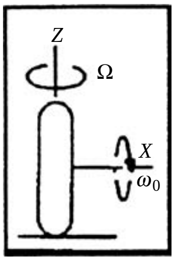
$$= \begin{bmatrix} 0 \\ 0 \\ -\Omega_y \omega_x I_{xx} \end{bmatrix},$$

from which $\mathbf{M}_0 = -\Omega_y \omega_x I_{xx} \mathbf{k}$. Substitute:

$$\boxed{|\mathbf{M}| = (0.7854)(100)2 = 157 \text{ N-m}}$$

Problem 20.59 If the rider turns to his left, will the couple exerted on the motorcycle by its wheels tend to cause the motorcycle to lean toward the rider's left side or his right side?

Solution: Choose a coordinate system as shown in the front view, with y positive into the paper. The Eqs. (20.19) in condensed notation are



$$\sum \mathbf{M} = \frac{d\mathbf{H}}{dt} + \mathbf{\Omega} \times \mathbf{H}.$$

For $\dfrac{d\mathbf{H}}{dt} = 0,$

$$\sum \mathbf{M} = \mathbf{\Omega} \times \mathbf{H}.$$

If the rider turns to his left, the angular velocity is $\mathbf{\Omega} = +\Omega\mathbf{k}$ rad/s. The angular momentum is $\mathbf{H} = \mathrm{H}_x\mathbf{i} + \mathrm{H}_z\mathbf{k}$, where $H_x > 0$. The cross product

$$\mathbf{\Omega} \times \mathbf{H} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & +\Omega \\ \mathrm{H}_x & 0 & \mathrm{H}_z \end{bmatrix} = +\Omega \mathrm{H}_x\mathbf{j}.$$

For a left turn the moment about y is positive, causing the cycle *to lean to the left*.

Problem 20.60* By substituting the components of H_O from Eqs. (20.9) into the equation

$$\Sigma \mathbf{M}_O = \frac{dH_{Ox}}{dt}\mathbf{i} + \frac{dH_{Oy}}{dt}\mathbf{j} + \frac{dH_{Oz}}{dt}\mathbf{k} + |\boldsymbol{\Omega}| \times \mathbf{H}_O$$

derive Eqs. (20.12).

Solution:

$$\sum \mathbf{M}_0 = \frac{dH_{Ox}}{dt}\mathbf{i} + \frac{dH_{Oy}}{dt}\mathbf{j} + \frac{dH_{Oz}}{dt}\mathbf{k}$$

$$+ \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \Omega_x & \Omega_y & \Omega_z \\ H_{Ox} & H_{Oy} & H_{Oz} \end{vmatrix}.$$

The components of this equation are

$$\sum M_{Ox} = \frac{dH_{Ox}}{dt} + \Omega_y H_{Oz} - \Omega_z H_{Oy},$$

$$\sum M_{Oy} = \frac{dH_{Oy}}{dt} - \Omega_x H_{Oz} + \Omega_z H_{Ox},$$

$$\sum M_{Oz} = \frac{dH_{Oz}}{dt} + \Omega_x H_{Oy} - \Omega_y H_{Ox}.$$

Substituting Eqs. (20.9) and *assuming that the moments and products of inertia are constants*, we obtain Eqs. (20.12):

$$\sum M_{Ox} = I_{xx}\frac{d\omega_x}{dt} - I_{xy}\frac{d\omega_y}{dt} - I_{xz}\frac{d\omega_z}{dt}$$

$$+ \Omega_y(-I_{zx}\omega_x - I_{zy}\omega_y + I_{zz}\omega_z)$$

$$- \Omega_z(-I_{yx}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z),$$

$$\sum M_{Oy} = -I_{yx}\frac{d\omega_x}{dt} + I_{yy}\frac{d\omega_y}{dt} - I_{yz}\frac{d\omega_z}{dt}$$

$$- \Omega_x(-I_{zx}\omega_x - I_{zy}\omega_y + I_{zz}\omega_z)$$

$$+ \Omega_z(I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z),$$

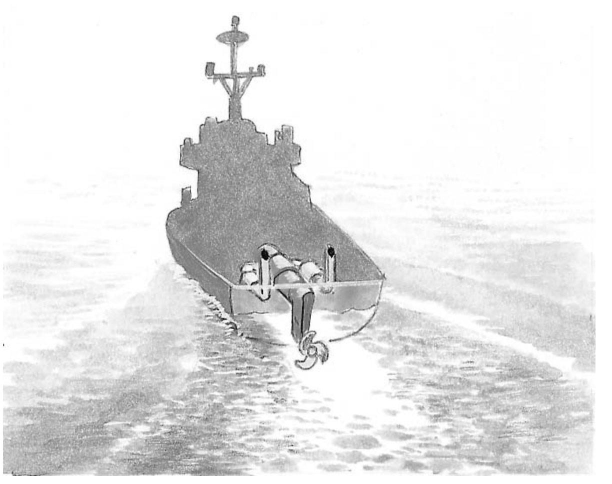
$$\sum M_{Oz} = -I_{zx}\frac{d\omega_x}{dt} - I_{zy}\frac{d\omega_y}{dt} + I_{zz}\frac{d\omega_z}{dt}$$

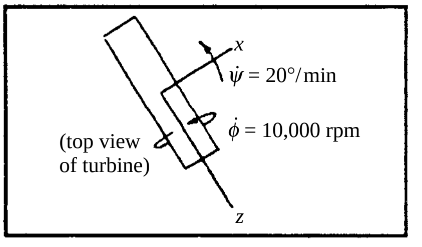
$$+ \Omega_x(-I_{yx}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z)$$

$$- \Omega_y(I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z).$$

Problem 20.61 A ship has a turbine engine. The spin axis of the axisymmetric turbine is horizontal and aligned with the ship's longitudinal axis. The turbine rotates at 10,000 rpm. Its moment of inertia about its spin axis is 1000 kg-m^2. If the ship turns at a constant rate of 20 degrees per minute, what is the magnitude of the moment exerted on the ship by the turbine?

Strategy: Treat the turbine's motion as steady precession with nutation angle $\theta = 90^\circ$.





Solution: Choose a coordinate system with the z axis parallel to the axis of the turbine, and y positive upward. From Eq. (20.29),

$$\sum M_x = (I_{zz} - I_{xx})\dot{\psi}^2 \sin\theta\cos\theta + I_{zz}\dot{\phi}\dot{\psi}\sin\theta,$$

where $I_{xx} = \frac{1}{2} I_{zz} = 500$ kg-m^2

$$\dot{\psi} = 20\left(\frac{\pi}{180}\right)\left(\frac{1}{60}\right) = 0.005818 \text{ (rad/s)},$$

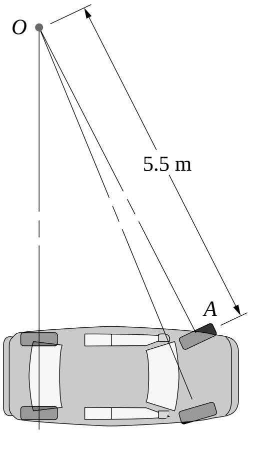
$$\dot{\phi} = 10000(2\pi/60) = 1047.2 \text{ rad/s},$$

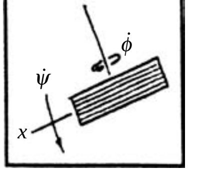
$$\theta = 90^\circ.$$

$$M_x = 6092 \text{ N-m}$$

Problem 20.62 The center of the car's wheel A travels in a circular path about O at 24.1 km/h. The wheel's radius is 0.31 m, and the moment of inertia of the wheel about its axis of rotation is 1.08 kg-m^2. What is the magnitude of the total external moment about the wheel's center of mass?

Strategy: Treat the wheel's motion as steady precession with nutation angle $\theta = 90^\circ$.





Solution: From Eq. (20.29)

$$\sum M_x = (I_{zz} - I_{xx})\dot{\psi}^2 \sin\theta\cos\theta + I_{zz}\dot{\phi}\dot{\psi}\sin\theta,$$

where the spin is

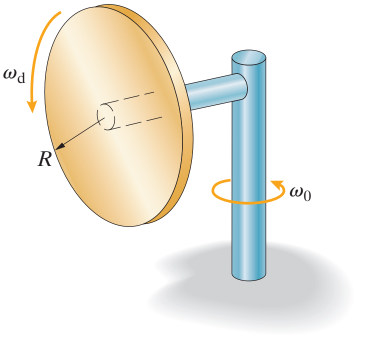
$$\dot{\phi} = \frac{v}{R} = \left(\frac{24.1 \times 1000}{5.5 \times 3600}\right) = 1.222 \text{ rad/s}$$

the precession rate is

$$\dot{\psi} = \frac{v}{R_w} = \frac{6.7}{0.31} = 22 \text{ rad/s},$$

and the nutation angle is $\theta = 90^\circ$. Using $I_{xx} = \frac{1}{2} I_{zz} = 0.542$ kg-m^2, from which $M_x = 29.2$ N-m

Problem 20.63 The radius of the 5-kg disk is $R = 0.2$ m. The disk is pinned to the horizontal shaft and rotates with constant angular velocity $\omega_d = 6$ rad/s relative to the shaft. The vertical shaft rotates with constant angular velocity $\omega_0 = 2$ rad/s. By treating the motion of the disk as steady precession, determine the magnitude of the couple exerted on the disk by the horizontal shaft.



Solution: In Problem 20.50 a thin circular disk of mass m is mounted on a horizontal shaft and rotates relative to the shaft with constant angular velocity ω_d. The horizontal shaft is rigidly attached to the vertical shaft rotating with constant angular velocity ω_0. The magnitude of the couple exerted on the disk by the horizontal shaft is to be determined. The nutation angle is $\theta = 90^\circ$. The precession rate is $\dot{\psi} = \omega_0$, and the spin rate is $\dot{\phi} = \omega_d$. The moments and products of inertia of the disk:

$$I_{xx} = I_{zz} = \frac{mR^2}{2},$$

$$I_{yy} = \frac{mR^2}{4},$$

$$I_{xy} = I_{xz} = I_{yz} = 0.$$

Eq. (20.29) is

$$M_y = (I_{xx} - I_{yy})\dot{\psi}^2 \sin\theta \cos\theta + I_{xx}\dot{\psi}\dot{\phi}\sin\theta,$$

from which

$$M_y = \frac{mR^2}{2}\omega_0\omega_d = 1.2 \text{ N-m}.$$

Problem 20.64 The helicopter is stationary. The z axis of the body-fixed coordinate system points downward and is coincident with the axis of the helicopter's rotor. The moment of inertia of the rotor about the z axis is 8600 kg-m^2. Its angular velocity is $-258\mathbf{k}$ (rpm). If the helicopter begins a pitch maneuver during which its angular velocity is $0.02\mathbf{j}$ (rad/s), what is the magnitude of the gyroscopic moment exerted on the helicopter by the rotor? Does the moment tend to cause the helicopter to roll about the x axis in the clockwise direction (as seen in the photograph) or the counterclockwise direction?

Solution: The spin rate is $\dot{\phi} = -258 \text{ rpm} = -27.0$ rad/s.

The pitch rate is $\dot{\psi} = 0.02$ rad/s.

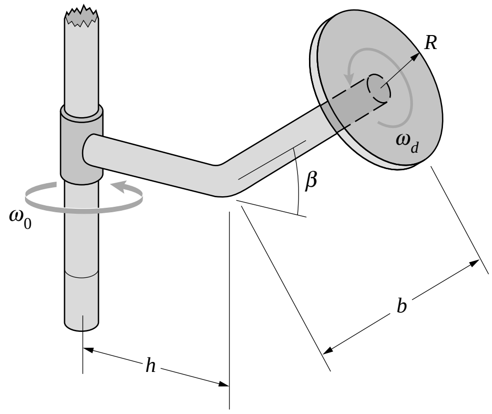
In eq. 20.29, the moment exerted *on the rotor* is

$$M = I_{zz}\dot{\phi}\dot{\psi} = (8600 \text{ kg-m}^2)(-27.0 \text{ rad/s})(0.02 \text{ rad/s}) = -4650 \text{ N-m}$$

The motor exerts a moment on the helicopter in the opposite direction which tends to roll the helicopter in the counterclockwise direction.

Answer: $\boxed{M = 4650 \text{ N-m counterclockwise}}$

Problem 20.65 The bent bar is rigidly attached to the vertical shaft, which rotates with constant angular velocity ω_0. The disk of mass m and radius R is pinned to the bent bar and rotates with constant angular velocity ω_d relative to the bar. Determine the magnitudes of the force and couple exerted on the disk by the bar.



Solution:

(a) The center of mass of the disk moves in a horizontal circular path of radius $h + b\cos\beta$ with angular velocity ω_0. The acceleration normal to the circular path is $a_N = \omega_0^2(h + b\cos\beta)$, so the bar exerts a horizontal force of magnitude $ma_N = m\omega_0^2(h + b\cos\beta)$. The bar also exerts on upward force equal to the weight of the disk, so the magnitude of the total force is

$$\sqrt{(ma_N)^2 + (mg)^2} = m\sqrt{\omega_0^4(h + b\cos\beta)^2 + g^2}.$$

(b) By orienting a coordinate system as shown, with the z axis normal to the disk and the x axis horizontal, the disk is in steady precession with precession rate $\dot{\psi} = \omega_0$, spin rate $\dot{\phi} = \omega_d$, and nutation angle

$$\theta = \frac{\pi}{2} - \beta.$$

The plate's moments of inertia are

$$I_{xx} = I_{yy} = \tfrac{1}{4}mR^2,$$

$$I_{zz} = \frac{1}{2}mR^2.$$

From Equation (20.29), the magnitude of the moment is

$$(I_{zz} - I_{xx})\dot{\psi}^2 \sin\theta\cos\theta + I_{zz}\dot{\phi}\dot{\psi}\sin\theta$$

$$= \frac{1}{4}mR^2\omega_0^2 \sin\left(\frac{\pi}{2} - \beta\right)\cos\left(\frac{\pi}{2} - \beta\right)$$

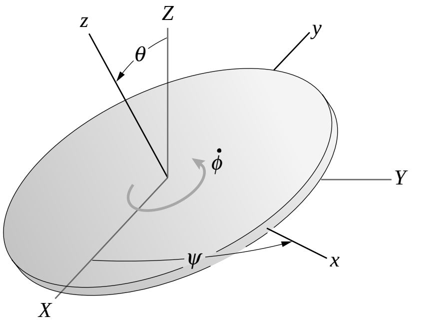
$$+\frac{1}{2}mR^2\omega_d\omega_0 \sin\left(\frac{\pi}{2} - \beta\right)$$

$$= R^2\omega_0 m\left(\frac{1}{4}\omega_0\cos\beta\sin\beta + \frac{1}{2}\omega_d\cos\beta\right)$$

Problem 20.66 The bent bar is rigidly attached to the vertical shaft, which rotates with constant angular velocity ω_0. The disk of mass m and radius R is pinned to the bent bar and rotates with constant angular velocity ω_d relative to the bar. Determine the value of ω_d for which no couple is exerted on the disk by the bar.

Solution: From the result for the magnitude of the moment in the solution of Problem 20.65 the moment equals zero if $\frac{1}{4}\omega_0\sin\beta + \frac{1}{2}\omega_d = 0$, so $\omega_d = -\frac{1}{2}\omega_0\sin\beta$.

Problem 20.67 A thin circular disk undergoes moment-free steady precession. The z axis is perpendicular to the disk. Show that the disk's precession rate is $\psi = -2\phi/\cos\theta$. (Notice that when the nutation angle is small, the precession rate is approximately two times the spin rate.)



Solution: Moment free steady precession is described by Eq. (20.33), $(I_{zz} - I_{xx})\dot{\psi}\cos\theta + I_{zz}\dot{\phi} = 0$, where $\dot{\psi}$ is the precession rate, $\dot{\phi}$ is the spin rate, and θ is the nutation angle. For a thin circular disk, the moments and products of inertia are

$$I_{xx} = I_{yy} = \frac{mR^2}{4},$$

$$I_{zz} = \frac{mR^2}{2},$$

$$I_{xy} = I_{xz} = I_{yz} = 0.$$

Substitute:

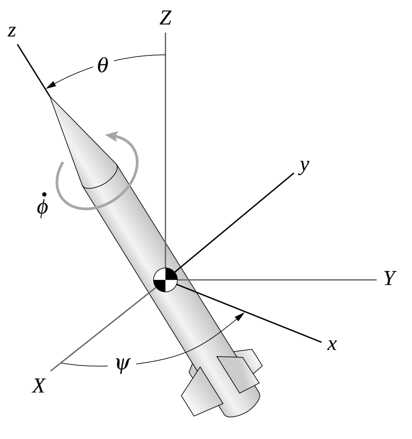
$$mR^2\left(\frac{1}{2} - \frac{1}{4}\right)\dot{\psi}\cos\theta + \left(\frac{mR^2}{2}\right)\dot{\phi} = 0.$$

Reduce, to obtain

$$\dot{\psi} = -\frac{2\dot{\phi}}{\cos\theta}.$$

When the nutation angle is small, $\theta \to 0$, $\cos\theta \to 1$, and $\dot{\psi} \cong -2\dot{\phi}$.

Problem 20.68 The rocket is in moment-free steady precession with nutation angle $\theta = 40^\circ$ and spin rate $\dot{\phi} = 4$ revolutions per second. Its moments of inertia are $I_{xx} = 10{,}000$ kg-m^2 and $I_{zz} = 2000$ kg-m^2. What is the rocket's precession rate $\dot{\psi}$ in revolutions per second?



Solution: Moment-free steady precession is described by Eq. (20.33), $(I_{zz} - I_{xx})\dot{\psi}\cos\theta + I_{zz}\dot{\phi} = 0$, where $\dot{\psi}$ is the precession rate, $\dot{\phi}$ is the spin rate, and θ is the nutation angle. Solve for the precession rate:

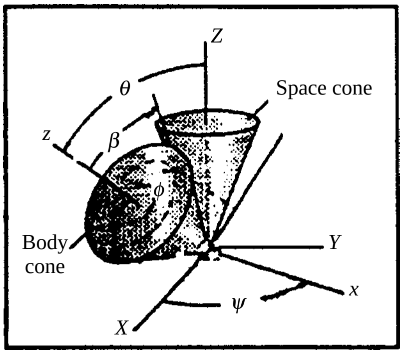
$$\dot{\psi} = \frac{I_{zz}\dot{\phi}}{(I_{xx} - I_{zz})\cos\theta} = 1.31 \text{ rev/s}.$$

Problem 20.69 Sketch the body and space cones for the motion of the rocket in Problem 20.68.

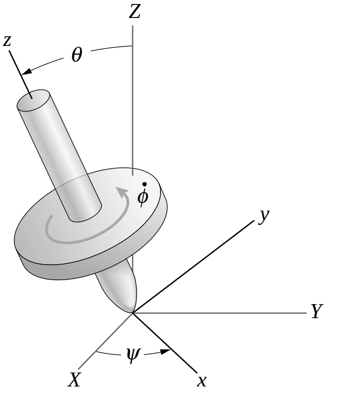
Solution: The angle $\theta = 40^\circ$. The angle β defined by

$$\beta = \tan^{-1}\left[\left(\frac{I_{zz}}{I_{xx}}\right)\tan\theta\right] = 9.53^\circ$$

satisfies the condition $\beta < \theta$. The body cone with an axis along the z axis, rolls on a space cone with axis on the Z axis. The result is shown.

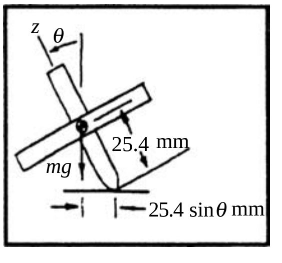


Problem 20.70 The top is in steady precession with nutation angle $\theta = 15^\circ$ and precession rate $\dot{\psi} = 1$ revolution per second. The mass of the top is 0.012 kg, its center of mass is 25.4 mm from the point, and its moments of inertia are $I_{xx} = 8.13 \times 10^{-6}$ kg-m^2 and $I_{zz} = 2.71 \times 10^{-6}$ kg-m^2. What is the spin rate ϕ of the top in revolutions per second?



Solution: The steady precession is not moment-free, since the weight of the top exerts a moment

$$M_x = 0.0254 \text{ mg} \sin\theta.$$



The motion of a spinning top is described by Eq. (20.32),

$$mgh = (I_{zz} - I_{xx})\dot{\psi}^2\cos\theta + I_{zz}\dot{\psi}\dot{\phi},$$

where $\dot{\psi}$ is the rate of precession, $\dot{\phi}$ is the spin rate, and θ is the nutation angle and

$$h = 0.0254 \text{ m}$$

is the distance from the point to the center of mass. Solve:

$$\dot{\phi} = \frac{0.0254 \text{ mg} - (I_{zz} - I_{xx})\dot{\psi}^2\cos\theta}{I_{zz}\dot{\psi}}.$$

Substitute numerical values (using $\dot{\psi} = 2\pi$ rad/s for dimensional consistency) to obtain $\dot{\phi} = 182.8$ rad/s, from which $\dot{\phi} = 29.1$ rev/s.

Problem 20.71 Suppose that the top described in Problem 20.70 has a spin rate $\dot{\phi} = 15$ revolutions per second. Draw a graph of the precession rate (in revolutions per second) as a function of the nutation angle θ for values of θ from zero to 45°.

