Stability theory

Some systems are stable, while others are not. In this chapter, we'll investigate the effects of the equations of motion and the potential function on stability.

1 Definitions

1.1 Equilibrium points

Let's consider a system with a **state** $\mathbf{x}$. This system is described by the **evolution equations**

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, t). \tag{1.1}$$

Once an **initial condition** $\mathbf{x^0} = \mathbf{x}(t_0)$ is specified, then the **solution** $\mathbf{x}(t)$ is uniquely determined.

A point $\mathbf{x}^*$ is called an **equilibrium point** or a **singular point** if

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}^*, t) = \mathbf{0}. \tag{1.2}$$

In other words, if $\mathbf{x^0} = \mathbf{x}^*$, then the system state remains constant.

1.2 Definition of stability

Let's examine an equilibrium point $\mathbf{x}^*$. We want to know whether this point is **stable**. To find this out, we take an initial point $\mathbf{x^0}$ close to $\mathbf{x}^*$. With close, we mean that it is less than a distance δ away:

$$|\mathbf{x^0} - \mathbf{x}^*| < \delta. \tag{1.3}$$

Now we examine the solution $\mathbf{x}(t)$. If it remains close to the equilibrium point $\mathbf{x}^*$, then the equilibrium point is called **stable**. With close, we again mean that it stays within a certain bound ε:

$$|\mathbf{x}(t) - \mathbf{x}^*| < \varepsilon. \tag{1.4}$$

(If there is no such ε, then the equilibrium point is called **unstable**.) We can examine the above difference more closely. We say the equilibrium point is **asymptotically stable** if this difference converges to zero:

$$\lim_{t\to\infty} |\mathbf{x}(t) - \mathbf{x}^*| = 0. \tag{1.5}$$

2 Examining matrices

2.1 Linear systems

We say that a system is linear, if we can write the evolution equations as

$$\dot{\mathbf{x}} = A(t)\mathbf{x}. \tag{2.1}$$

It directly follows that $\mathbf{x} = \mathbf{0}$ is an equilibrium point. Often, the matrix A does not depend on time. In this case, it is not very hard to determine the stability of the system. For this, we assume $\mathbf{x}$ has a solution of the form

$$\mathbf{x}(t) = \sum \mathbf{c_i} e^{\lambda_i t}. \tag{2.2}$$

It follows that λ_i must be an **eigenvalue** of A, with $\mathbf{c_i}$ the corresponding **eigenvector**. The real part of these eigenvalues determine the stability of the system. If $\mathrm{Re}(\lambda_i) \leq 0$, for all eigenvalues λ_i, then the system is **stable**. (If also $\mathrm{Re}(\lambda_i) < 0$, for all eigenvalues λ_i, then the system is **asymptotically stable**.) If there, however, is an eigenvalue with a real part bigger than zero (so $\mathrm{Re}(\lambda_i) > 0$), then the system is **unstable**.

2.2 Nonlinear systems

Of course not all systems are linear. And not all systems have $\mathbf{x} = \mathbf{0}$ as an equilibrium point either. Let's examine a nonlinear system of equations $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, t)$ with equilibrium point $\mathbf{x}^*$. Now let's take an initial point $\mathbf{x^0}$ close to $\mathbf{x}^*$. The corresponding solution is $\mathbf{x}(t)$. We now define the **perturbation vector** $\eta(t)$ such that

$$\mathbf{x}(t) = \mathbf{x}^* + \eta(t). \tag{2.3}$$

Let's consider this perturbation $\eta(t)$. By applying a linearization about the equilibrium point, we can find that

$$\frac{d\eta}{dt} = DF(\mathbf{x}^*, t)\eta(t) + \mathbf{g}(\eta, \mathbf{x}^*, t). \tag{2.4}$$

In this equation, $\mathbf{g}(\eta, \mathbf{x}^*, t)$ is the part taking into account nonlinear terms. When examining stability, it is usually neglected. Also, $DF(\mathbf{x}^*, t)$ is the **derivative** of $\mathbf{F}$ at $\mathbf{x}^*$. It is a matrix with components

$$DF_{ij}(t) = \left.\frac{\partial F_i}{\partial x_j}\right|_{\mathbf{x}=\mathbf{x}^*}. \tag{2.5}$$

If the nonlinear system does not depend on time, then we have reduced our system to $\dot{\eta} = DF(\mathbf{x}^*)\eta$. The stability of the system thus depends on the eigenvalues of $DF(\mathbf{x}^*)$, just like in the previous paragraph.

3 Stability in Lagrangian systems

3.1 Stability and the potential function

Let's examine a Lagrangian system with generalized coordiantes q_i and a potential function $V(\mathbf{q}, \dot{\mathbf{q}}, t)$. The forces Q_i acting on the system are given by $Q_i = -\partial V/\partial q_i$. We have an **equilibrium configuration** $\mathbf{q}^*$ if the forces acting on the system are zero. So, an equilibrium configuration must satisfy

$$\left.\frac{\partial V}{\partial q_i}\right|_{q_i=q_i^*} = 0 \quad \text{for all } i. \tag{3.1}$$

Now let's examine the stability of an equilibrium configuration $\mathbf{q}^*$. We give the system a small deviation from the equilibrium configuration. If all forces Q_i push the configuration back, then the system is **stable**. This is the case if

$$\left.\frac{\partial^2 V}{\partial q_i^2}\right|_{q_i=q_i^*} > 0 \quad \text{for all } i. \tag{3.2}$$

In other words, the potential function is at a **minimum**. If, however, we have $\partial^2 V/\partial q_i^2 < 0$ at $q_i = q_i^*$ for at least one i, then the system is **unstable**. (By the way, if $\partial^2 V/\partial q_i^2 < 0$ for all i, then the potential function V is at a **maximum**. If $\partial^2 V/\partial q_i^2 > 0$ for some i, and $\partial^2 V/\partial q_i^2 < 0$ for other i, then V has a **saddle point**. In both cases, the system is unstable.)

3.2 Stability and the Jacobi energy function

Now let's examine a Lagrangian system with a Jacobi energy function h. We know that $h = T_2 - T_0 + V =$ constant. In an equilibrium configuration, we have $\dot{q}_i = 0$ for all i. This means that also $T_2 = 0$, and thus $h = V - T_0 =$ constant. This trick also works the other way around. If $V - T_0$ is constant, and thus

$$\left.\frac{\partial (V - T_0)}{\partial q_i}\right|_{q_i=q_i^*} = 0 \tag{3.3}$$

for all i, then $\mathbf{q}^*$ is an equilibrium configuration.

We can combine this rule with what we've seen in the previous paragraph. To do this, we define the **effective potential** $V_{eff} = V - T_0$. It follows that, if $\partial V_{eff}/\partial q_i = 0$, for all i, then we have an equilibrium configuration. This means that, if we use the effective potential, all the rules of the previous paragraph still hold.