Solution: The behavior of the top is described in Eq. (20.32),

$$mgh = (I_{xx} - I_{yy})\dot{\psi}^2 \cos\theta + I_{xx}\dot{\psi}\dot{\phi},$$

where $\dot{\psi}$ is the rate of precession, $\dot{\phi}$ is the spin rate, and θ is the nutation angle and $h = 0.254$ m is the distance from the point to the center of mass. Rearrange: $(I_{zz} - I_{xx})\dot{\psi}^2 \cos\theta + I_{zz}\dot{\psi}\dot{\phi} - mgh = 0$. The velocity of the center of the base is

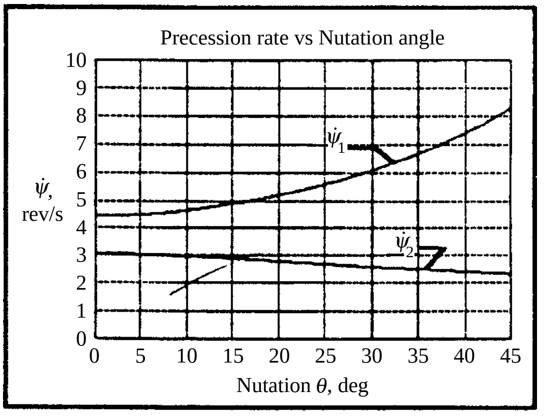
$$v = -\frac{\omega_0 h}{4},$$

from which the spin axis is the z axis and the spin *rate is*

$$\dot{\phi} = \frac{v}{R} = -\frac{\omega_0 h}{4R}.$$

The solution, $\dot{\psi}_{1,2} = -b \pm \sqrt{b^2 - c}$.

The two solutions, which are real over the interval, are graphed as a function of θ over the range $0 \le \theta \le 45°$. The graph is shown.



Problem 20.72 The rotor of a tumbling gyroscope can be modeled as being in moment-free steady precession. The moments of inertia of the gyroscope are $I_{xx} = I_{yy} = 0.04$ kg-m^2 and $I_{zz} = 0.18$ kg-m^2. The gyroscope's spin rate is $\dot{\phi} = 1500$ rpm and its nutation angle is $\theta = 20°$.

(a) What is the precession rate of the gyroscope in rpm?
(b) Sketch the body and space cones.

Solution:

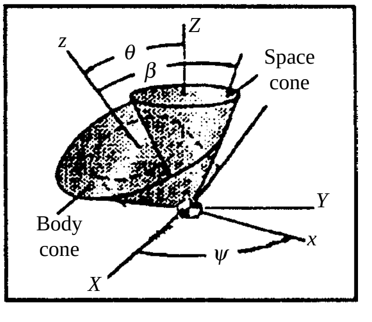
(a) The motion in moment-free, steady precession is described by Eq. (20.33), $(I_{zz} - I_{xx})\dot{\psi}\cos\theta + I_{zz}\dot{\phi} = 0$, where $\dot{\psi}$ is the precession rate, $\dot{\phi} = 1500$ rpm is the spin rate, and $\theta = 20°$ is the nutation angle.

Solve: $\dot{\psi} = -\dfrac{I_{zz}\dot{\phi}}{(I_{zz} - I_{xx})\cos\theta} = -2050$ rpm.

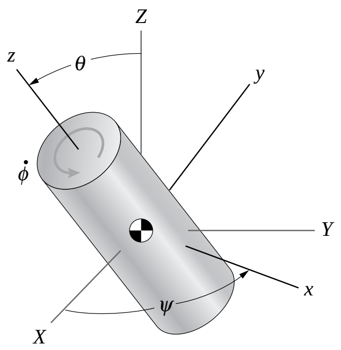
(b) The apex angle for the body cone is given by

$$\tan\beta = \left(\frac{I_{zz}}{I_{xx}}\right)\tan\theta,$$

from which $\beta = 58.6°$. Since $\beta > \theta$, the space cone lies inside the body cone as the figure.



Problem 20.73 A satellite can be modeled as an 800-kg cylinder 4 m in length and 2 m in diameter. If the nutation angle is $\theta = 20^\circ$ and the spin rate $\dot{\phi}$ is one revolution per second, what is the satellite's precession rate $\dot{\psi}$ in revolutions per second?



Solution: From Appendix C, the moments and products of inertia of a homogenous cylinder are

$$I_{xx} = I_{yy} = m\left(\frac{L^2}{12} + \frac{R^2}{4}\right) = 1267 \text{ kg-m}^2,$$

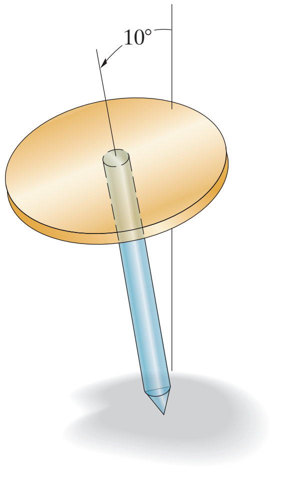
$$I_{zz} = mR^2/2 = 400 \text{ kg-m}^2.$$

$$I_{xy} = I_{xz} = I_{yz} = 0.$$

The angular motion of an axisymmetric moment-free object in steady precession is described by Eq. (20.33), $(I_{zz} - I_{xx})\dot{\psi}\cos\theta + I_{zz}\dot{\phi} = 0$, where $\dot{\psi}$ is the precession rate, $\theta = 20^\circ$ is the nutation angle, and $\dot{\phi} = 1$ rps is the spin rate. Solve:

$$\dot{\psi} = -\frac{I_{zz}\dot{\phi}}{(I_{zz} - I_{xx})\cos\theta} = 0.49 \text{ rps.}$$

Problem 20.74 The top consists of a thin disk bonded to a slender bar. The radius of the disk is 30 mm and its mass is 0.008 kg. The length of the bar is 80 mm and its mass is negligible compared to the disk. When the top is in steady precession with a nutation angle of 10°, the precession rate is observed to be 2 revolutions per second in the same direction the top is spinning. What is the top's spin rate?



Solution: The inertias about the base are

$$I_{zz} = \frac{1}{2}(0.008 \text{ kg})(0.03 \text{ m})^2 = 3.6 \times 10^{-6} \text{ kg-m}^2$$

$$I_{xx} = I_{yy} = \frac{1}{2}(0.008 \text{ kg})(0.03 \text{ m})^2$$

$$+ (0.008 \text{ kg})(0.08 \text{ m})^2 = 53 \times 10^{-6} \text{ kg-m}^2$$

The precession rate is

$$\dot{\psi} = 2(2\pi) = 12.6 \text{ rad/s}$$

The moment about the base is

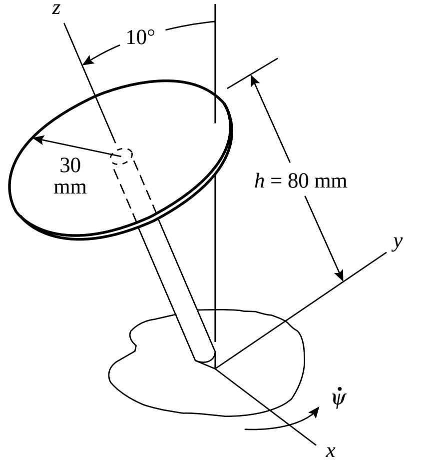
$$\mathbf{M} = M_x\mathbf{i} = (0.008 \text{ kg})(9.81 \text{ m/s}^2)(0.08 \text{ m})\mathbf{i} = (6.28 \times 10^{-3} \text{ N-m})\mathbf{i}$$

Eq. 20.32 is

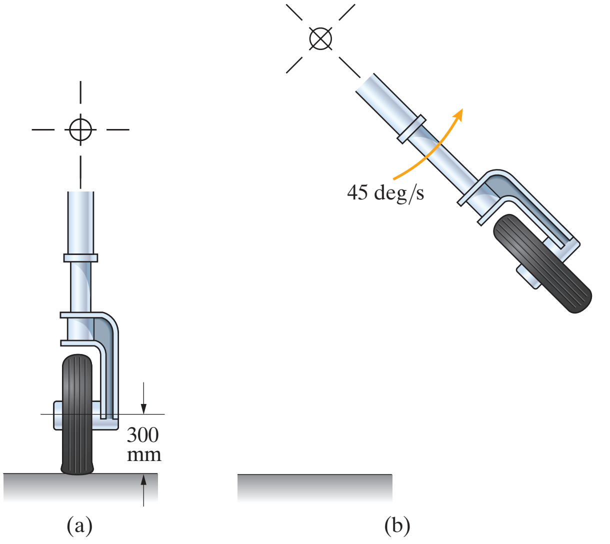
$$M_x = (I_{zz} - I_{xx})\dot{\psi}^2\cos 10^\circ + I_{zz}\dot{\psi}\dot{\phi}$$

Solving we find

$$\boxed{\dot{\phi} = 309 \text{ rad/s} \quad (49.1 \text{ rev/s})}$$

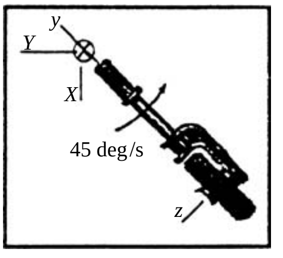


Problem 20.75 Solve Problem 20.58 by treating the motion as steady precession.



Solution: The view of an airplane's landing gear looking from behind the airplane is shown in Fig. (a). The radius of the wheel is 300 mm and its moment of inertia is 2 kg-m^2. The airplane takes off at 30 m/s. After takeoff, the landing gear retracts by rotating toward the right side of the airplane as shown in Fig. (b). The magnitude of the couple exerted by the wheel on its support is to be determined.

Choose X, Y, Z with the Z axis parallel to the runway, X perpendicular to the runway, and Y parallel to the runway. Choose the x, y, z coordinate system with the origin at the center of mass of the wheel and the z axis aligned with the direction of the axis of rotation of the wheel and the y axis positive upward. The Eq. (20.29) is



$$\sum M_x = (I_{zz} - I_{xx})\dot{\psi}^2 \sin\theta \cos\theta + I_{zz}\dot{\phi}\dot{\psi} \sin\theta.$$

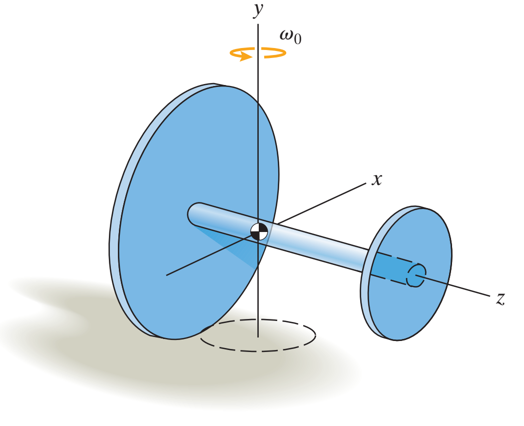
The nutation angle and rates of precession: The nutation angle is the angle between Z and z, $\theta = 90^\circ$. The precession angle is the angle between the X and x, which is increasing in value, from which $\dot{\psi} = 45^\circ/\text{s} = 0.7853$ rad/s. The spin vector is aligned with the z axis, from which

$$\dot{\phi} = \left(\frac{v}{R}\right) = \left(\frac{30}{0.3}\right) = 100 \text{ rad/s}$$

The moments and products of inertia of the wheel are $I_{zz} = mR^2/2 = 2$ kg-m^2. The moment is

$$M_x = I_{zz}\dot{\psi}\dot{\phi} \sin 90^\circ = 2(0.7854)(100) = 157 \text{ N-m}.$$

Problem 20.76* The two thin disks are rigidly connected by a slender bar. The radius of the large disk is 200 mm and its mass is 4 kg. The radius of the small disk is 100 mm and its mass is 1 kg. The bar is 400 mm in length and its mass is negligible. The composite object undergoes a steady motion in which it spins about the vertical y axis through its center of mass with angular velocity ω_0. The bar is horizontal during this motion and the large disk rolls on the floor. Determine ω_0 by treating the motion as steady precession.



Solution: Use the data from 20.57

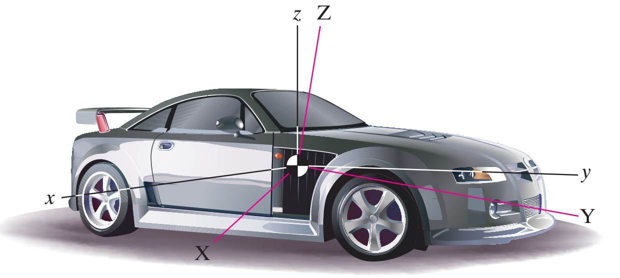
Set the nutation angle to 90° and the precession rate to ω_0

$$3.924 \text{ N-m} = (0.085 \text{ kg-m}^2)(0.4\omega_0)\omega_0$$

Solving we obtain

$$\boxed{\omega_0 = 10.7 \text{ rad/s}}$$

Problem 20.77* Suppose that you are testing a car and use accelerometers and gyroscopes to measure its Euler angles and their derivatives relative to a reference coordinate system. At a particular instant, $\psi = 15^\circ$, $\theta = 4^\circ$, $\phi = 15^\circ$, the rates of change of the Euler angles are zero, and their second derivatives with respect to time are $\ddot{\psi} = 0$, $\ddot{\theta} = 1$ rad/s^2, and $\ddot{\phi} = -0.5$ rad/s^2. The car's principal moments of inertia, in kg-m^2, are $I_{xx} = 2200$, $I_{yy} = 480$, and $I_{zz} = 2600$. What are the components of the total moment about the car's center of mass?



Solution: The description of the motion of an arbitrarily shaped object is given by the Eqs. (20.36):

$$M_x = I_{xx}(\ddot{\psi}\sin\theta\sin\phi + \ddot{\theta}\cos\phi + \dot{\psi}\dot{\theta}\cos\theta\sin\phi$$

$$+ \dot{\psi}\dot{\phi}\sin\theta\cos\phi - \dot{\theta}\dot{\phi}\sin\phi)$$

$$- (I_{yy} - I_{zz})(\dot{\psi}\sin\theta\cos\phi - \dot{\theta}\sin\phi)(\dot{\psi}\cos\theta + \dot{\phi}),$$

$$M_y = I_{yy}(\ddot{\psi}\sin\theta\cos\phi - \ddot{\theta}\sin\phi + \dot{\psi}\dot{\theta}\cos\theta\cos\phi$$

$$- \dot{\psi}\dot{\phi}\sin\theta\sin\phi - \dot{\theta}\dot{\phi}\cos\phi)$$

$$- (I_{zz} - I_{xx})(\dot{\psi}\sin\theta\sin\phi + \dot{\theta}\cos\phi)(\dot{\psi}\cos\theta + \dot{\phi}),$$

$$M_z = I_{zz}(\ddot{\psi}\cos\theta + \ddot{\phi} - \dot{\psi}\dot{\theta}\sin\theta) - (I_{xx} - I_{yy})(\dot{\psi}\sin\theta\sin\phi$$

$$+ \dot{\theta}\cos\phi)(\dot{\psi}\sin\theta\cos\phi - \dot{\theta}\sin\phi).$$

Substitute $\ddot{\psi} = 0$, $\dot{\psi} = \dot{\theta} = \dot{\phi} = 0$, to obtain $M_x = I_{xx}\ddot{\theta}\sin\phi$, $M_y = -I_{yy}\ddot{\theta}\sin\phi$, $M_z = I_{zz}\ddot{\phi}$. Substitute values:

$M_x = 2125$ N-m,

$M_y = -124.2$, $M_z = -1300$ N-m

Problem 20.78* If the Euler angles and their second derivatives for the car described in Problem 20.77 have the given values, but their rates of change are $\dot{\psi} = 0.2$ rad/s, $\dot{\theta} = -2$ rad/s, and $\dot{\phi} = 0$, what are the components of the total moment about the car's center of mass?

Solution: Use Eqns. (20.36)

$$M_x = I_{xx}(\ddot{\psi}\sin\theta\sin\phi + \ddot{\theta}\cos\phi + \dot{\psi}\dot{\theta}\cos\theta\sin\phi$$

$$+ \dot{\psi}\dot{\phi}\sin\theta\cos\phi - \dot{\theta}\dot{\phi}\sin\phi)$$

$$- (I_{yy} - I_{zz})(\dot{\psi}\sin\theta\cos\phi - \dot{\theta}\sin\phi)(\dot{\psi}\cos\theta + \dot{\phi}),$$

$$M_y = I_{yy}(\ddot{\psi}\sin\theta\cos\phi - \ddot{\theta}\sin\phi + \dot{\psi}\dot{\theta}\cos\theta\cos\phi$$

$$- \dot{\psi}\dot{\phi}\sin\theta\sin\phi - \dot{\theta}\dot{\phi}\cos\phi)$$

$$- (I_{zz} - I_{xx})(\dot{\psi}\sin\theta\sin\phi + \dot{\theta}\cos\phi)(\dot{\psi}\cos\theta + \dot{\phi}),$$

$$M_z = I_{zz}(\ddot{\psi}\cos\theta + \ddot{\phi} - \dot{\psi}\dot{\theta}\sin\theta) - (I_{xx} - I_{yy})(\dot{\psi}\sin\theta\sin\phi$$

$$+ \dot{\theta}\cos\phi)(\dot{\psi}\sin\theta\cos\phi - \dot{\theta}\sin\phi)$$

Substitute values:

$M_x = 2123$ N-m,

$M_y = -155.4$ N-m,

$M_z = 534$ N-m.

Problem 20.79* Suppose that the Euler angles of the car described in Problem 20.77 are $\psi = 40^\circ$, $\theta = 20^\circ$, and $\phi = 5^\circ$, their rates of change are zero, and the components of the total moment about the car's center of mass are

$$\sum M_x = -400 \text{ N-m},$$

$$\sum M_y = 200 \text{ N-m},$$

$$\sum M_z = 0.$$

What are the x, y, and z components of the car's angular acceleration?

Solution: Eq. (20.36) simplifies when $\dot{\psi} = \dot{\phi} = \dot{\theta} = 0$ to

$$M_x = I_{xx}(\ddot{\psi} \sin\theta \sin\phi + \ddot{\theta} \cos\phi),$$

$$M_y = I_{yy}(\ddot{\psi} \sin\theta \cos\phi - \ddot{\theta} \sin\phi),$$

$$M_z = I_{zz}(\ddot{\psi} \cos\theta + \ddot{\phi}).$$

These three simultaneous equations have the solutions,

$$\ddot{\theta} = \frac{M_x}{I_{xx}} \cos\phi - \frac{M_y}{I_{yy}} \sin\phi = -0.2174 \text{ rad/s}^2,$$

$$\ddot{\psi} = \left(\frac{M_{xx}}{I_{xx}}\right) \frac{\sin\phi}{\sin\theta} + \left(\frac{M_y}{I_{yy}}\right) \frac{\cos\phi}{\sin\theta} = 1.167 \text{ rad/s}^2,$$

$$\ddot{\phi} = (M_z/I_{zz}) - \ddot{\psi} \cos\theta = -1.097 \text{ rad/s}^2.$$

From Eq. (20.35), when

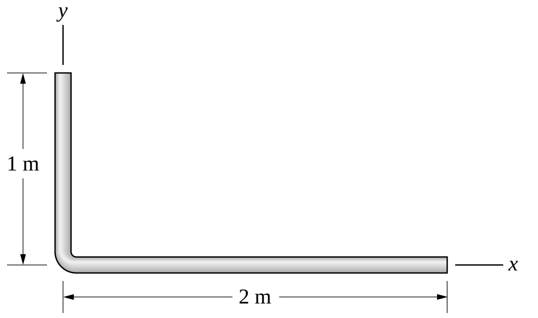
$$\dot{\psi} = \dot{\phi} = \dot{\theta} = 0:$$

$$\frac{d\omega_x}{dt} = \ddot{\psi} \sin\theta \sin\phi + \ddot{\theta} \cos\phi = -0.1818 \text{ rad/s}^2,$$

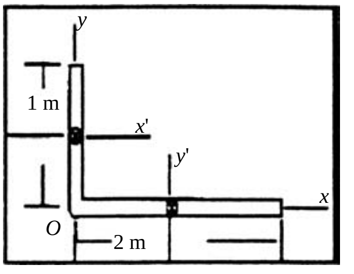
$$\frac{d\omega_y}{dt} = \ddot{\psi} \sin\theta \cos\phi - \ddot{\theta} \sin\phi = 0.417 \text{ rad/s},$$

$$\frac{d\omega_z}{dt} = \ddot{\psi} \cos\phi + \ddot{\phi} = 0.$$

Problem 20.80 The mass of the bar is 6 kg. Determine the moments and products of inertia of the bar in terms of the coordinate system shown.



Solution: One strategy (not the simplest, see *Check* note following the solution) is to determine the moment of inertia matrix for each element of the bar, and then to use the parallel axis theorem to transfer each to the coordinate system shown.



(a) *The vertical element Oy of the bar*. The mass density per unit volume is $\rho = \dfrac{6}{3A}$ kg/m^3, where A is the (unknown) cross section of the bar, from which $\rho A = 2$ kg/m. The element of mass is $dm = \rho A dL$, where dL is an element of length. The mass of the vertical element is $m_v = \rho A L_v = 2$ kg, where $L_v = 1$ m. From Appendix C the moment of inertia about an x' axis passing through the center of mass is

$$I^{(1)}_{x'x'} = \frac{m_v L_v^2}{12} = 0.1667 \text{ kg-m}^2.$$

Since the bar is slender, $I^{(1)}_{y'y'} = 0$.

$$I^{(1)}_{z'z'} = \frac{m_v L_v^2}{12} = 0.1667 \text{ kg-m}^2.$$

Since the bar is slender, the products of inertia vanish:

$$I^{(1)}_{x'y'} = \int_m x'y'dm = 0, \; I^{(1)}_{x'z'} = \int_m x'z'dm = 0,$$

$$I^{(1)}_{y'z'} = \int_m y'z'dm = 0,$$

from which the inertia matrix for the element Oy about the x' axis is

$$[I^{(1)}] = \begin{bmatrix} 0.1667 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.1667 \end{bmatrix} \text{ kg-m}^2.$$

(b) *The horizontal element Ox of the bar*. The mass of the horizontal element is $m_h = \rho A L_h = 4$ kg, where $L_h = 2$ m. From Appendix C the moments and products of inertia about the y' axis passing through the center of mass of the horizontal element are:

$$I^{(2)}_{x'x'} = 0,$$

$$I^{(2)}_{y'y'} = \frac{m_h L_h^2}{12} = 1.333 \text{ kg-m}^2,$$

$$I^{(2)}_{z'z'} = \frac{m_h L_h^2}{12} = 1.333 \text{ kg-m}^2.$$

Since the bar is slender, the cross products of inertia about the y' axis through the center of mass of the horizontal element of the bar vanish:

$$I^{(2)}_{x'y'} = 0, \; I^{(2)}_{x'z'} = 0, \; I^{(2)}_{y'z'} = 0.$$

The inertia matrix is

$$[I^{(2)}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1.333 & 0 \\ 0 & 0 & 1.333 \end{bmatrix} \text{ kg-m}^2.$$

Use the parallel axis theorem to transfer the moment of inertia matrix to the origin O: For the vertical element the coordinates of the center of mass O are $(d_x, d_y, d_z) = (0, 0.5, 0)$ m. Use the parallel axis theorem (see Eq. (20.42)).

$$I_{xx}^{(1)} = I_{x'x'}^{(1)} + (d_y^2 + d_z^2)m_v = \frac{m_v L_v^2}{12} + m_v(0.5^2) = 0.6667 \text{ kg-m}^2.$$

$$I_{yy}^{(1)} = I_{y'y'}^{(1)} + (d_x^2 + d_z^2)m_v = 0.$$

$$I_{zz}^{(1)} = I_{z'z'}^{(1)} + (d_x^2 + d_y^2)mv = 0.6667 \text{ kg-m}^2.$$

The products of inertia are

$$I_{xy}^{(1)} = I_{x'y'}^{(1)} + d_x d_y \rho A(1) = 0,$$

$$I_{xz}^{(1)} = I_{x'z'}^{(1)} + d_x d_z \rho A(1) = 0,$$

$$I_{yz}^{(1)} = I_{y'z'}^{(1)} + d_y d_z \rho A(1) = 0.$$

The inertia matrix for the vertical element:

$$[I^{(1)}] = \begin{bmatrix} \frac{\rho A}{3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\rho A}{3} \end{bmatrix}.$$

For the horizontal element, the coordinates of the center of mass relative to O are $(d_x, d_y, d_z) = (1, 0, 0)$ m. From the parallel axis theorem,

$$I_{xx}^{(2)} = I_{x'x'}^{(2)} + (d_y^2 + d_z^2)m_h = 0.$$

$$I_{yy}^{(2)} = I_{y'y'}^{(2)} + (d_x^2 + d_z^2)m_h = 5.333 \text{ kg-m}^2.$$

$$I_{zz}^{(2)} = I_{z'z'}^{(2)} + (d_x^2 + d_y^2)m_h = 5.333 \text{ kg-m}^2.$$

By inspection, the products of inertia vanish. The inertia matrix is

$$[I^{(2)}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 5.333 & 0 \\ 0 & 0 & 5.333 \end{bmatrix}.$$

Sum the two inertia matrices:

$$\boxed{[I]_O = [I^{(1)}] + [I^{(2)}] = \begin{bmatrix} 0.6667 & 0 & 0 \\ 0 & 5.333 & 0 \\ 0 & 0 & 6 \end{bmatrix} \text{ kg-m}^2.}$$

[*Check*: The moment of inertia in the coordinate system shown can be derived by insepection by taking the moment of inertia of each element about the origin: From Appendix C the moments of inertia about the origin of the slender bars are

$$I_{xx} = \frac{m_v L_v^3}{3}, I_{yy} = \frac{m_h L_h^2}{3},$$

and $I_{zz} = I_{xx} + I_{yy}$, where the subscripts v and h denote the vertical and horizontal bars respectively. Noting that the masses are

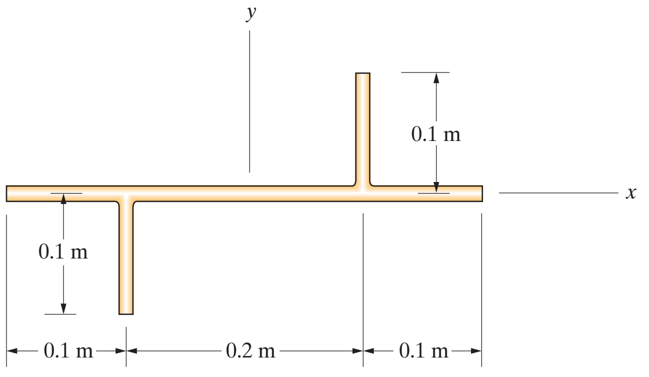
$$m_v = \frac{mL_v}{L_v + L_h}, m_h = \frac{mL_h}{L_v + L_h},$$

the moment of inertia matrix becomes:

$$[I] = \begin{bmatrix} \frac{6(1)^3}{3(3)} & 0 & 0 \\ 0 & 0 & \frac{6(2)^3}{3(3)} \\ 0 & 0 & 6.0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.6667 & 0 & 0 \\ 0 & 5.333 & 0 \\ 0 & 0 & 6 \end{bmatrix} \text{ kg-m}^2. \textit{ check.}$$

Problem 20.81 The object consists of two 1-kg vertical slender bars welded to a 4-kg horizontal slender bar. Determine its moments and products of inertia in terms of the coordinate system shown.



Solution:

$$I_{xx} = 2\frac{1}{3}(1\text{ kg})(0.1\text{ m})^2 = 0.00667\text{ kg-m}^2,$$

$$I_{yy} = \frac{1}{12}(4\text{ kg})(0.4\text{ m})^2 + 2(1\text{ kg})(0.1\text{ m})^2 = 0.0733\text{ kg-m}^2,$$

$$I_{zz} = I_{yy} + I_{zz} = 0.08\text{ kg-m}^2,$$

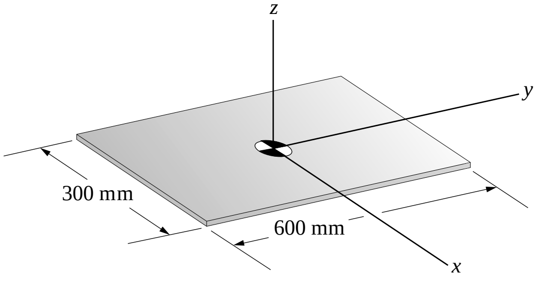
$$I_{xy} = 2(1\text{ kg})(0.1\text{ m})(0.05\text{ m}) = 0.01\text{ kg-m}^2,$$

$$I_{xz} = 0,$$

$$I_{yz} = 0.$$

$$\boxed{\begin{array}{lll} I_{xx} = 0.00667\text{ kg-m}^2, & I_{yy} = 0.0733\text{ kg-m}^2, & I_{zz} = 0.08\text{ kg-m}^2, \\ I_{xy} = 0.01\text{ kg-m}^2, & I_{xz} = 0, & I_{yz} = 0. \end{array}}$$

Problem 20.82 The 4-kg thin rectangular plate lies in the x-y plane. Determine the moments and products of inertia of the plate in terms of the coordinate system shown.



Solution: From Appendix B, the moments of inertia of the plate's area are

$$I_x = \frac{1}{12}(0.3)(0.6)^3 = 0.00540\text{ m}^4,$$

$$I_y = \frac{1}{12}(0.6)(0.3)^3 = 0.00135\text{ m}^4,$$

$$I_{xy}^A = 0.$$

Therefore the moments of inertia of the plate are

$$I_{xx} = \frac{m}{A}I_x = \frac{4}{(0.3)(0.6)}(0.00540)$$

$$= 0.12\text{ kg-m}^2,$$

$$I_{yy} = \frac{m}{A}I_y = \frac{4}{(0.3)(0.6)}(0.00135)$$

$$= 0.03\text{ kg-m}^2,$$

$$I_{zz} = I_x + I_y = 0.15\text{ kg-m}^2,$$

$$I_{xy} = I_{yz} = I_{zx} = 0.$$

Problem 20.83 If the 4-kg plate is rotating with angular velocity $\boldsymbol{\omega} = 6\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ (rad/s), what is its angular momentum about its center of mass?

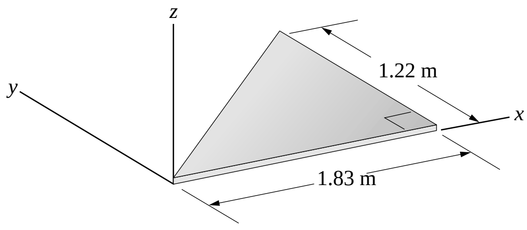
Solution: Angular momentum is

$$[H] = \begin{bmatrix} 0.12 & 0 & 0 \\ 0 & 0.03 & 0 \\ 0 & 0 & 0.15 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 0.72 \\ 0.12 \\ -0.3 \end{bmatrix}\text{ kg-m}^2/\text{s},$$

from which $\boxed{\mathbf{H} = 0.72\mathbf{i} + 0.12\mathbf{j} - 0.3\mathbf{k}\ (\text{kg-m}^2/\text{s})}$

Problem 20.84 The 133.4 N triangular plate lies in the $x-y$ plane. Determine the moments and products of inertia of the plate in terms of the coordinate system shown.



Solution: From Appendix B, the moments of inertia of the plate's area are

$$I_x = \frac{1}{12}(1.83)(1.22)^3 = 0.276 \text{ m}^4,$$

$$I_y = \frac{1}{4}(1.22)(1.83)^3 = 1.86 \text{ m}^4,$$

$$I_{xy}^A = \frac{1}{8}(1.83)^2(1.22)^2 = 0.62 \text{ m}^4.$$

The plate's area and mass are

$$A = \frac{1}{2}(1.83)(1.22) = 1.11 \text{ m}^2$$

and $m = 133.4/9.81 = 13.6$ kg. The plate's moments and products of inertia are

$$I_{xx} = \frac{m}{A}I_x = 3.36 \text{ kg-m}^2,$$

$$I_{yy} = \frac{m}{A}I_y = 22.73 \text{ kg-m}^2,$$

$$I_{xy} = \frac{m}{A}I_{xy}^A = 7.58 \text{ kg-m}^2,$$

$$I_{yz} = I_{zx} = O,$$

$$I_{zz} = I_{xx} + I_{yy} = 26.1 \text{ kg-m}^2.$$

Problem 20.85 The 133.4 N triangular plate lies in the $x-y$ plane.

(a) Determine its moments and products of inertia in terms of a parallel coordinate system $x'y'z'$ with its origin at the plate's center of mass.

(b) If the plate is rotating with angular velocity $\boldsymbol{\omega} = 20\mathbf{i} - 12\mathbf{j} + 16\mathbf{k}$ (rad/s), what is its angular momentum about its center of mass?

Solution: See the solution of Problem 20.84

(a) The coordinates of the plate's center of mass are

$$d_x = \frac{2}{3}(1.83) = 1.22 \text{ m},$$

$$d_y = \frac{1}{3}(1.22) = 0.41 \text{ m},$$

$$d_z = 0.$$

From the parallel-axis theorems (Eq. 20.42), we obtain

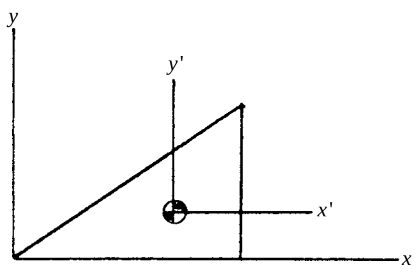
$$I_{x'x'} = 3.26 - (0.41)^2(13.6) = 1.12 \text{ kg-m}^2,$$

$$I_{y'y'} = 22.73 - (1.22)^2(13.6) = 2.53 \text{ kg-m}^2,$$

$$I_{z'z'} = 26.1 - [(0.41)^2 + (1.22)^2](13.6) = 3.65 \text{ kg-m}^2,$$

$$I_{x'y'} = 7.58 - (0.41)(1.22)(13.6) = 0.842 \text{ kg-m}^2,$$

$$I_{y'z'} = I_{z'x'} = 0$$

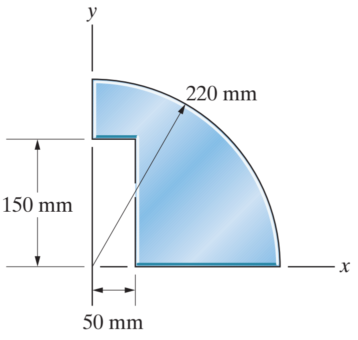


(b) $$[I]\omega = \begin{bmatrix} 1.12 & -0.842 & 0 \\ -0.842 & 2.53 & 0 \\ 0 & 0 & 3.65 \end{bmatrix} \begin{bmatrix} 20 \\ -12 \\ 16 \end{bmatrix}$$

$$= \begin{bmatrix} 32.5 \\ -47.2 \\ 58.4 \end{bmatrix}$$

$$\mathbf{H} = 32.5\mathbf{i} - 47.2\mathbf{j} + 58.4\mathbf{k} \text{ (kg-m}^2).$$

Problem 20.86 Determine the inertia matrix of the 2.4-kg steel plate in terms of the coordinate system shown.



Solution: Equation (20.39) gives the plate's moments and products of inertia in terms of the moments and product of inertia of its area. Treating the area as a quarter-circle using Appendix B, the moments and products of inertia of the area are

$$I_x = \frac{1}{16}\pi(0.22)^4 - \frac{1}{3}(0.05)(0.15)^3 = 0.000404 \text{ m}^4$$

$$I_y = \frac{1}{16}\pi(0.22)^4 - \frac{1}{3}(0.15)(0.05)^3 = 0.000454 \text{ m}^4$$

$$I_{xy}^A = \frac{1}{8}(0.22)^4 - \frac{1}{4}(0.05)^2(0.15)^2 = 0.000279 \text{ m}^4.$$

The area is

$$A = \tfrac{1}{4}\pi(0.22)^2 - (0.05)(0.15) = 0.0305 \text{ m}^2.$$

The moments of inertia of the plate are

$$I_{xx} = \frac{m}{A}I_x = 0.0318 \text{ kg-m}^2,$$

$$I_{yy} = \frac{m}{A}I_y = 0.0357 \text{ kg-m}^2,$$

$$I_{zz} = I_{xx} + I_{yy} = 0.0674 \text{ kg-m}^2,$$

$$I_{xy} = \frac{m}{A}I_{xy}^A = 0.0219 \text{ kg-m}^2, \text{ and } I_{yz} = I_{zx} = 0.$$

Problem 20.87 The mass of the steel plate is 2.4 kg.

(a) Determine its moments and products of inertia in terms of a parallel coordinate system $x'y'z'$ with its origin at the plate's center of mass.

(b) If the plate is rotating with angular velocity $\omega = 20\mathbf{i} + 10\mathbf{j} - 10\mathbf{k}$ (rad/s), what is its angular momentum about its center of mass?

Solution:

(a) The x and y coordinates of the center of mass coincide with the centroid of the area:

$$A_1 = \frac{1}{4}\pi(0.22)^2 = 0.0380 \text{ m}^2,$$

$$A_2 = (0.05)(0.15) = 0.0075 \text{ m}^2$$

$$\mathbf{x} = \frac{\dfrac{4(0.22)}{3\pi}A_1 - (0.025)A_2}{A_1 - A_2} = 0.1102 \text{ m},$$

$$\mathbf{y} = \frac{\dfrac{4(0.22)}{3\pi}A_1 - (0.075)A_2}{A_1 - A_2} = 0.0979 \text{ m}.$$

Using the results of the solution of Problem 20.86 and the parallel axis theorems,

$$I_{x'x'} = I_{xx} - m\mathbf{y}^2 = 0.00876 \text{ kg-m}^2:$$

$$I_{y'y'} = I_{yy} - m\mathbf{x}^2 = 0.00655 \text{ kg-m}^2:$$

$$I_{z'z'} = I_{x'x'} + I_{y'y'} = 0.01531 \text{ kg-m}^2$$

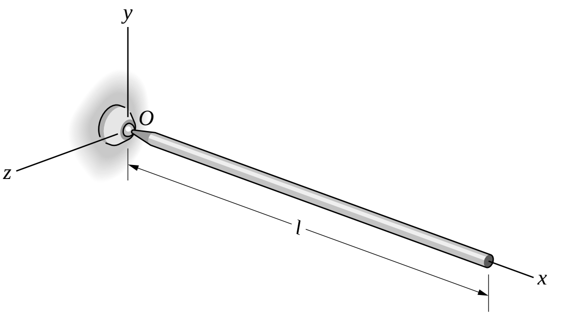
$$I_{x'y'} = I_{xy} - m\mathbf{xy} = -0.00396 \text{ kg-m}^2, \text{ and } I_{y'z'} = I_{z'x'} = 0.$$

The angular momentum is

$$\begin{bmatrix} H_{x'} \\ H_{y'} \\ H_{z'} \end{bmatrix} = \begin{bmatrix} I_{x'x'} & -I_{x'y'} & 0 \\ -I_{x'y'} & I_{y'y'} & 0 \\ 0 & 0 & I_{z'z'} \end{bmatrix} \begin{bmatrix} 20 \\ 10 \\ -10 \end{bmatrix}$$

$$= \begin{bmatrix} 0.215 \\ 0.145 \\ -0.153 \end{bmatrix} (\text{kg-m}^2/\text{s}).$$

Problem 20.88 The slender bar of mass m rotates about the fixed point O with angular velocity $\boldsymbol{\omega} = \omega_y \mathbf{j} + \omega_z \mathbf{k}$. Determine its angular momentum (a) about its center of mass and (b) about O.



Solution:

(a) From Appendix C and by inspection, the moments and products of inertia about the center of mass of the bar are:

$$I_{xx} = 0,$$

$$I_{yy} = I_{zz} = \frac{mL^2}{12},$$

$$I_{xy} = I_{xz} = I_{yz} = 0.$$

The angular momentum about its center of mass is

$$[H]_G = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} 0 \\ \omega_y \\ \omega_z \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \dfrac{mL^2}{12} & 0 \\ 0 & 0 & \dfrac{mL^2}{12} \end{bmatrix} \begin{bmatrix} 0 \\ \omega_y \\ \omega_z \end{bmatrix}$$

Alternatively, in terms of the **i**, **j**, **k**,

$$\boxed{\mathbf{H}_G = \frac{mL^2}{12}(\omega_y \mathbf{j} + \omega_z \mathbf{k})}.$$

(b) From Appendix C and by inspection, the moments and products of inertia about O are

$$I_{xx} = 0,$$

$$I_{yy} = I_{zz} = \frac{mL^2}{3},$$

$$I_{xy} = I_{xz} = I_{yz} = 0$$

The angular momentum about O is

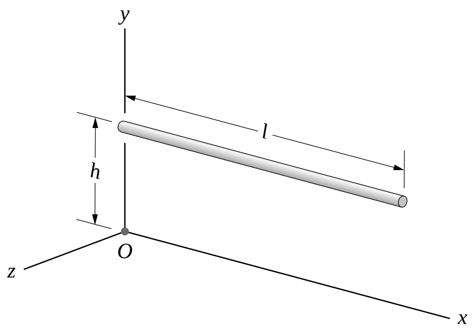
$$[H]_O = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} 0 \\ \omega_y \\ \omega_z \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \dfrac{mL^2}{3} & 0 \\ 0 & 0 & \dfrac{mL^2}{3} \end{bmatrix} \begin{bmatrix} 0 \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} 0 \\ \dfrac{mL^2\omega_y}{3} \\ \dfrac{mL^2\omega_z}{3} \end{bmatrix},$$

or, alternatively, in terms of the unit vectors **i**, **j**, **k**,

$$\boxed{\mathbf{H}_O = \frac{mL^2}{3}(\omega_y \mathbf{j} + \omega_z \mathbf{k})}$$

Problem 20.89 The slender bar of mass m is parallel to the x axis. If the coordinate system is body fixed and its angular velocity about the fixed point O is $\boldsymbol{\omega} = \omega_y \mathbf{j}$, what is the bar's angular momentum about O?



Solution: From Appendix C and by inspection, the moments and products of inertia about the center of mass of the bar are:

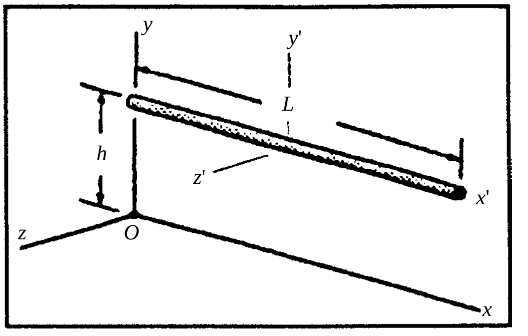
$$I_{x'x'} = 0,$$

$$I_{y'y'} = I_{z'z'} = \frac{mL^2}{12},$$

$$I_{x'y'} = I_{x'z'} = I_{y'z'} = 0.$$

The coordinates of the center of mass are

$$(d_x, d_y, d_z) = \left(\frac{L}{2}, h, 0\right).$$



From Eq. (20.42),

$$I_{xx} = I_{x'x'} + (d_y^2 + d_z^2)m = \text{mh}^2$$

$$I_{yy} = I_{y'y'} + (d_x^2 + d_z^2)m = \frac{mL^2}{12} + \frac{mL^2}{4} = \frac{mL^2}{3}$$

$$I_{zz} = I_{z'z'} + (d_x^2 + d_y^2)m = m\left(h^2 + \frac{L^2}{3}\right),$$

$$I_{xy} = I_{x'y'} + d_x d_y m = 0 + \frac{mLh}{2},$$

$$I_{xz} = I_{x'z'} + d_x d_z m = 0$$

$$I_{yz} = I_{y'z'} + d_y d_z m = 0$$

The angular momentum about O is

$$[H]_O = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} 0 \\ \omega_y \\ 0 \end{bmatrix}$$

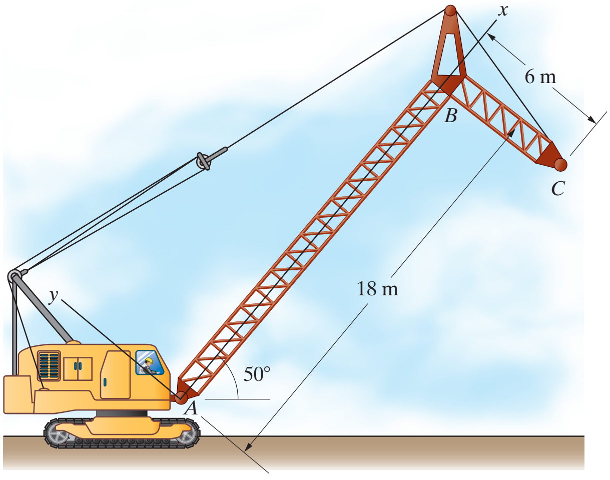
$$= \begin{bmatrix} mh^2 & -\dfrac{mLh}{2} & 0 \\ -\dfrac{mLh}{2} & \dfrac{mL^2}{3} & 0 \\ 0 & 0 & m\left(h^2 + \dfrac{L^2}{3}\right) \end{bmatrix} \begin{bmatrix} 0 \\ \omega_y \\ 0 \end{bmatrix}$$

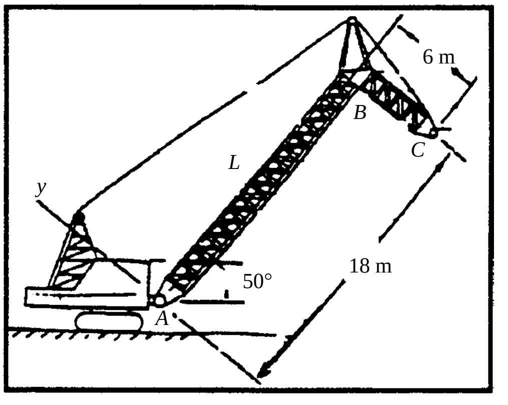
$$= \begin{bmatrix} -\dfrac{mLh\omega_y}{2} \\ \dfrac{mL^2\omega_y}{3} \\ 0 \end{bmatrix}.$$

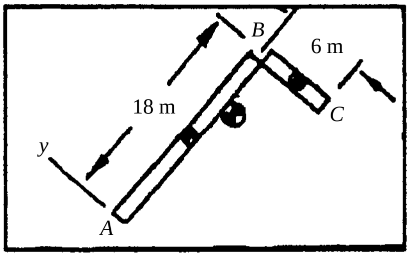
Alternatively,

$$\boxed{\mathbf{H}_O = -\frac{mLh}{2}\omega_y \mathbf{i} + \frac{mL^2}{3}\omega_y \mathbf{j}}$$

Problem 20.90 In Example 20.8, the moments and products of inertia of the object consisting of the booms AB and BC were determined in terms of the coordinate system shown in Fig. 20.34. Determine the moments and products of inertia of the object in terms of a parallel coordinate system $x'y'z'$ with its origin at the center of mass of the object.







Solution: From Example 20.8, the inertia matrix for the two booms in the x, y, z system is

$$[I]_O = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix}$$

$$= \begin{bmatrix} 19{,}200 & 86{,}400 & 0 \\ 86{,}400 & 1{,}036{,}800 & 0 \\ 0 & 0 & 1{,}056{,}000 \end{bmatrix} \text{ kg-m}^2.$$

The mass of the boom AB is $m_{AB} = 4800$ kg. The mass of the boom BC is $m_{BC} = 1600$ kg. The coordinates of the center of mass of the two booms are

$$x' = \frac{m_{AB}\dfrac{L}{2} + m_{BC}L}{m_{AB} + m_{BC}} = 11.25 \text{ m}.$$

$$y' = \frac{m_{AB}(0) + m_{BC}(-3)}{m_{AB} + m_{BC}} = -0.75 \text{ m}.$$

$z' = 0$, from which $(d_x, d_y, d_z) = (11.25, -0.75, 0)$ m. Algebraically rearrange Eq. (20.42) to obtain the moments and products of inertia about the parallel axis passing through the center of mass of the two booms when the moments and products of inertia in the x, y, z system are known:

$$\boxed{I_{x'x'}^{(G)} = I_{xx}^{(0)} - (d_y^2 + d_x^2)m = 15600 \text{ kg-m}^2}.$$

$$\boxed{I_{y'y'}^{(G)} = I_{yy}^{(o)} - (d_x^2 + d_z^2)m = 226800 \text{ kg-m}^2}.$$

$$\boxed{I_{z'z'}^{(G)} = I_{zz}^{(o)} - (d_x^2 + d_y^2)m = 242400 \text{ kg-m}^2}$$

$$\boxed{I_{x'y'}^{(G)} = I_{xy}^{(0)} - d_x d_y m = -32400 \text{ kg-m}^2},$$

$$\boxed{I_{x'z'}^{(G)} = 0}, \quad \boxed{I_{y'z'}^{(G)} = 0}.$$

The inertia matrix for the x', y', z' system is

$$[I]_G = \begin{bmatrix} 15{,}600 & 32{,}400 & 0 \\ 32{,}400 & 226{,}800 & 0 \\ 0 & 0 & 242{,}400 \end{bmatrix} \text{ kg-m}^2.$$

Problem 20.91 Suppose that the crane described in Example 20.8 undergoes a rigid-body rotation about the vertical axis at 0.1 rad/s in the counterclockwise direction when viewed from above.

(a) What is the crane's angular velocity vector $\boldsymbol{\omega}$ in terms of the body-fixed coordinate system shown in Fig. 20.34?

(b) What is the angular momentum of the object consisting of the booms AB and BC *about its center of mass*?

Solution: The unit vector parallel to vertical axis in the x', y', z' system is

$$\mathbf{e} = \mathbf{i}\sin 50^\circ + \mathbf{j}\cos 50^\circ = 0.7660\mathbf{i} + 0.6428\mathbf{j}.$$

The angular velocity vector is

$$\boxed{\boldsymbol{\omega} = (0.1)\mathbf{e} = 0.07660\mathbf{i} + 0.06428\mathbf{j}}$$

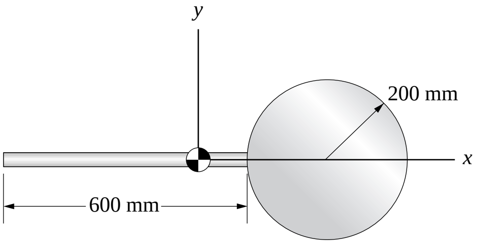
From the inertia matrix given in the solution of Problem 20.90, the angular moment about the center of mass is

$$[H]_G = \begin{bmatrix} 15{,}600 & 32{,}400 & 0 \\ 32{,}400 & 226{,}800 & 0 \\ 0 & 0 & 242{,}400 \end{bmatrix} \begin{bmatrix} 0.07660 \\ 0.06428 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3277.7 \\ 17060.4 \\ 0 \end{bmatrix} \text{ kg-m}^2/\text{s},$$

or, $\boxed{\mathbf{H}_G = 3277.7\mathbf{i} + 17060.4\mathbf{j}}$ (kg-m^2/s).

Problem 20.92 A 3-kg slender bar is rigidly attached to a 2-kg thin circular disk. In terms of the body-fixed coordinate system shown, the angular velocity of the composite object is $\boldsymbol{\omega} = 100\mathbf{i} - 4\mathbf{j} + 6\mathbf{k}$ (rad/s). What is the object's angular momentum about its center of mass?



Solution: Choose a coordinate system x, y, z originating at the left end of the bar. From Appendix C and by inspection, the inertia matrix for the bar about its left end is

$$[I]_B = \begin{bmatrix} 0 & 0 & \\ 0 & \dfrac{m_B L^2}{3} & 0 \\ 0 & 0 & \dfrac{m_B L^2}{3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.36 & 0 \\ 0 & 0 & 0.36 \end{bmatrix} \text{ kg-m}^2.$$

From Eq. (20.42) the inertia matrix of the disk about the left end of the bar is

$$[I]_D = \begin{bmatrix} \dfrac{m_D R^2}{4} & 0 & 0 \\ 0 & \dfrac{m_D R^2}{4} & 0 \\ 0 & 0 & \dfrac{m_D R^2}{2} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & m_D(L+R)^2 & 0 \\ 0 & 0 & m_D(L+R)^2 \end{bmatrix}$$

$$= \begin{bmatrix} 0.02 & 0 & 0 \\ 0 & 1.3 & 0 \\ 0 & 0 & 1.32 \end{bmatrix} \text{ kg-m}^2.$$

The inertia matrix of the composite is

$$[I]_{\text{left\_end}} = [I]_B + [I]_D = \begin{bmatrix} 0.02 & 0 & 0 \\ 0 & 1.66 & 0 \\ 0 & 0 & 1.68 \end{bmatrix}.$$

The coordinates of the center of mass of the composite in the x, y, z system are

$$x' = \frac{m_B(L/2) + m_D(R+L)}{m_B + m_D} = 0.5 \text{ m}.$$

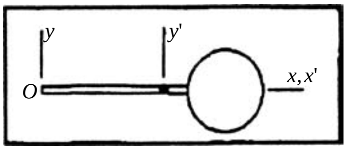
$$y' = 0, \quad z' = 0,$$

from which $(d_x, d_y, d_z) = (0.5, 0, 0)$ m. Rearrange Eq. (20.42) to yield the inertia matrix in the x', y', z' system when the inertia matrix in the x, y, z system is known:

$$[I]_G = \begin{bmatrix} 0.02 & 0 & 0 \\ 0 & 1.66 & 0 \\ 0 & 0 & 1.68 \end{bmatrix}$$

$$- \begin{bmatrix} 0 & 0 & 0 \\ 0 & d_x^2(m_B + m_D) & 0 \\ 0 & 0 & d_x^2(m_B + m_D) \end{bmatrix}$$

$$= \begin{bmatrix} 0.02 & 0 & 0 \\ 0 & 0.41 & 0 \\ 0 & 0 & 0.43 \end{bmatrix} \text{ kg-m}^2.$$

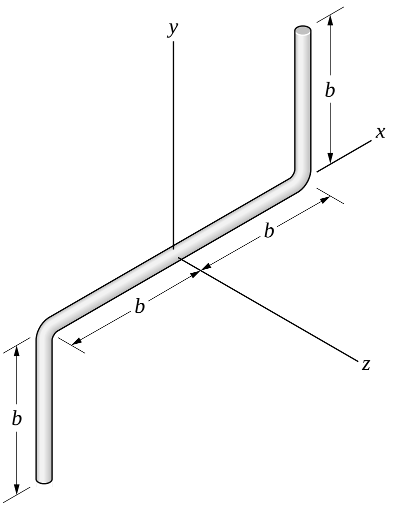


The angular momentum is

$$[H]_G = \begin{bmatrix} 0.02 & 0 & 0 \\ 0 & 0.41 & 0 \\ 0 & 0 & 0.43 \end{bmatrix} \begin{bmatrix} 100 \\ -4 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ -1.64 \\ 2.58 \end{bmatrix},$$

$$\boxed{\mathbf{H} = 2\mathbf{i} - 1.64\mathbf{j} + 2.58\mathbf{k}(\text{kg-m}^2/\text{s})}$$

Problem 20.93* The mass of the homogeneous slender bar is m. If the bar rotates with angular velocity $\boldsymbol{\omega} = \omega_0(24\mathbf{i} + 12\mathbf{j} - 6\mathbf{k})$, what is its angular momentum about its center of mass?



Solution: The strategy is to transfer the moments of inertia of the ends about their attached ends to the center of mass, and sum the resulting moments and products of inertia. The mass of the central element is $m_C = \dfrac{m}{2}$, and the mass of each end element is $m_E = \dfrac{m}{4}$. For the central element about its center of mass:

$$I_{xx} = 0,$$

$$I_{yy} = I_{zz} = \frac{m_C(2b)^2}{12} = \frac{mb^2}{6},$$

$$I_{xy} = I_{xz} = I_{yz} = 0.$$

For each end element about its center of mass:

$$I_{xx} = \frac{m_E b^2}{12} = \frac{mb^2}{48},$$

$$I_{yy} = 0,$$

$$I_{zz} = \frac{m_E b^2}{12} = \frac{mb^2}{48},$$

$$I_{xy} = I_{xz} = I_{yz} = 0.$$

The coordinates of the center of mass (at the origin) relative to the center of mass of the left end is $(d_x, d_y, d_z) = \left(-b, -\dfrac{b}{2}, 0\right)$, and from the right end $(d_x, d_y, d_z) = \left(b, \dfrac{b}{2}, 0\right)$. From Eq. (20.42), the moments of inertia of each the end pieces about the center of mass (at the origin) are

$$I^{(G)}_{x'x'} = I^{(L)}_{xx} + (d_y^2 + d_z^2)m_E = \frac{mb^2}{48} + \frac{m_E b^2}{4} = \frac{mb^2}{12}.$$

$$I^{(G)}_{y'y'} I^{(L)}_{yy} + (d_x^2 + d_z^2)m_E = \frac{mb^2}{4}.$$

$$I^{(G)}_{z'z'} = I^{(L)}_{zz} + (d_x^2 + d_y^2)m_E = \frac{mb^2}{48} + m_E\left(b^2 + \frac{b^2}{4}\right) = \frac{mb^2}{3}$$

$$I^{(G)}_{x'y'} = I^{(L)}_{xy} + d_x d_y m_E = mb^2/8,\ I^{(G)}_{x'z'} = 0,\ I^{(G)}_{y'z'} = 0.$$

The sum of the matrices:

$$[I]_G = [I]_{GC} + 2[I]_{\text{end}}$$

$$= mb^2 \begin{bmatrix} 0.1667 & -0.25 & 0 \\ -0.25 & 0.6667 & 0 \\ 0 & 0 & 0.8333 \end{bmatrix}.$$

The angular momentum about the center of mass is

$$[H]_G = \omega_0 mb^2 \begin{bmatrix} 0.1667 & -0.25 & 0 \\ -0.25 & 0.6667 & 0 \\ 0 & 0 & 0.8333 \end{bmatrix} \begin{bmatrix} 24 \\ 12 \\ -6 \end{bmatrix}$$

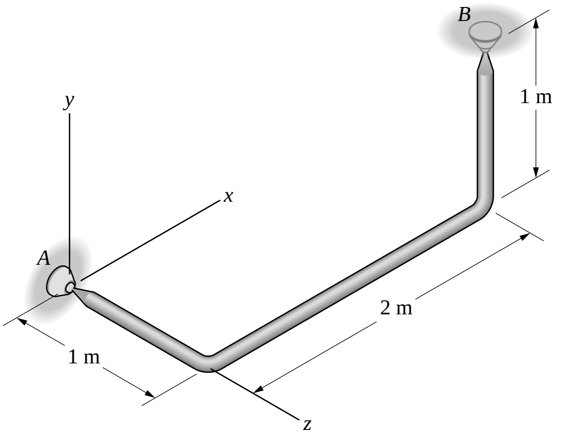
$$= \omega_0 mb^2 \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}.$$

$$\boxed{\mathbf{H} = \omega_0 mb^2(\mathbf{i} + 2\mathbf{j} - 5\mathbf{k})}$$

Problem 20.94* The 8-kg homogeneous slender bar has ball-and-socket supports at A and B.

(a) What is the bar's moment of inertia about the axis AB?

(b) If the bar rotates about the axis AB at 4 rad/s, what is the magnitude of its angular momentum about its axis of rotation?



Solution: Divide the bar into three elements: the central element, and the two end element. The strategy is to find the moments and products of inertia in the x, y, z system shown, and then to use Eq. (20.43) to find the moment of inertia about the axis AB. Denote the total mass of the bar by $m = 8$ kg, the mass of each end element by $m_E = \frac{m}{4} = 2$ kg, and the mass of the central element by $m_C = \frac{m}{2} = 4$ kg.

The left end element: The moments and products of inertia about point A are:

$$I_{xx}^{(LA)} = \frac{m_E(1^2)}{3} = \frac{m}{12} = 0.6667 \text{ kg-m}^2,$$

$$I_{yy}^{(LA)} = \frac{m_E(1^2)}{3} = \frac{m}{12} = 0.6667 \text{ kg-m}^2,$$

$$I_{zz}^{(LA)} = 0,$$

$$I_{xy}^{(LA)} = I_{xz}^{(LA)} = I_{yz}^{(LA)} = 0.$$

The right end element: The moments and products of inertia about its center of mass are

$$I_{xx}^{(RG)} = \frac{m_E(1^2)}{12} = \frac{m}{48} = 0.1667 \text{ kg-m}^2,$$

$$I_{yy}^{(RG)} = 0,$$

$$I_{zz}^{(RG)} = \frac{m_E(1^2)}{12} = \frac{m}{48} = 0.1667 \text{ kg-m}^2,$$

$$I_{xx}^{(RG)} = I_{xz}^{(RG)} = I_{yz}^{(RG)} = 0.$$

The coordinates of the center of mass of the right end element are $(d_x, d_y, d_z) = (2, 0.5, 1)$. From Eq. (20.42), the moments and products of inertia in the x, y, z system are

$$I_{xx}^{(RA)} = I_{xx}^{(RG)} + (d_y^2 + d_z^2)\frac{m}{4} = 2.667 \text{ kg-m}^2,$$

$$I_{yy}^{(RA)} = I_{yy}^{(RG)} + (d_x^2 + d_z^2)\frac{m}{4} = 10.0 \text{ kg-m}^2$$

$$I_{zz}^{(RA)} = I_{zz}^{(RG)} + (d_x^2 + d_y^2)\frac{m}{4} = 8.667 \text{ kg-m}^2,$$

$$I_{xy}^{(RA)} = I_{xy}^{(RG)} + d_x d_y \frac{m}{4} = 2.0 \text{ kg-m}^2$$

$$I_{xz}^{(RA)} = I_{xz}^{(RG)} + d_x d_z \frac{m}{4} = 4.0 \text{ kg-m}^2,$$

$$I_{yz}^{(RA)} = I_{yz}^{(RG)} + d_y d_z \frac{m}{4} = 1.0 \text{ kg-m}^2$$

Sum the two inertia matrices:

$$[I]_{RLA} = [I]_{RA} + [I]_{LA} = \begin{bmatrix} 3.333 & -2 & -4 \\ -2 & 10.67 & -1 \\ -4 & -1 & 8.667 \end{bmatrix} \text{ kg-m}^2,$$

where the negative signs are a consequence of the definition of the inertia matrix.

The central element: The moments and products of inertia of the central element about its center of mass are:

$$I_{xx}^{(CG)} = 0,\ I_{yy}^{(CG)} = \frac{m_C(2^2)}{12} = \frac{m}{6} = 1.333 \text{ kg-m}^2,$$

$$I_{zz}^{(CG)} = \frac{m_C(2^2)}{12} = \frac{m}{6} = 1.333 \text{ kg-m}^2.$$

$$I_{xy}^{(CG)} = I_{xz}^{(CG)} = I_{yz}^{(CG)} = 0.$$

The coordinates of the center of mass of the central element are $(d_x, d_y, d_z) = (1, 0, 1)$. From Eq. (20.42) the moments and products of inertia in the x, y, z system are:

$$I_{xx}^{(CA)} = I_{xx}^{(CG)} + (d_y^2 + d_z^2)\frac{m}{2} = 4 \text{ kg-m}^2,$$

$$I_{yy}^{(CA)} = I_{yy}^{(CG)} + (d_x^2 + d_z^2)\frac{m}{2} = 9.333 \text{ kg-m}^2$$

$$I_{zz}^{(CA)} = I_{zz}^{(CG)} + (d_x^2 + d_y^2)\frac{m}{2} = 5.333 \text{ kg-m}^2,$$

$$I_{xy}^{(CA)} = I_{xy}^{(CG)} + d_x d_y \frac{m}{2} = 0,$$

$$I_{xz}^{(CA)} = I_{xz}^{(CG)} + d_x d_z \frac{m}{2} = 4 \text{ kg-m}^2,$$

$$I_{yz}^{(CA)} = I_{yz}^{(CG)} + d_y d_z \frac{m}{2} = 0$$

Sum the inertia matrices:

$$[I]_A = [I]_{RLA} + [I]_{CA} = \begin{bmatrix} 7.333 & -2 & -8 \\ -2 & 20.00 & -1 \\ -8 & -1 & 14.00 \end{bmatrix} \text{ kg-m}^2.$$

(a) *The moment of inertia about the axis AB*: The distance AB is parallel to the vector $\mathbf{r}_{AB} = 2\mathbf{i} + 1\mathbf{j} + 1\mathbf{k}$ (m). The unit vector parallel to $\mathbf{r}_{AB}$ is

$$\mathbf{e}_{AB} = \frac{\mathbf{r}_{AB}}{|\mathbf{r}_{AB}|} = 0.8165\mathbf{i} + 0.4082\mathbf{j} + 4082\mathbf{k}.$$

From Eq. (20.43), the moment of inertia about the AB axis is

$$I_{AB} = I_{xx}^{(A)} e_x^2 + I_{yy}^{(A)} e_y^2 + I_{zz}^{(A)} e_z^2 - 2I_{xy}^{(A)} e_x e_y - 2I_{xz}^{(A)} e_x e_z$$

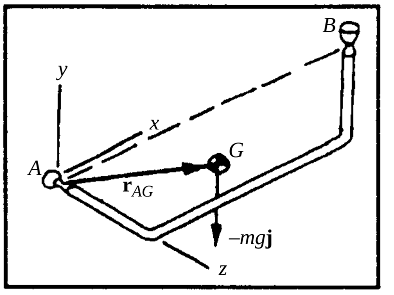
$$- 2I_{yz}^{(A)} e_y e_z,\ \boxed{I_{AB} = 3.56 \text{ kg-m}^2}.$$

(b) The angular momentum about the AB axis is

$$\boxed{H_{AB} = I_{AB}\omega = 3.56(4) = 14.22 \text{ kg-m}^2/\text{s}}$$

Problem 20.95* The 8-kg homogeneous slender bar in Problem 20.94 is released from rest in the position shown. (The x-z plane is horizontal.) What is the magnitude of the bar's angular acceleration about the axis AB at the instant of release?

Solution: The center of mass of the bar has the coordinates:



$$x_G = \frac{1}{m}\left(\left(\frac{m}{4}\right)0 + \left(\frac{m}{2}\right)(1) + \left(\frac{m}{4}\right)(2)\right) = 1 \text{ m.}$$

$$y_G = \frac{1}{m}\left(\left(\frac{m}{4}\right)(0) + \left(\frac{m}{2}\right)(0) + \left(\frac{m}{4}\right)(0.5)\right) = 0.125 \text{ m.}$$

$$z_G = \frac{1}{m}\left(\left(\frac{m}{4}\right)(0.5) + \left(\frac{m}{2}\right)(1) + \left(\frac{m}{4}\right)(1)\right) = 0.875 \text{ m.}$$

The line from A to the center of mass is parallel to the vector $\mathbf{r}_{AG} = \mathbf{i} + 0.125\mathbf{j} + 0.875\mathbf{k}$ (m). From the solution to Problem 20.94 the unit vector parallel to the line AB is $\mathbf{e}_{AB} = 0.8165\mathbf{i} + 0.4082\mathbf{j} + 0.4082\mathbf{k}$. The magnitude of the moment about line AB due to the weight is

$$e[\mathbf{r}_{AB} \times (-mg\mathbf{j})] = \begin{vmatrix} 0.8165 & 0.4082 & 0.4082 \\ 1.000 & 0.125 & 0.875 \\ 0 & -78.48 & 0 \end{vmatrix} = 24.03 \text{ N-m.}$$

From the solution to Problem 20.94, $I_{AB} = 3.556$ kg-m^2. From the equation of angular motion about axis AB, $M_{AB} = I_{AB}\alpha$, from which

$$\boxed{\alpha = \frac{24.03}{3.556} = 6.75 \text{ rad/s}^2}.$$

Problem 20.96 In terms of a coordinate system $x'y'z'$ with its origin at the center of mass, the inertia matrix of a rigid body is

$$[I'] = \begin{bmatrix} 20 & 10 & -10 \\ 10 & 60 & 0 \\ -10 & 0 & 80 \end{bmatrix} \text{ kg-m}^2.$$

Determine the principal moments of inertia and unit vectors parallel to the corresponding principal axes.

Solution: *Principal Moments of Inertia*: The moments and products:

$I_{x'x'} = 20 \text{ kg-m}^2$,

$I_{y'y'} = 60 \text{ kg-m}^2$,

$I_{z'z'} = 80 \text{ kg-m}^2$,

$I_{x'y'} = -10 \text{ kg-m}^2$,

$I_{x'z'} = 10 \text{ kg-m}^2$,

$I_{y'z'} = 0$.

From Eq. (20.45), the principal values are the roots of the cubic equation. $AI^3 + BI^2 + CI + D = 0$, where

$A = +1$,

$B = -(I_{x'x'} + I_{y'y'} + I_{z'z'}) = -160$,

$C = (I_{x'x'}I_{y'y'} + I_{y'y'}I_{z'z'} + I_{z'z'}I_{x'x'} - I_{x'y'}^2 - I_{x'z'}^2 - I_{y'z'}^2)$

$= 7400$,

$D = -(I_{x'x'}I_{y'y'}I_{z'z'} - I_{x'x'}I_{y'z'}^2 - I_{y'y'}I_{x'z'}^2$

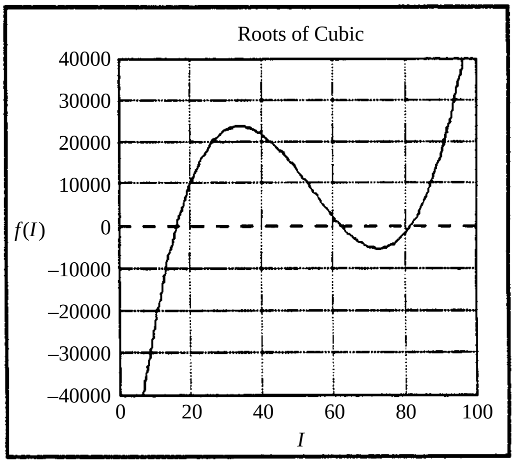
$- I_{z'z'}I_{x'y'}^2 - 2I_{x'y'}I_{y'z'}I_{x'z'})$

$= -82000$.

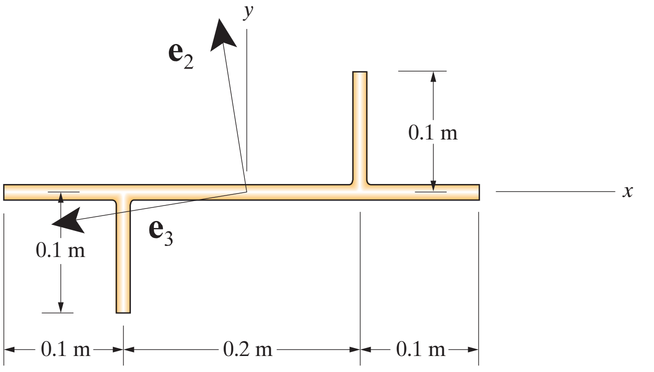
The function $f(I) = AI^3 + BI^2 + CI + D$ is graphed to find the zero crossings, and these values are refined by iteration. The graph is shown. The principal moments of inertia are:

$\boxed{I_1 = 16.15 \text{ kg-m}^2}$,

$\boxed{I_2 = 62.10 \text{ kg-m}^2}$, $\boxed{I_3 = 81.75 \text{ kg-m}^2}$.



Problem 20.97 For the object in Problem 20.81, determine the principal moments of inertia and unit vectors parallel to the corresponding principal axes. Draw a sketch of the object showing the principal axes.



Solution: In Problem 20.81, we found the inertia matrix to be

$$[I] = \begin{bmatrix} 0.00667 & -0.01 & 0 \\ -0.01 & 0.0733 & 0 \\ 0 & 0 & 0.08 \end{bmatrix} \text{ kg-m}^2$$

To find the principal inertia we expand the determinant to produce the characteristic equation

$$\det \begin{vmatrix} 0.00667 - I & -0.01 & 0 \\ -0.01 & 0.0733 - I & 0 \\ 0 & 0 & 0.08 - I \end{vmatrix}$$

$$= (0.08 - I)[(0.00667 - I)(0.0733 - I) - (0.01)^2] = 0$$

Solving this equation, we have

$$\boxed{I_1 = 0.08 \text{ kg-m}^2,\ I_2 = 0.0748 \text{ kg-m}^2,\ I_3 = 0.0052 \text{ kg-m}^2.}$$

Substituting these principal inertias into Eqs. (20.46) and dividing the resulting vector **V** by its magnitude, we find a unit vector parallel to the corresponding principal value.

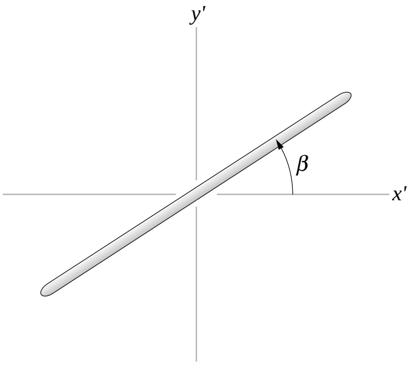
$$\boxed{\mathbf{e}_1 = 0.989\mathbf{i} + 0.145\mathbf{j},\ \mathbf{e}_2 = -0.145\mathbf{i} + 0.989\mathbf{j}, \mathbf{e}_3 = \mathbf{k}.}$$

The principal axes are shown on the figure at the top of the page with $\mathbf{e}_1$ pointing out of the paper. The axes are rotated counterclockwise through the angle $\theta = \cos^{-1}(0.989) = 8.51°$.

Problem 20.98 The 1-kg, 1-m long slender bar lies in the x'-y' plane. Its moment of inertia matrix is

$$[I'] = \begin{bmatrix} \frac{1}{12}\sin^2\beta & -\frac{1}{12}\sin\beta\cos\beta & 0 \\ -\frac{1}{12}\sin\beta\cos\beta & \frac{1}{12}\cos^2\beta & 0 \\ 0 & 0 & \frac{1}{12} \end{bmatrix}.$$

Use Eqs. (20.45) and (20.46) to determine the principal moments of inertia and unit vectors parallel to the corresponding principal axes.



Solution: [*Preliminary Discussion:* The moment of inertia about an axis coinciding with the slender rod is zero; it follows that one principal value will be zero, and the associated principal axis will coincide with the slender bar. Since the moments of inertia about the axes normal to the slender bar will be equal, there will be two equal principal values, and Eq. (20.46) will fail to yield unique solutions for the associated characteristic vectors. However the problem can be solved by inspection: the unit vector parallel to the axis of the slender rod will be $\mathbf{e}_1 = \mathbf{i}\cos\beta + \mathbf{j}\sin\beta$. A unit vector orthogonal to $\mathbf{e}_1$ is $\mathbf{e}_2 = -\mathbf{i}\sin\beta + \mathbf{j}\cos\beta$. A third unit vector orthogonal to these two is $\mathbf{e}_3 = \mathbf{k}$. The solution based on Eq. (20.46) must agree with these preliminary results.]

Principal Moments of Inertia: The moments and products:

$$I_{x'x'} = \frac{\sin^2\beta}{12},\ I_{y'y'} = \frac{\cos^2\beta}{12},\ I_{z'z'} = \frac{1}{12},$$

$$I_{x'y'} = +\frac{\sin\beta\cos\beta}{12},\ I_{x'z'} = 0,\ I_{y'z'} = 0.$$

From Eq. (20.45), the principal values are the roots of the cubic, $AI^3 + BI^2 + CI + D = 0$. The coefficients are:

$$A = +1,$$

$$B = -(I_{x'x'} + I_{y'y'} + I_{z'z'}) = -\left(\frac{\sin^2\beta}{12} + \frac{\cos^2\beta}{12} + \frac{1}{12}\right) = -\frac{1}{6},$$

$$C = (I_{x'x'}I_{y'y'} + I_{y'y'}I_{z'z'} + I_{z'z'}I_{x'x'} - I_{x'y'}^2 - I_{x'z'}^2 - I_{y'z'}^2),$$

$$C = \frac{\sin^2\beta\cos^2\beta}{144} + \frac{\cos^2\beta}{144} + \frac{\sin^2\beta}{144} - \frac{\sin^2\beta\cos^2\beta}{144} = \frac{1}{144}.$$

$$D = -(I_{x'x'}I_{y'y'}I_{z'z'} - I_{x'x'}I_{y'z'}^2 - I_{y'y'}I_{x'z'}^2 - I_{z'z'}I_{x'y'}^2$$
$$- 2I_{x'y'}I_{y'z'}I_{x'z'}),$$

$$D = -\left(\frac{\sin^2\beta\cos^2\beta}{12^3} - \frac{\sin^2\beta\cos^2\beta}{12^3}\right) = 0.$$

The cubic equation reduces to

$$I^3 - \left(\frac{1}{6}\right)I^2 + \left(\frac{1}{144}\right)I = \left(I^2 - \left(\frac{1}{6}\right)I + \left(\frac{1}{144}\right)\right)I = 0.$$

By inspection, the least root is $\boxed{I_1 = 0}$. The other two roots are the solution of the quadratic $I^2 + 2bI + c = 0$ where $b = -\frac{1}{12}$, $c = \frac{1}{144}$, from which $I_{1,2} = -b \pm \sqrt{b^2 - c} = \frac{1}{12}$, from which $\boxed{I_2 = \frac{1}{12}}$, $\boxed{I_3 = \frac{1}{12}}$

Principal axes: The characteristic vectors parallel to the principal axes are obtained from Eq. (20.46),

$$V_{x'}^{(j)} = (I_{y'y'} - I_j)(I_{z'z'} - I_j) - I_{y'z'}^2$$

$$V_{y'}^{(j)} = I_{x'y'}(I_{z'z'} - I_j) + I_{x'z'}I_{y'z'}$$

$$V_{z'}^{(j)} = I_{x'z'}(I_{y'y'} - I_j) + I_{x'y'}I_{y'z'}.$$

For the first root,

$$I_1 = 0,\ \mathbf{V}^{(1)} = \frac{\cos^2\beta}{144}\mathbf{i} + \frac{\cos\beta\sin\beta}{144}\mathbf{j} = \frac{\cos\beta}{144}(\cos\beta\mathbf{i} + \sin\beta\mathbf{j}).$$

The magnitude:

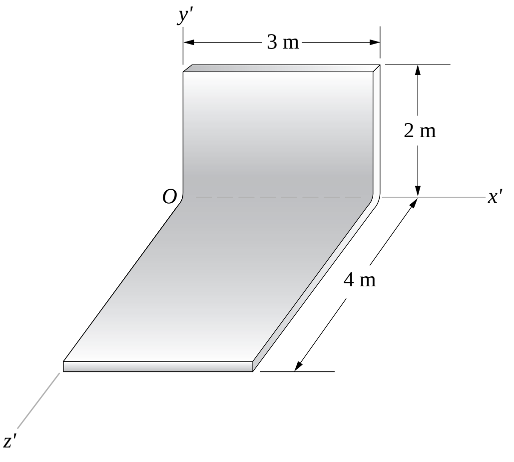
$$|\mathbf{V}^{(1)}| = \frac{|\cos\beta|}{144}\sqrt{\cos^2\beta + \sin^2\beta} = \frac{|\cos\beta|}{144},$$

and the unit vector is $\mathbf{e}_1 = \mathrm{sgn}(\cos\beta)(\cos\beta\mathbf{i} + \sin\beta\mathbf{j})$, where $\mathrm{sgn}(\cos\beta)$ is equal to the sign of $\cos\beta$. Without loss of generality, β can be restricted to lie in the first or fourth quadrants, from which

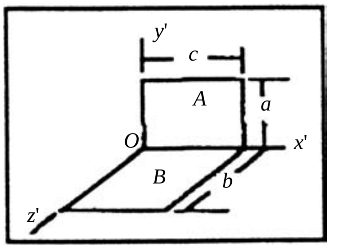
$$\boxed{\mathbf{e}_1 = (\cos\beta\mathbf{i} + \sin\beta\mathbf{j})}$$

For $I_2 = I_3 = \left(\frac{1}{12}\right)$, $\mathbf{V}^{(2)} = 0$, and $\mathbf{V}^{(3)} = 0$, from which *the equation* Eq. (20.46) *fails for the repeated principal values, and the characteristic vectors are to determined from the condition of orthogonality with* $\mathbf{e}_1$. From the preliminary discussion, $\boxed{\mathbf{e}_2 = -\mathbf{i}\sin\beta + \mathbf{j}\cos\beta}$ and $\boxed{\mathbf{e}_3 = 1\mathbf{k}}$.

Problem 20.99* The mass of the homogeneous thin plate is 3 kg. For a coordinate system with its origin at O, determine the plate's principal moments of inertia and unit vectors parallel to the corresponding principal axes.



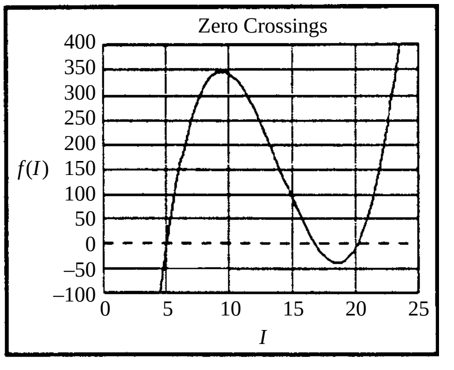
Solution: Divide the plate into A and B sheets, as shown. Denote $m = 3$ kg, $a = 2$ m, $b = 4$ m, and $c = 3$ m. The mass of plate A is $m_A = \frac{6m}{18} = \frac{m}{3} = 1$ kg. The mass of plate B is $m_B = \frac{12m}{18} = 2$ kg. The coordinates of the center of mass of A are



$$(d_x^{(A)}, d_y^{(A)}, d_z^{(A)}) = \left(\frac{c}{2}, \frac{a}{2}, 0\right) = (1.5, 1, 0) \text{ m}.$$

The coordinates of the center of mass of B are

$$(d_x^{(B)}, d_y^{(B)}, d_z^{(B)}) = \left(\frac{c}{2}, 0, \frac{b}{2}\right) = (1.5, 0, 2) \text{ m}.$$



Principal Values: From Appendix C, the moments and products of inertia for plate A are

$$I_{x'x'}^{(A)} = \frac{m_A a^2}{12} + (d_y^2 + d_z^2)m_A = 1.333 \text{ kg-m}^2,$$

$$I_{y'y'}^{(A)} = \frac{m_A c^2}{12} + (d_x^2 + d_z^2)m_A = 3 \text{ kg-m}^2,$$

$$I_{z'z'}^{(A)} = \frac{m_A(c^2 + a^2)}{12} + (d_x^2 + d_y^2)m_A = 4.333 \text{ kg-m}^2,$$

$$I_{x'y'}^{(A)} = d_x d_y m_A = 1.5 \text{ kg-m}^2,$$

$$I_{x'z'}^{(A)} = 0,\ I_{y'z'}^{(A)} = 0.$$

The moments and products of inertia for plate B are

$$I_{x'x'}^{(B)} = \frac{m_B b^2}{12} + (d_y^2 + d_z^2)m_B = 10.67 \text{ kg-m}^2,$$

$$I_{y'y'}^{(B)} = \frac{m_B(c^2 + b^2)}{12} + (d_x^2 + d_z^2)m_B = 16.67 \text{ kg-m}^2,$$

$$I_{z'z'}^{(B)} = \frac{m_B c^2}{12} + (d_x^2 + d_y^2)m_B = 6 \text{ kg-m}^2,\ I_{x'y'}^{(B)} = 0,$$

$$I_{x'z'}^{(B)} = d_x d_z m_B = 6 \text{ kg-m}^2,\ I_{y'z'}^{(B)} = 0.$$

The inertia matrix is the sum of the two matrices:

$$[I'] = \begin{bmatrix} I_{x'x'} & -I_{x'y'} & -I_{x'z'} \\ -I_{y'x'} & I_{y'y'} & -I_{y'z'} \\ -I_{z'x'} & -I_{z'y'} & I_{z'z'} \end{bmatrix} = \begin{bmatrix} 12 & -15 & -6 \\ -1.5 & 19.67 & 0 \\ -6 & 0 & 10.33 \end{bmatrix}.$$

From Eq. (20.45), the principal values are the roots of the cubic equation $AI^3 + BI^2 + CI + D = 0$, where

$$A = +1,$$

$$B = -(I_{x'x'} + I_{y'y'} + I_{z'z'}) = -42,$$

$$C = (I_{x'x'}I_{y'y'} + I_{y'y'}I_{z'z'} + I_{z'z'}I_{x'x'} - I_{x'y'}^2 - I_{x'z'}^2 - I_{y'z'}^2)$$

$$= 524.97,$$

$$D = -(I_{x'x'}I_{y'y'}I_{z'z'} - I_{x'x'}I_{y'z'}^2 - I_{y'y'}I_{x'z'}^2 - I_{z'z'}I_{x'y'}^2$$

$$- 2I_{x'y'}I_{y'z'}I_{x'z'}) = -1707.4.$$

The function $f(I) = AI^3 + BI^2 + CI + D$ is graphed to determine the zero crossings, and the values refined by iteration. The graph is shown. The principal values are

$\boxed{I_1 = 5.042 \text{ kg-m}^2}$, $\boxed{I_2 = 16.79 \text{ kg-m}^2}$,

$\boxed{I_3 = 20.17 \text{ kg-m}^2}$.

Principal axes: The characteristic vectors parallel to the principal axes are obtained from Eq. (9.20),

$$V_{x'}^{(j)} = (I_{y'y'} - I_j)(I_{z'z'} - I_j) - I_{y'z'}^2$$

$$V_{y}^{(j)} = I_{x'y'}(I_{z'z'} - I_j) + I_{x'z'}I_{y'z'}$$

$$V_{z'}^{(j)} = I_{x'z'}(I_{y'y'} - I_j) + I_{x'y'}I_{y'z'}.$$

For $I_1 = 5.042$, $\mathbf{V}^{(1)} = 77.38\mathbf{i} + 7.937\mathbf{j} + 87.75\mathbf{k}$,

and $\boxed{\mathbf{e}_1 = 0.6599\mathbf{i} + 0.06768\mathbf{j} + 0.7483\mathbf{k}}$.

For $I_2 = 16.79$, $\mathbf{V}^{(2)} = -18.57\mathbf{i} - 9.687\mathbf{j} - 17.25\mathbf{k}$,

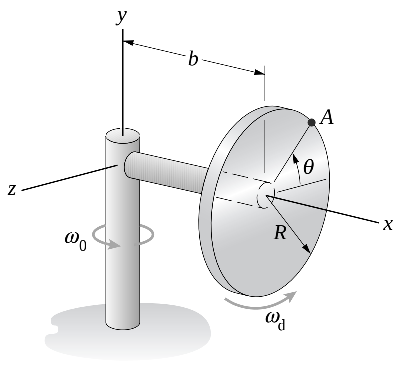
and $\boxed{\mathbf{e}_2 = -0.6843\mathbf{i} - 0.3570\mathbf{j} + 0.6358\mathbf{k}}$.

For $I_3 = 20.17$, $\mathbf{V}^{(3)} = 4.911\mathbf{i} - 14.75\mathbf{j} - 2.997\mathbf{k}$,

and $\mathbf{e}_3 = +0.3102\mathbf{i} - 0.9316\mathbf{j} - 0.1893\mathbf{k}$.

Problem 20.100 The disk is pinned to the horizontal shaft and rotates relative to it with angular velocity ω_0. Relative to an earth-fixed reference frame, the vertical shaft rotates with angular velocity ω_0.

(a) Determine the disk's angular velocity vector $\boldsymbol{\omega}$ relative to the earth-fixed reference frame.
(b) What is the velocity of point A of the disk relative to the earth-fixed reference frame?



Solution: From Problem 20.43, the inertia matrix is

$$[I] = \begin{bmatrix} 4.167 \times 10^5 & 2.5 \times 10^5 & 0 \\ 2.5 \times 10^5 & 4.167 \times 10^5 & 0 \\ 0 & 0 & 8.333 \times 10^5 \end{bmatrix} \text{ kg-m}^2.$$

For rotation at a constant rate, the angular acceleration is zero, $\boldsymbol{\alpha} = 0$. The body-fixed coordinate system rotates with angular velocity Eq. (20.19) reduces to:

$$\begin{bmatrix} \sum M_{Ox} \\ \sum M_{Oy} \\ \sum M_{Oz} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -0.01047 \\ 0 & 0.01047 & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} 4.167 \times 10^5 & 2.5 \times 10^5 & 0 \\ 2.5 \times 10^5 & 4.167 \times 10^5 & 0 \\ 0 & 0 & 8.333 \times 10^5 \end{bmatrix}$$

$$\times \begin{bmatrix} 0.01047 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -8726.7 \\ 2618.0 & 4363.3 & 0 \end{bmatrix} \begin{bmatrix} 0.01047 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 27.42 \end{bmatrix} \text{ N-m}, \quad |\mathbf{M}_0| = 27.4 \text{ (N-m)}$$

Problem 20.101 The disk is pinned to the horizontal shaft and rotates relative to it with constant angular velocity ω_0. Relative to an earth-fixed reference frame, the vertical shaft rotates with constant angular velocity ω_0. What is the acceleration of point A of the disk relative to the earth-fixed reference frame?

Solution: (See Figure in solution to Problem 20.100.) The angular acceleration of the disk is given by

$$\frac{d\boldsymbol{\omega}}{dt} = \frac{d}{dt}(\omega_d\mathbf{i} + \omega_O\mathbf{j}) + \boldsymbol{\omega}_O \times \boldsymbol{\omega}_d = 0 + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \omega_O & 0 \\ \omega_d & 0 & 0 \end{bmatrix}$$

$$= -\omega_O\omega_d\mathbf{k}.$$

The velocity of point A relative to O is

$$\mathbf{a}_{A/O} = \boldsymbol{\alpha} \times \mathbf{r}_{A/O} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{A/O})$$

$$= (-\omega_O\omega_d)(\mathbf{k} \times \mathbf{r}_{A/O}) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{A/O}).$$

Term by term:

$$-\omega_O\omega_d(\mathbf{k} \times \mathbf{r}_{A/O}) = -\omega_O\omega_d \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ b & R\sin\theta & -R\cos\theta \end{bmatrix}$$

$$= \omega_O\omega_d R\sin\theta\mathbf{i} - \omega_O\omega_d b\mathbf{j},$$

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{A/O}) = \boldsymbol{\omega} \times \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_d & \omega_O & 0 \\ b & R\sin\theta & -R\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_d & \omega_O & 0 \\ -R\omega_O\cos\theta & R\omega_d\cos\theta & R\omega_d\sin\theta - b\omega_O \end{bmatrix}$$

$$= (R\omega_d\sin\theta - b\omega_O)(\omega_O\mathbf{i} - \omega_d\mathbf{j}) + (R\cos\theta)(\omega_d^2 + \omega_O^2)\mathbf{k}.$$

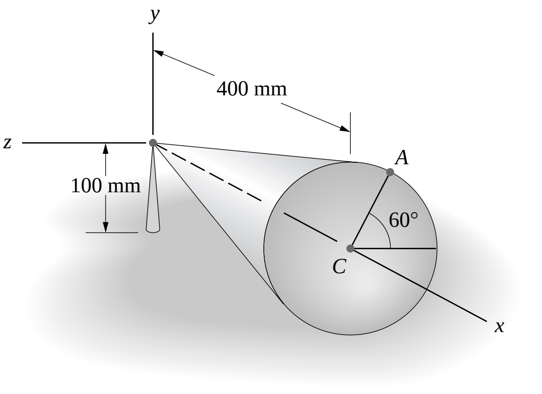
Collecting terms:

$$\mathbf{a}_{A/O} = (2R\omega_O\omega_d\sin\theta - b\omega_O^2)\mathbf{i} - (R\omega_d^2\sin\theta)\mathbf{j}$$

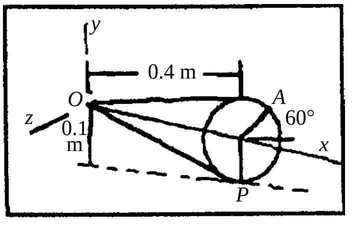
$$+ (R\omega_d^2\cos\theta + R\omega_O^2\cos\theta)\mathbf{k}.$$

Problem 20.102 The cone is connected by a ball-and-socket joint at its vertex to a 100-mm post. The radius of its base is 100 mm, and the base rolls on the floor. The velocity of the center of the base is $\mathbf{v}_C = 2\mathbf{k}$ (m/s).

(a) What is the cone's angular velocity vector $\boldsymbol{\omega}$?
(b) What is the velocity of point A?



Solution: Denote $\theta = 60^\circ$, $h = 0.1$ m, $L = 0.4$ m, $R = 0.1$ m.



(a) The strategy is to express the velocity of the center of the base of the cone, point C, in terms of the known (zero) velocities of O and P to formulate simultaneous equations for the angular velocity vector components.

The line OC is parallel to the vector $\mathbf{r}_{C/O} = \mathbf{i}L$ (m). The line PC is parallel to the vector $\mathbf{r}_{C/P} = R\mathbf{j}$ (m). The velocity of the center of the cone is given by the two expressions: $\mathbf{v}_C = \mathbf{v}_O + \boldsymbol{\omega} \times \mathbf{r}_{C/O}$, and $\mathbf{v}_C = \mathbf{v}_P + \boldsymbol{\omega} \times \mathbf{r}_{C/P}$, where $\mathbf{v}_C = 2\mathbf{k}$ (m/s), and $\mathbf{v}_O = \mathbf{v}_P = 0$.

Expanding:

$$\mathbf{v}_C = \boldsymbol{\omega} \times \mathbf{r}_{C/O} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_x & \omega_y & \omega_z \\ L & 0 & 0 \end{bmatrix} = \omega_z L\mathbf{j} - \omega_y L\mathbf{k}.$$

$$\mathbf{v}_C = \boldsymbol{\omega} \times \mathbf{r}_{C/P} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_x & \omega_y & \omega_z \\ 0 & R & 0 \end{bmatrix} = -R\omega_z\mathbf{i} + R\omega_x\mathbf{k}.$$

Solve by inspection:

$$\omega_x = \frac{v_C}{R} = 20 \text{ rad/s}, \ \omega_y = -\frac{v_C}{L} = -5 \text{ m/s},$$

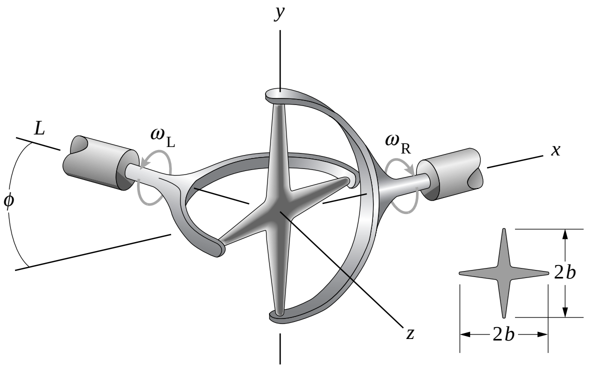
$\omega_z = 0.$ $\boxed{\boldsymbol{\omega} = 20\mathbf{i} - 5\mathbf{j} \text{ (rad/s)}}$

(b) The line OA is parallel to the vector $\mathbf{r}_{A/O} = \mathbf{i}L + \mathbf{j}R\sin\theta - \mathbf{k}R\cos\theta$. The velocity of point A is given by: $\mathbf{v}_A = \mathbf{v}_O + \boldsymbol{\omega} \times \mathbf{r}_{A/O}$, where $\mathbf{v}_O = 0$.

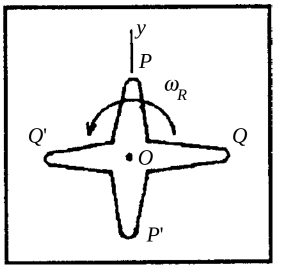
$$\mathbf{v}_A = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 20 & -5 & 0 \\ L & R\sin\theta & -R\cos\theta \end{bmatrix},$$

$\boxed{\mathbf{v}_A = 0.25\mathbf{i} + 1\mathbf{j} + 3.73\mathbf{k} \text{ (m/s)}}$

Problem 20.103 The mechanism shown is a type of universal joint called a yoke and spider. The axis L lies in the x-z plane. Determine the angular velocity ω_L and the angular velocity vector $\boldsymbol{\omega}_S$ of the cross-shaped "spider" in terms of the angular velocity ω_R at the instant shown.



Solution: Denote the center of mass of the spider by point O, and denote the line coinciding with the vertical arms of the spider (the y axis) by $P'P$, and the line coinciding with the horizontal arms by $Q'Q$. The line $P'P$ is parallel to the vector $\mathbf{r}_{P/O} = b\mathbf{j}$. The angular velocity of the right hand yoke is positive along the x axis, $\boldsymbol{\omega}_R = \omega_R \mathbf{i}$, from which the angular velocity $\boldsymbol{\omega}_L$ is positive toward the right, so that $\boldsymbol{\omega}_L = \omega_L(\mathbf{i}\cos\phi + \mathbf{k}\sin\phi)$. The velocity of the point P on the extremities of the line $P'P$ is $\mathbf{v}_P = \mathbf{v}_O + \boldsymbol{\omega}_R \times \mathbf{r}_{P/O}$, where $\mathbf{v}_O = 0$, from which



$$\mathbf{v}_P = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_R & 0 & 0 \\ 0 & b & 0 \end{bmatrix} = b\omega_R \mathbf{k}.$$

The velocity v_P is also given by

$$\mathbf{v}_P = \mathbf{v}_O + \boldsymbol{\omega}_S \times \mathbf{r}_{P/O} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_{Sx} & \omega_{Sy} & \omega_{Sz} \\ 0 & b & 0 \end{bmatrix}$$

$$= -\mathbf{i}b\omega_{Sz} + \mathbf{k}b\omega_{Sx},$$

from which $\omega_{Sz} = 0$, $\omega_{Sx} = \omega_R$. The line $Q'Q$ is parallel to the vector $\mathbf{r}_{Q/O} = \mathbf{i}b\sin\phi - \mathbf{k}b\cos\phi$. The velocity of the point Q is

$$\mathbf{v}_Q = \mathbf{v}_O + \boldsymbol{\omega}_S \times \mathbf{r}_{Q/O} = 0 + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_{Sx} & \omega_{Sy} & 0 \\ b\sin\phi & 0 & -b\cos\phi \end{bmatrix},$$

$$\mathbf{v}_Q = -\mathbf{i}(\omega_{Sy} b\cos\phi) + \mathbf{j}(\omega_{Sx} b\cos\phi) - \mathbf{k}(\omega_{Sy} b\sin\phi).$$

The velocity $\mathbf{v}_Q$ is also given by $\mathbf{v}_Q = \mathbf{v}_O + \boldsymbol{\omega}_L \times \mathbf{r}_{Q/O}$, from which

$$\mathbf{v}_Q = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_L\cos\phi & 0 & \omega_L\sin\phi \\ b\sin\phi & 0 & -b\cos\phi \end{bmatrix} = \mathbf{j}b\omega_L(\cos^2\phi + \sin^2\phi)$$

$$= \mathbf{j}b\omega_L,$$

from which $\omega_{Sy} = 0$, from which

$\omega_S = \omega_R$,

$\boxed{\boldsymbol{\omega}_S = \omega_R \mathbf{i}}$, $\boxed{\omega_L = \omega_{Sx}\cos\phi = \omega_R\cos\phi}$.

Problem 20.104 The inertia matrix of a rigid body in terms of a body-fixed coordinate system with its origin at the center of mass is

$$[I] = \begin{bmatrix} 4 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 6 \end{bmatrix} \text{ kg-m}^2.$$

If the rigid body's angular velocity is $\boldsymbol{\omega} = 10\mathbf{i} - 5\mathbf{j} + 10\mathbf{k}$ (rad/s), what is its angular momentum about its center of mass?

Solution: The angular momentum is

$$\begin{bmatrix} H_{Ox} \\ H_{Oy} \\ H_{Oz} \end{bmatrix} = \begin{bmatrix} 4 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 6 \end{bmatrix} \begin{bmatrix} 10 \\ -5 \\ 10 \end{bmatrix} = \begin{bmatrix} 25 \\ 0 \\ 50 \end{bmatrix} \text{ kg-m}^2/\text{s}$$

In terms of the unit vectors **i**, **j**, **k**, $\boxed{\mathbf{H} = 25\mathbf{i} + 50\mathbf{k} \text{ kg-m}^2/\text{s}}$

Problem 20.105 What is the moment of inertia of the rigid body in Problem 20.104 about the axis that passes through the origin and the point (4, −4, 7) m?

Strategy: Determine the components of a unit vector parallel to the axis, and use Eq. (20.43).

Solution: The unit vector parallel to the line passing through (0, 0, 0) and (4, −4, 7) is

$$\mathbf{e} = \frac{4\mathbf{i} - 4\mathbf{j} + 7\mathbf{k}}{\sqrt{4^2 + 4^2 + 7^2}} = 0.4444\mathbf{i} - 0.4444\mathbf{j} + 0.7778\mathbf{k}.$$

The inertia matrix in Problem 20.104 is

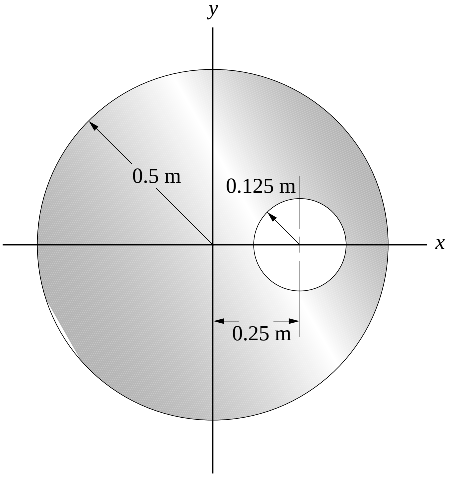
$$[I] = \begin{bmatrix} 4 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 6 \end{bmatrix} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix},$$

where advantage is taken of the symmetric property of the inertia matrix. From Eq. (20.43), the new moment of inertia about the line through (0, 0, 0) and (4, −4, 7) is

$$I_O = 4e_x^2 + 2e_y^2 + 6e_z^2 + 2(1)(e_x e_y) + 2(-1)(e_x e_z) + 2(0)(e_y e_z),$$

$$\boxed{I_O = 3.728 \text{ kg-m}^2}$$

Problem 20.106 Determine the inertia matrix of the 0.6 kg thin plate in terms of the coordinate system shown.



Solution: The strategy is to determine the moments and products for a solid thin plate about the origin, and then subtract the moments and products of the cut-out. The mass density is

$$\rho = \frac{0.6}{\left(\pi(0.5^2) - \pi\left(\frac{3}{24}\right)^2\right)T} = \frac{0.8149}{T}\ \text{kg/m}^3,$$

from which $\rho T = 0.8149$ kg/m^2, where T is the (unknown) thickness of the plate. From Appendix C and by inspection, the moments and products of inertia for a thin plate of radius R are:

$$I_{xx} = I_{yy} = \frac{mR^2}{4}, I_{zz} = \frac{mR^2}{2}, I_{xy} = I_{xz} = I_{yz} = 0.$$

For a 6 *in.* radius *solid thin plate*, $m = \rho T\pi(0.5^2) = 0.64$ kg. $I_{xx} = I_{yy} = 0.04$ kg-m^2, $I_{zz} = 0.08$ kg-m^2, $I_{xy} = I_{xz} = I_{yz} = 0$. The coordinates of the 0.125 m radius cut-out are $(d_x, d_y, d_x) = (3, 0, 0)$. The mass removed by the cut-out is $m_C = \rho T\pi R_C^2 = 0.04$ kg. The moments and products of inertia of the cut-out are

$$I_{xx}^C = \frac{m_C R_C^2}{4} = 1.563 \times 10^{-4}\ \text{kg-m}^2,$$

$$I_{yy}^C = \frac{m_C R_C^2}{4} + \left(\frac{3}{12}\right)^2 m_C = 2.656 \times 10^{-3}\ \text{kg-m}^2,$$

$$I_{zz}^C = \frac{m_C R_C^2}{2} + d_x^2 m_C = 2.813 \times 10^{-3}\ \text{kg-m}^2,$$

$$I_{xy}^C = 0, I_{xz}^C = 0, I_{yz}^C = 0.$$

The inertia matrix of the plate with the cut-out is

$$[I]_0 = \begin{bmatrix} 0.04 & 0 & 0 \\ 0 & 0.04 & 0 \\ 0 & 0 & 0.08 \end{bmatrix}$$

$$- \begin{bmatrix} 1.563 \times 10^{-4} & 0 & 0 \\ 0 & 2.656 \times 10^{-3} & 0 \\ 0 & 0 & 2.813 \times 10^{-3} \end{bmatrix},$$

$$\boxed{[I]_0 = \begin{bmatrix} 0.0398 & 0 & 0 \\ 0 & 0.0373 & 0 \\ 0 & 0 & 0.0772 \end{bmatrix}\ \text{kg-m}^2}$$

Problem 20.107 At $t = 0$, the plate in Problem 20.106 has angular velocity $\boldsymbol{\omega} = 10\mathbf{i} + 10\mathbf{j}$ (rad/s) and is subjected to the force $\mathbf{F} = -10\mathbf{k}$ (N) acting at the point $(0, 0.5, 0)$ m. No other forces or couples act on the plate. What are the components of its angular acceleration at that instant?

Solution: The coordinates of the center of mass are $(-0.01667, 0, 0)$ m. The vector from the center of mass to the point of application of the force is $\mathbf{r}_{F/G} = 0.01667\mathbf{i} + 0.5\mathbf{j}$ (m). The moment about the center of mass of the plate is

$$\mathbf{M}_G = \mathbf{r}_{F/G} \times \mathbf{F} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.01667 & 0.5 & 0 \\ 0 & 0 & -10 \end{bmatrix}$$

$$= -5\mathbf{i} + 0.1667\mathbf{j} \text{ (N-m)}.$$

Eq. (20.19) reduces to

$$\begin{bmatrix} -5 \\ 0.1667 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.03984 & 0 & -0 \\ 0 & 0.03718 & 0 \\ 0 & 0 & 0.07702 \end{bmatrix} \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -0.267 \end{bmatrix}.$$

Carry out the matrix multiplication to obtain the three equations:

$0.03984\alpha_x = -5$, $0.03718\alpha_y = 0.1667$, $0.07702\alpha_z - 0.267 = 0$.

Solve: $\boxed{\boldsymbol{\alpha} = -125.5\mathbf{i} + 4.484\mathbf{j} + 3.467\mathbf{k} \text{ rad/s}^2}$

Problem 20.108 The inertia matrix of a rigid body in terms of a body-fixed coordinate system with its origin at the center of mass is

$$[I] = \begin{bmatrix} 4 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 6 \end{bmatrix} \text{ kg-m}^2.$$

If the rigid body's angular velocity is $\boldsymbol{\omega} = 10\mathbf{i} - 5\mathbf{j} + 10\mathbf{k}$ (rad/s) and its angular acceleration is zero, what are the components of the total moment about its center of mass?

Solution: Use general motion, Eq. (20.19),

$$\begin{bmatrix} \sum M_{\phi x} \\ \sum M_{\phi y} \\ \sum M_{\phi z} \end{bmatrix} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix} \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

with $\boldsymbol{\alpha} = \dfrac{d\boldsymbol{\omega}}{dt} = 0$. The coordinate system is rotating with angular velocity $\boldsymbol{\omega}$, from which $\boldsymbol{\omega} = \boldsymbol{\omega}$. Eq. (20.19) reduces to

$$\begin{bmatrix} \sum M_{\phi x} \\ \sum M_{\phi y} \\ \sum M_{\phi z} \end{bmatrix} = \begin{bmatrix} 0 & -10 & -5 \\ 10 & 0 & -10 \\ 5 & 10 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 6 \end{bmatrix} \begin{bmatrix} 10 \\ -5 \\ 10 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & -20 & -30 \\ 50 & 10 & -70 \\ 30 & 25 & -5 \end{bmatrix} \begin{bmatrix} 10 \\ -5 \\ 10 \end{bmatrix} \text{ N-m,}$$

$$\boxed{\mathbf{M} = -250\mathbf{i} - 250\mathbf{j} + 125\mathbf{k} \text{ (N-m)}}$$

Problem 20.109 If the total moment about the center of mass of the rigid body described in Problem 20.108 is zero, what are the components of its angular acceleration?

Solution: Use general motion, Eq. (20.19), with the moment components equated to zero,

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix} \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 6 \end{bmatrix} \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{bmatrix} + \begin{bmatrix} 0 & -10 & -5 \\ 10 & 0 & -10 \\ 5 & 10 & 0 \end{bmatrix}$$

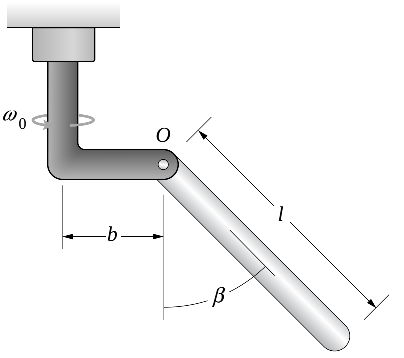
$$\times \begin{bmatrix} 4 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 6 \end{bmatrix} \begin{bmatrix} 10 \\ -5 \\ 10 \end{bmatrix},$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 6 \end{bmatrix} \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{bmatrix} + \begin{bmatrix} -250 \\ -250 \\ 125 \end{bmatrix}.$$

Carry out the matrix multiplication to obtain the three simultaneous equations in the unknowns: $4\alpha_x + \alpha_y - \alpha_z = 250$, $\alpha_x + 2\alpha_y + 0 = 250$, $-\alpha_x + 0 + 6\alpha_z = -125$.

Solve: $\boxed{\boldsymbol{\alpha} = 31.25\mathbf{i} + 109.4\mathbf{j} - 15.63\mathbf{k} \text{ (rad/s}^2)}$

Problem 20.110 The slender bar of length l and mass m is pinned to the L-shaped bar at O. The L-shaped bar rotates about the vertical axis with a constant angular velocity ω_0. Determine the value of ω_0 necessary for the bar to remain at a constant angle β relative to the vertical.



Solution: Since the point O is not fixed, this is general motion, in which Eq. (20.19) applies. Choose a coordinate system with the origin at O and the x axis parallel to the slender bar.

The moment exerted by the bar. Since axis of rotation is fixed, the acceleration must be taken into account in determining the moment. The vector distance from the axis of rotation in the coordinates system shown is

$$\mathbf{r}_0 = b(\mathbf{i}\cos(90^\circ - \beta) + \mathbf{j}\sin(90^\circ - \beta)) = b(\mathbf{i}\sin\beta + \mathbf{j}\cos\beta).$$

The vector distance to the center of mass of the slender bar is $\mathbf{r}_{G/O} = \mathbf{i}\left(\frac{L}{2}\right)$. The angular velocity is a constant and the coordinate system is rotating with an angular velocity $\boldsymbol{\omega} = \omega_0(-\mathbf{i}\cos\beta + \mathbf{j}\sin\beta)$. The acceleration of the center of mass relative to O is

$$\mathbf{a}_G = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times (\mathbf{r}_0 + \mathbf{r}_{G/O}))$$

$$= \boldsymbol{\omega} \times \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\omega_0\cos\beta & \omega_0\sin\beta & 0 \\ b\sin\beta + \dfrac{L}{2} & b\cos\beta & 0 \end{bmatrix},$$

$$\mathbf{a}_G = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\omega_0\cos\beta & \omega_0\sin\beta & 0 \\ 0 & 0 & -\omega_0 b - \dfrac{\omega_0 L\sin\beta}{2} \end{bmatrix}$$

$$= a_{Gx}\mathbf{i} + a_{Gy}\mathbf{j}$$

$$\mathbf{a}_G = -\omega_0^2\left(+b\sin\beta + \frac{L}{2}\sin^2\beta\right)\mathbf{i}$$

$$-\omega_0^2\left(b\cos\beta + \frac{L}{2}\sin\beta\cos\beta\right)\mathbf{j}.$$

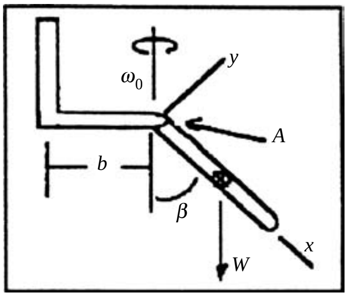
From Newton's second law, $m\mathbf{a}_G = \mathbf{A} + \mathbf{W}$, from which $\mathbf{A} = m\mathbf{a}_G - \mathbf{W}$. The weight is $\mathbf{W} = mg(\mathbf{i}\cos\beta - \mathbf{j}\sin\beta)$. The moment about the center of mass is

$$\mathbf{M}_G = \mathbf{r}_{O/G} \times \mathbf{A} = \mathbf{r}_{O/G} \times (m\mathbf{a}_G - \mathbf{W})$$

$$= \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\dfrac{L}{2} & 0 & 0 \\ ma_{Gx} - mg\cos\beta & ma_{Gy} + mg\sin\beta & 0 \end{bmatrix}.$$

$$\mathbf{M}_G = +\left(\frac{m\omega_o^2 bL}{2}\cos\beta - \frac{mgL}{2}\sin\beta + \frac{m\omega_0^2 L^2}{4}\sin\beta\cos\beta\right)\mathbf{k}$$

$$= M_z\mathbf{k}$$



The Euler Equations: The moments of inertia of the bar about the center of mass are $I_{xx} = 0$, $I_{yy} = I_{zz} = \dfrac{mL^2}{12}$, $I_{xy} = I_{xz} = I_{yz} = 0$. Eq. (20.19) becomes:

$$\begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} = \omega_o^2 \begin{bmatrix} 0 & 0 & +\sin\beta \\ 0 & 0 & \cos\beta \\ -\sin\beta & -\cos\beta & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \dfrac{mL^2}{12} & 0 \\ 0 & 0 & \dfrac{mL^2}{12} \end{bmatrix}$$

$$\times \begin{bmatrix} -\cos\beta \\ \sin\beta \\ 0 \end{bmatrix}.$$

Carry out the matrix multiplication:

$$\begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\dfrac{\omega_0^2 mL^2}{12}\cos\beta\sin\beta \end{bmatrix}.$$

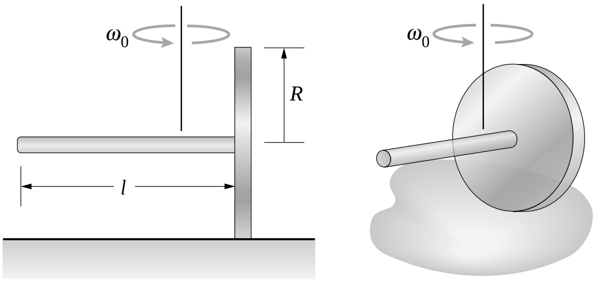
Substitute:

$$\frac{m\omega_0^2 bL}{2}\cos\beta - \frac{mgL}{2}\sin\beta + \frac{m\omega_0^2 L^2}{4}\sin\beta\cos\beta$$

$$= -\omega_0^2\frac{mL^2}{12}\sin\beta\cos\beta.$$

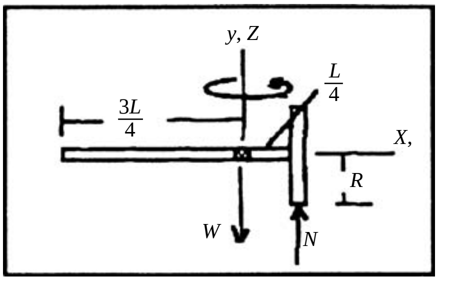
Solve : $\boxed{\omega_0 = \sqrt{\dfrac{g\sin\beta}{\left(\dfrac{2}{3}\right)L\sin\beta\cos\beta + b\cos\beta}}}$

Problem 20.111 A slender bar of length l and mass m is rigidly attached to the center of a thin circular disk of radius R and mass m. The composite object undergoes a motion in which the bar rotates in the horizontal plane with constant angular velocity ω_0 about the center of mass of the composite object and the disk rolls on the floor. Show that $\omega_0 = 2\sqrt{g/R}$.



Solution: Measuring from the left end of the slender bar, the distance to the center of mass is

$$d_G = \frac{\left(\frac{L}{2}\right)m + mL}{2m} = \frac{3L}{4}.$$



Choose an X, Y, Z coordinate system with the origin at the center of mass, the Z axis parallel to the vertical axis of rotation and the X axis parallel to the slender bar. Choose an x, y, z coordinate system with the origin at the center of mass, the z axis parallel to the slender bar, and the y axis parallel to the Z axis. By definition, the nutation angle is the angle between Z and z, $\theta = 90^\circ$. The precession rate is the rotation about the Z axis, $\dot{\psi} = \omega_0$ rad/s. The velocity of the center of mass of the disk is $v_G = (L/4)\dot{\psi}$, from which the spin rate is $\dot{\phi} = \frac{v}{R} = \left(\frac{L}{4R}\right)\dot{\psi}$. From Eq. (20.29), the moment about the x-axis is

$$M_x = (I_{zz} - I_{xx})\dot{\psi}^2 \sin\theta\cos\theta + I_{zz}\dot{\psi}\dot{\phi}\sin\theta = I_{zz}\dot{\phi}\dot{\psi} = \frac{\omega_0^2 L}{4R} I_{zz}.$$

The moment of inertia is $I_{zz} = \frac{mR^2}{2}$, from which

$$M_x = \left(\frac{mRL}{8}\right)\omega_0^2.$$

The normal force acting at the point of contact is $N = 2mg$.
The moment exerted about the center of mass

$$M_G = N\frac{L}{4} = mg\left(\frac{L}{2}\right).$$

Equate the moments: $mg\left(\frac{L}{2}\right) = m\left(\frac{RL}{8}\right)\omega_0^2$, from which

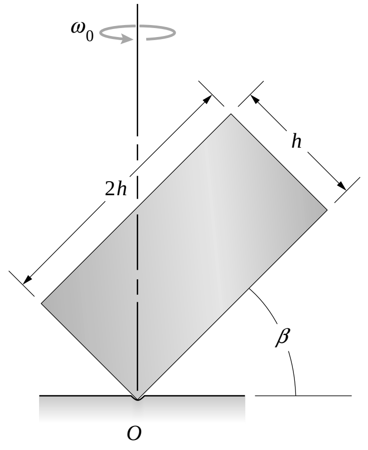
$$\boxed{\omega_0 = 2\sqrt{\frac{g}{R}}}.$$

Problem 20.112* The thin plate of mass m spins about a vertical axis with the plane of the plate perpendicular to the floor. The corner of the plate at O rests in an indentation, so that it remains at the same point on the floor. The plate rotates with constant angular velocity ω_0 and the angle β is constant.

(a) Show that the angular velocity ω_0 is related to the angle β by

$$\frac{h\omega_0^2}{g} = \frac{2\cos\beta - \sin\beta}{\sin^2\beta - 2\sin\beta\cos\beta - \cos^2\beta}$$

(b) The equation you obtained in (a) indicates that $\omega_0 = 0$ when $2\cos\beta - \sin\beta = 0$. What is the interpretation of this result?



Solution:

Choose a body-fixed coordinate system with its origin at the fixed point O and the axes aligned with the plate's edges. Using the moments of intertia for a rectangular area,

$$I_{xx} = \frac{mh^2}{3},\ I_{yy} = \frac{4mh^2}{3},\ I_{zz} = I_{xx} + I_{yy} = \frac{5mh^2}{3},$$

$$I_{xy} = \frac{mh^2}{2},\ I_{xz} = I_{yz} = 0.$$

The plate's angular velocity is $\boldsymbol{\omega} = \omega_0 \sin\beta\mathbf{i} + \omega_0\cos\beta\mathbf{j}$, and the moment about O due to the plate's weight is $\sum M_x = 0$, $\sum M_y = 0$, $\sum M_z = \frac{h}{2}mg\sin\beta - hmg\cos\beta$.

Choose a coordinate system with the x axis parallel to the right lower edge of the plate and the y axis parallel to the left lower edge of the plate, as shown. The body fixed coordinate system rotates with angular velocity $\boldsymbol{\omega} = \omega_0(\mathbf{j}\sin\beta + \mathbf{k}\cos\beta)$. From Eq. (20.13),

$$\begin{bmatrix} 0 \\ 0 \\ M_z \end{bmatrix} = \begin{bmatrix} 0 & 0 & \omega_0\cos\beta \\ 0 & 0 & -\omega_0\sin\beta \\ -\omega_0\cos\beta & \omega_0\sin\beta & 0 \end{bmatrix}$$

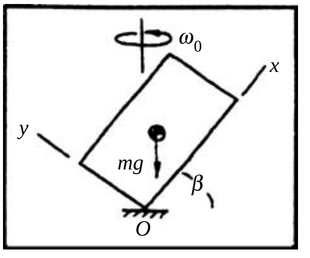
$$\times \begin{bmatrix} \frac{mh^2}{3} & -\frac{mh^2}{2} & 0 \\ -\frac{mh^2}{2} & \frac{4mh^2}{3} & 0 \\ 0 & 0 & \frac{5m^2}{3} \end{bmatrix} \begin{bmatrix} \omega_0\cos\beta \\ \omega_0\cos\beta \\ 0 \end{bmatrix}.$$

Expand, $M_z = hmg\left(\frac{\sin\beta}{2} - \cos\beta\right)$

$$= mh^2\omega_a^2\left(-\frac{\sin\beta\cos\beta}{3} + \frac{\cos^2\beta}{2} - \frac{\sin^2\beta}{2}\right.$$

$$\left. + \frac{4\sin\beta\cos\beta}{3}\right)$$

Solve: $\boxed{\frac{\omega_0^2 h}{g} = \frac{2\cos\beta - \sin\beta}{\sin^2\beta - 2\sin\beta\cos\beta - \cos^2\beta}}$



(b) The perpendicular distance from the axis of rotation to the center of mass of the plate is

$$d = \left|\mathbf{r}_{O/G} \times \frac{\boldsymbol{\omega}}{|\boldsymbol{\omega}|}\right| = \left\|\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -\frac{h}{2} & -h \\ 0 & \cos\beta & \sin\beta \end{bmatrix}\right\|$$

$$= h\left(\cos\beta - \frac{\sin\beta}{2}\right).$$

If this distance is zero, $\beta = \tan^{-1}(2) = 63.43^\circ$, the accelerations of the center of mass and the external moments are zero (see equations above, where for convenience the term $\cos\beta - \frac{\sin\beta}{2}$ has been kept as a factor) and the plate is balanced.

The *angular velocity of rotation is zero* (the plate is stationary) *if* $\beta = \tan^{-1}(2) = 63.435^\circ$, since the numerator of the right hand term in the boxed expression vanishes (the balance at this point would be very unstable, since an infinitesimally small change in β would induce a destabilizing moment.).

Problem 20.113* In Problem 20.112, determine the range of values of the angle β for which the plate will remain in the steady motion described.

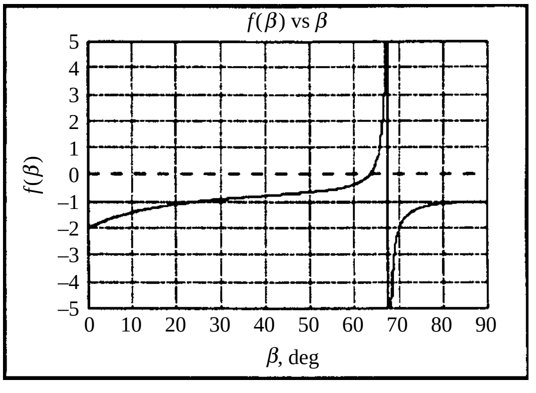
Solution: From the solution to Problem 20.112

$$\frac{\omega_0^2 h}{g} = \frac{2\cos\beta - \sin\beta}{\sin^2\beta - 2\sin\beta\cos\beta - \cos^2\beta}.$$

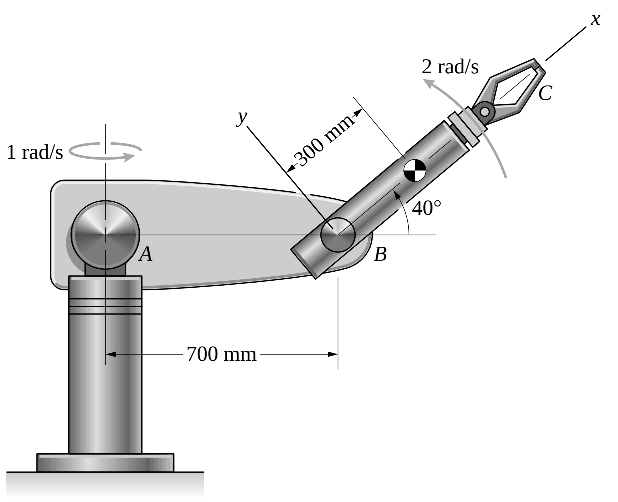
The angular velocity is a real number, from which $\omega_0^2 \geq 0$, from which

$$f(\beta) = \frac{2\cos\beta - \sin\beta}{\sin^2\beta - 2\cos\beta\sin\beta - \cos^2\beta} \geq 0.$$

A graph of $f(\beta)$ for values of $0 \leq \beta \leq 90^\circ$ is shown. The function is positive over the half-open interval $63.4348^\circ \leq \beta < 67.50^\circ$. The angular velocity is zero at the lower end of the interval, and "blows up" (becomes infinite) when the denominator vanishes, which occurs at *exactly* $\beta = \frac{3\pi}{8} = 67.50^\circ$.



Problem 20.114* Arm BC has a mass of 12 kg, and its moments and products of inertia, in terms of the coordinate system shown, are $I_{xx} = 0.03$ kg-m^2, $I_{yy} = I_{zz} = 4$ kg-m^2, and $I_{xy} = I_{yz} = I_{xz} = 0$. At the instant shown, arm AB is rotating in the horizontal plane with a constant angular velocity of 1 rad/s in the counterclockwise direction viewed from above. Relative to arm AB, arm BC is rotating about the z axis with a constant angular velocity of 2 rad/s. Determine the force and couple exerted on arm BC at B.



Solution: In terms of the body-fixed coordinate system shown, the moments and products of inertia are

$$I_{xx} = 0.03 \text{ kg-m}^2,$$

$$I_{yy} = I_{zz} = 4 - (0.3)^2(12) = 2.92 \text{ kg-m}^2,$$

$$I_{xy} = I_{yz} = I_{zx} = 0.$$

In terms of the angle θ, the angular velocity of the coordinate system is

$$\boldsymbol{\omega} = (1)\sin\theta\mathbf{i} + (1)\cos\theta\mathbf{j} + (2)\mathbf{k} \text{ (rad/s)},$$

so its angular acceleration is Eq. (20.4)

$$\boldsymbol{\alpha} = \frac{d\boldsymbol{\omega}}{dt} = \cos\theta\frac{d\theta}{dt}\mathbf{i} - \sin\theta\frac{d\theta}{dt}\mathbf{j}.$$

Setting $\theta = 40^\circ$ and $d\theta/dt = 2$ rad/s, we obtain

$$\boldsymbol{\omega} = 0.643\mathbf{i} + 0.766\mathbf{j} + 2\mathbf{k} \text{ (rad/s)}, \quad \textbf{(1)}$$

$$\boldsymbol{\alpha} = 1.532\mathbf{i} - 1.286\mathbf{j} \text{ (rad/s}^2). \qquad \textbf{(2)}$$

Equation (20.19) is

$$\begin{bmatrix} M_{Bx} \\ M_{By} + 0.3F_{Bz} \\ M_{Bz} - 0.3F_{By} \end{bmatrix} = \begin{bmatrix} 0.03 & 0 & 0 \\ 0 & 2.92 & 0 \\ 0 & 0 & 2.92 \end{bmatrix} \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \begin{bmatrix} 0.03 & 0 & 0 \\ 0 & 2.92 & 0 \\ 0 & 0 & 2.92 \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}.$$

Using Eqs. (1) and (2), this gives the equations

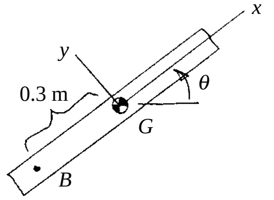
$$M_{Bx} = 0.046,$$

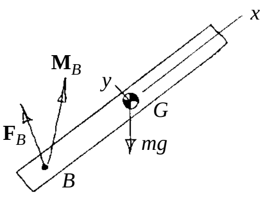
$$M_{By} + 0.3F_{Bz} = -7.469,$$

$$M_{Bz} - 0.3F_{By} = 1.423,$$

From which

$$\underline{\mathbf{M}_B = 0.046\mathbf{i} - 10.25\mathbf{j} + 30.63\mathbf{k} \text{ (N-m)}}.$$





The acceleration of B toward A is $(1 \text{ rad/s})^2(0.7 \text{ m}) = 0.7$ m/s^2, so

$$\mathbf{a}_B = -0.7\cos 40^\circ\mathbf{i} + 0.7\sin 40^\circ\mathbf{j}$$

$$= -0.536\mathbf{i} + 0.450\mathbf{j} \text{ (m/s}^2).$$

The acceleration of the center of mass is

$$\mathbf{a}_G = \mathbf{a}_B + \boldsymbol{\alpha} \times \mathbf{r}_{G/B} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{G/B})$$

$$= -1.912\mathbf{i} + 0.598\mathbf{j} + 0.771\mathbf{k} \text{ (m/s}^2).$$

From Newton's second law (see free-body diagram),

$$F_{Bx} - mg\sin 40^\circ = m(-1.912),$$

$$F_{By} - mg\cos 40^\circ = m(0.598),$$

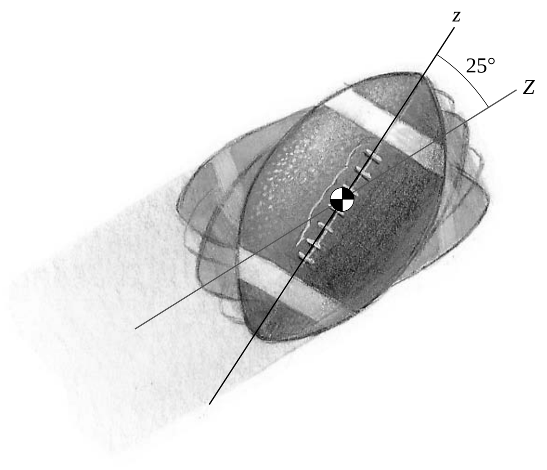
$$F_{Bz} = m(0.771).$$

Solving, $\underline{\mathbf{F}_B = 52.72\mathbf{i} + 97.35\mathbf{j} + 9.26\mathbf{k} \text{ (N)}}$. The moment about the center of mass is

$$\sum \mathbf{M} = \mathbf{M}_B + (-0.3\mathbf{i}) \times \mathbf{F}_B$$

$$= \mathbf{M}_B + 0.3F_{Bz}\mathbf{j} - 0.3F_{By}\mathbf{k}.$$

Problem 20.115 Suppose that you throw a football in a wobbly spiral with a nutation angle of 25°. The football's moments of inertia are $I_{xx} = I_{yy} = 0.0041$ kg-m^2 and $I_{zz} = 0.00136$ kg-m^2. If the spin rate is $\dot{\phi} = 4$ revolutions per second, what is the magnitude of the precession rate (the rate at which the football wobbles)?



Solution: This is modeled as moment-free, steady precession of an axisymmetric object. From Eq. (20.33), $(I_{zz} - I_{xx})\dot{\psi}\cos\theta + I_{zz}\dot{\phi} = 0$, from which

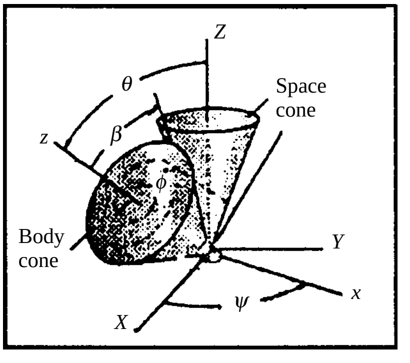
$$\dot{\psi} = -\frac{I_{zz}\dot{\phi}}{(I_{zz} - I_{xx})\cos\theta}.$$

Substitute $\dot{\psi} = -\dfrac{(0.00136\text{ kg-m}^2)(4)}{(0.00136\text{ kg-m}^2 - 0.0041\text{ kg-m}^2)\cos 25^\circ} = 2.21$ rev/s.

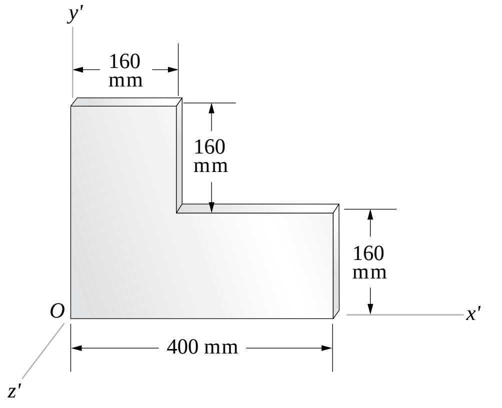
$$\boxed{|\dot{\psi}| = 2.21\text{ rev/s}}$$

Problem 20.116 Sketch the body and space cones for the motion of the football in Problem 20.115.

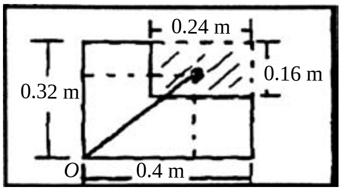
Solution: The angle β is given by $\tan\beta = \left(\dfrac{I_{zz}}{I_{xx}}\right)\tan\theta$, from which $\beta = 8.8^\circ$. $\beta < \theta$, and the body cone revolves outside the space cone. The sketch is shown. The angle $\theta = 25^\circ$.



Problem 20.117 The mass of the homogeneous thin plate is 1 kg. For a coordinate system with its origin at O, determine the plate's principal moments of inertia and the directions of unit vectors parallel to the corresponding principal axes.



Solution: The moment of inertia is determined by the strategy of determining the moments and products of a larger plate, and then subtracting the moments and products of inertia of a cutout, as shown in the sketch. Denote $h = 0.32$ m, $b = 0.4$ m, $c = 0.16$ m, $d = b - c = 0.24$ m. The area of the plate is $A = hb - cd = 0.0896$ m^2. Denote the mass density by ρ kg/m^3. The mass is $\rho AT = 1$ kg, from which $\rho T = \frac{1}{A} = 11.16$ kg/m^2, where T is the (unknown) thickness. *The moments and products of inertia of the large plate*: The moments and products of inertia about O are (See Appendix C)



$$I_{xx}^{(p)} = \frac{\rho T b h^3}{3}, I_{yy}^{(p)} = \frac{\rho T h b^3}{3},$$

$$I_{zz}^{(p)} = I_{yy}^{(p)} + I_{xx}^{(p)} = \frac{\rho T bh(h^2 + b^2)}{3},$$

$$I_{xy}^{(p)} = \frac{\rho T h^2 b^2}{4}, I_{xz}^{(p)} = I_{yz}^{(p)} = 0.$$

The moments and products of inertia for the cutout: The moments and products of inertia about the center of mass of the cutout are:

$$I_{xx}^{(c)} = \frac{\rho T d c^3}{12}, I_{yy}^{(c)} = \frac{\rho T c d^3}{12}, I_{zz}^{(c)} = \frac{\rho T cd(c^2 + d^2)}{12},$$

$$I_{xy}^{(c)} = I_{xz}^{(c)} = I_{yz}^{(c)} = 0.$$

The distance from O to the center of mass of the cutout is $(d_x, d_y, 0) = (0.28, 0.24, 0)$ m. The moments and products of inertia about the point O are

$$I_{xx}^{(0)} = I_{xx}^{(c)} + \rho T cd(d_y^2), I_{yy}^{(0)} = I_{yy}^{(c)} + \rho T cd(d_x^2),$$

$$I_{zz}^{(0)} = I_{zz}^{(c)} + \rho T cd(d_x^2 + d_y^2),$$

$$I_{xy}^{(0)} = I_{xy}^{(c)} + \rho T cd(d_x d_y), I_{xz}^{(0)} = I_{yz}^{(0)} = 0.$$

The moments and products of inertia of the object: The moments and products of inertia about O are

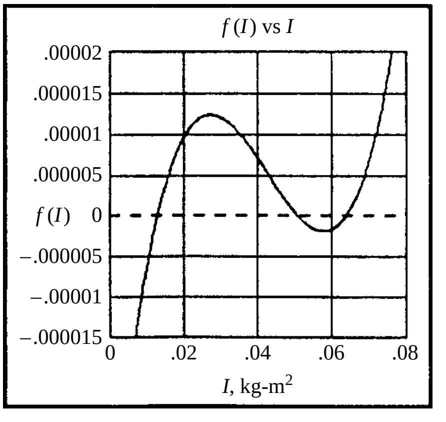
$$I_{xx} = I_{xx}^{(p)} - I_{xx}^{(0)} = 0.02316 \text{ kg-m}^2,$$

$$I_{yy} = I_{yy}^{(p)} - I_{yy}^{(0)} = 0.04053 \text{ kg-m}^2,$$

$$I_{zz} = I_{zz}^{(p)} - I_{zz}^{(0)} = 0.06370 \text{ kg-m}^2,$$

$$I_{xy} = I_{xy}^{(p)} - I_{xy}^{(0)} = 0.01691 \text{ kg-m}^2,$$

$$I_{xz} = I_{yz} = 0.$$



The principal moments of inertia. The principal values are given by the roots of the cubic equation $AI^3 + BI^2 + CI + D = 0$, where

$$A = 1,\ B = -(I_{xx} + I_{yy} + I_{zz}) = -0.1274,$$

$$C = I_{xx}I_{yy} + I_{yy}I_{zz} + I_{xx}I_{zz} - I_{xy}^2 - I_{xz}^2 - I_{yz}^2 = 4.71 \times 10^{-3},$$

$$D = -(I_{xx}I_{yy}I_{zz} - I_{xx}I_{yz}^2 - I_{yy}I_{xz}^2 - I_{zz}I_{xy}^2 - 2I_{xy}I_{yz}I_{xz})$$

$$= -4.158 \times 10^{-5}.$$

The function $f(I) = AI^3 + BI^2 + CI + D$ is graphed to get an estimate of the roots, and these estimates are refined by iteration. The graph is shown. The refined values of the roots are $I_1 = 0.01283$ kg-m^2, $I_2 = 0.05086$ kg-m^2, $I_3 = 0.06370$ kg-m^2.

The principal axes. The principal axes are obtained from a solution of the equations

$$V_x = (I_{yy} - I)(I_{zz} - I) - I_{yz}^2$$

$$V_y = I_{xy}(I_{zz} - I) + I_{xz}I_{yz}$$

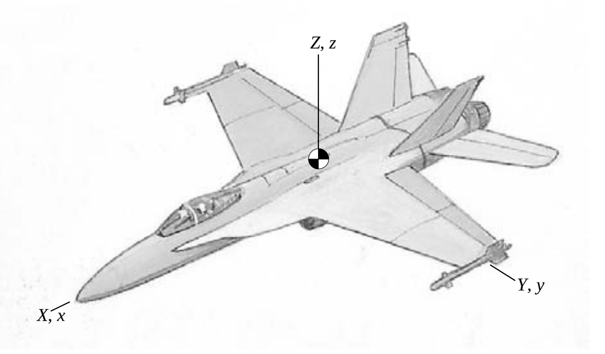
$$V_z = I_{xz}(I_{yy} - I) + I_{xz}I_{yz}.$$

Since $I_{xz} = I_{yz} = 0$, $V_z = 0$, and the solution fails for this axis, and the vector is to be determined from the orthogonality condition. Solving for V_x, V_y, the unit vectors are: for $I = I_1$, $V_x^{(1)} = 0.00141$, $V_y^{(1)} = 8.6 \times 10^{-4}$, from which the unit vectors are $\boxed{\mathbf{e}_1 = 0.8535\mathbf{i} + 0.5212\mathbf{j}}$. For $I = I_2$, $V_x^{(2)} = -1.325 \times 10^{-4}$, $V_y^{(2)} = 2.170 \times 10^{-4}$, from which $\boxed{\mathbf{e}_2 = -0.5212\mathbf{i} + 0.8535\mathbf{j}}$. The third unit vector is determined from orthogonality conditions: $\boxed{\mathbf{e}_3 = \mathbf{k}}$.

Problem 20.118 The airplane's principal moments of inertia, in kg-m^2, are $I_{xx} = 10844$, $I_{yy} = 65062$, and $I_{zz} = 67773$.

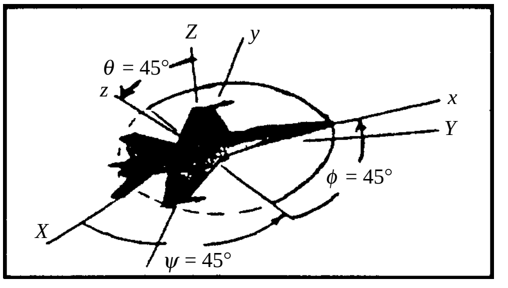
(a) The airplane begins in the reference position shown and maneuvers into the orientation $\psi = \theta = \phi = 45^\circ$. Draw a sketch showing the plane's orientation relative to the XYZ system.

(b) If the airplane is in the orientation described in (a), the rates of change of the Euler angles are $\dot{\psi} = 0$, $\dot{\theta} = 0.2$ rad/s, and $\dot{\phi} = 0.2$ rad/s, and the second derivatives of the angles with respect to time are zero, what are the components of the total moment about the airplane's center of mass?



Solution:

(a)



(b) The Eqs. (20.36) apply.

$$M_x = I_{xx}(\ddot{\psi}\sin\theta\sin\phi + \ddot{\theta}\cos\phi + \dot{\psi}\dot{\theta}\cos\theta\sin\phi$$

$$+ \dot{\psi}\dot{\phi}\sin\theta\cos\phi - \dot{\theta}\dot{\phi}\sin\phi) - (I_{yy} - I_{zz})$$

$$\times (\dot{\psi}\sin\theta\cos\phi - \dot{\theta}\sin\phi)(\dot{\psi}\cos\theta + \phi),$$

$$M_y = I_{yy}(\ddot{\psi}\sin\theta\cos\phi - \ddot{\theta}\sin\phi + \dot{\psi}\dot{\theta}\cos\theta\cos\phi$$

$$- \dot{\psi}\dot{\phi}\sin\theta\sin\phi - \dot{\theta}\dot{\phi}\cos\phi) - (I_{zz} - I_{xx})$$

$$\times (\dot{\psi}\sin\theta\sin\phi + \dot{\theta}\cos\phi)(\dot{\psi}\cos\theta + \dot{\phi}),$$

$$M_z = I_{zz}(\ddot{\psi}\cos\theta + \ddot{\phi} - \dot{\psi}\dot{\theta}\sin\theta) - (I_{xx} - I_{yy})(\dot{\psi}\sin\theta\sin\phi$$

$$+ \dot{\theta}\cos\phi)(\dot{\psi}\sin\theta\cos\phi - \dot{\theta}\sin\phi).$$

Substitute $I_{xx} = 10844$, $I_{yy} = 65062$, and $I_{zz} = 67773$, in kg-m^2, and $\dot{\psi} = 0$, $\dot{\theta} = 0.2$ rad/s, and $\dot{\phi} = 0.2$ rad/s, and $\ddot{\psi} = \ddot{\theta} = \ddot{\phi} = 0$. The moments:

$\boxed{M_x = -383.5 \text{ N-m}}$, $\boxed{M_y = -3451.2 \text{ N-m}}$,

$\boxed{M_z = -1084.6 \text{ N-m}}$

Problem 20.119 What are the x, y, and z components of the angular acceleration of the airplane described in Problem 20.118?

Solution: The angular accelerations are given by Eq. (20.35):

$$\frac{d\omega_x}{dt} = \ddot{\psi}\sin\theta\sin\phi + \dot{\psi}\dot{\theta}\cos\theta\sin\phi + \dot{\psi}\dot{\phi}\sin\theta\cos\phi$$

$$+\ddot{\theta}\cos\phi - \dot{\theta}\dot{\phi}\sin\phi,$$

$$\frac{d\omega_y}{dt} = \ddot{\psi}\sin\theta\cos\phi + \dot{\psi}\dot{\theta}\cos\theta\cos\phi - \dot{\psi}\dot{\phi}\sin\theta\sin\phi$$

$$-\ddot{\theta}\sin\phi - \dot{\theta}\dot{\phi}\cos\phi,$$

$$\frac{d\omega_z}{dt} = \ddot{\psi}\cos\theta - \dot{\psi}\dot{\theta}\sin\theta + \ddot{\phi}.$$

Substitute $I_{xx} = 10844$, $I_{yy} = 65062$, and $I_{zz} = 67773$, in kg-m^2, and $\dot{\psi} = 0$, $\dot{\theta} = 0.2$ rad/s, and $\dot{\phi} = 0.2$ rad/s, and $\ddot{\psi} = \ddot{\theta} = \ddot{\phi} = 0$, to obtain

$\boxed{\alpha_x = -0.0283 \text{ rad/s}^2}$, $\boxed{\alpha_y = -0.0283 \text{ rad/s}}$, $\boxed{\alpha_z = 0}$

Problem 20.120 If the orientation of the airplane in Problem 20.118 is $\psi = 45^\circ$, $\theta = 60^\circ$, and $\phi = 45^\circ$, the rates of change of the Euler angles are $\dot{\psi} = 0$, $\dot{\theta} = 0.2$ rad/s, and $\dot{\phi} = 0.1$ rad/s, and the components of the total moment about the center of mass of the plane are $\Sigma M_x = 542$ N-m, $\Sigma M_y = 1627$ N-m, and $\Sigma M_z = 0$, what are the x, y, and z components of the airplane's angular acceleration?

Solution: The strategy is to solve Eqs. (20.36) for $\ddot{\theta}$, $\ddot{\phi}$, and $\ddot{\psi}$, and then to use Eqs. (20.35) to determine the angular accelerations.

$$M_x = I_{xx}(\ddot{\psi}\sin\theta\sin\phi + \ddot{\theta}\cos\phi + \dot{\psi}\dot{\theta}\cos\theta\sin\phi$$

$$+\dot{\psi}\dot{\phi}\sin\theta\cos\phi - \dot{\theta}\dot{\phi}\sin\phi) - (I_{yy} - I_{zz})$$

$$\times(\dot{\psi}\sin\theta\cos\phi - \dot{\theta}\sin\phi)(\dot{\psi}\cos\theta + \dot{\phi}),$$

$$M_y = I_{yy}(\ddot{\psi}\sin\theta\cos\phi - \ddot{\theta}\sin\phi + \dot{\psi}\dot{\theta}\cos\theta\cos\phi$$

$$-\dot{\psi}\dot{\phi}\sin\theta\sin\phi - \dot{\theta}\dot{\phi}\cos\phi) - (I_{zz} - I_{xx})$$

$$\times(\dot{\psi}\sin\theta\sin\phi + \dot{\theta}\cos\phi)(\dot{\psi}\cos\theta + \dot{\phi}),$$

$$M_z = I_{zz}(\ddot{\psi}\cos\theta + \ddot{\phi} - \dot{\psi}\dot{\theta}\sin\theta) - (I_{xx} - I_{yy})(\dot{\psi}\sin\theta\sin\phi$$

$$+\dot{\theta}\cos\phi)(\dot{\psi}\sin\theta\cos\phi - \theta\sin\phi).$$

Substitute numerical values and solve to obtain $\ddot{\phi} = -0.03266$ rad/s^2, $\ddot{\theta} = 0.01143$ rad/s^2, $\ddot{\psi} = 0.09732$ rad/s^2. These are to be used (with the other data) in Eqs. (20.35),

$$\frac{d\omega_x}{dt} = \ddot{\psi}\sin\theta\sin\phi + \dot{\psi}\dot{\theta}\cos\theta\sin\phi + \dot{\psi}\dot{\phi}\sin\theta\cos\phi$$

$$+\ddot{\theta}\cos\phi - \dot{\theta}\dot{\phi}\sin\phi,$$

$$\frac{d\omega_y}{dt} = \ddot{\psi}\sin\theta\cos\phi + \dot{\psi}\dot{\theta}\cos\theta\cos\phi - \dot{\psi}\dot{\phi}\sin\theta\sin\phi$$

$$-\ddot{\theta}\sin\phi - \dot{\theta}\dot{\phi}\cos\phi,$$

$$\frac{d\omega_z}{dt} = \ddot{\psi}\cos\theta - \dot{\psi}\dot{\theta}\sin\theta + \ddot{\phi}.$$

Substitute, to obtain:

$\boxed{\alpha_x = 0.05354 \text{ rad/s}^2}$, $\boxed{\alpha_y = 0.03737 \text{ rad/s}^2}$,

$\boxed{\alpha_z = 0.016 \text{ rad/s}^2}$