Rigid bodies

Rigid bodies occur quite often in daily life. For that reason, it's worth while examining them. What relations apply to them? That's what we'll find out in this chapter.

1 Basic ideas of rigid bodies

1.1 The center of gravity

Previously, we have only examined particles. We will now examine rigid bodies. A **rigid body** is a collection of particles. The distance between the particles does not change.

Let's examine a rigid body. The position $\mathbf{r_C}$ of the **center of gravity** (CG) is defined as

$$\mathbf{r_C} = \frac{1}{m} \sum_{i=1}^{N} m_i \mathbf{r_i}. \tag{1.1}$$

In this equation, m_i denotes the mass of a certain particle i and $\mathbf{r_i}$ its position. Also, $m = \sum_{i=1}^{N} m_i$ is the **total mass** of the body. It can be shown that the acceleration $\mathbf{a_C} = \ddot{\mathbf{r}}_\mathbf{C}$ of the CG satisfies

$$m\mathbf{a_C} = \sum_{i=1}^{N} \mathbf{F_i} = \mathbf{F_R}. \tag{1.2}$$

So, by using the **resultant force** $\mathbf{F_R}$, we can examine the translation of a body just as if it was just a particle. Sadly, rotation isn't so easy. And that is exactly the reason why this chapter is so long.

1.2 Rigid body kinematics

Let's suppose we know the position $\mathbf{r_C}$ of the CG, with respect to some inertial reference system F_O. The position $\mathbf{r_O}$ of any point P on the body, with respect to F_O, now satisfies

$$\mathbf{r_O} = \mathbf{r_C} + \mathbf{r_{rel}}. \tag{1.3}$$

In this equation, $\mathbf{r_{rel}}$ is the position of P with respect to the CG of the body. Let's now use a coordinate system with as origin the CG of the body. It also rotates in the same way the body rotates. Using the basic concepts shown in the first chapter of this summary, we find that

$$\mathbf{v_O} = \mathbf{v_C} + \omega \times \mathbf{r_{rel}}, \tag{1.4}$$

$$\mathbf{a_O} = \mathbf{a_C} + \alpha \times \mathbf{r_{rel}} + \omega \times (\omega \times \mathbf{r_{rel}}). \tag{1.5}$$

Note that, since the body is rigid, we must have $\mathbf{v_{rel}} = \mathbf{a_{rel}} = \mathbf{0}$.

2 The inertia tensor

2.1 The definition of the inertia tensor

Let's examine a body. The distribution of mass in this body is quite important. To quantify this, we use the **inertia tensor** $[I_O]$ with respect to some point O. It is defined as

$$[I_O] = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} = \begin{bmatrix} \int (y^2+z^2)\,dm & -\int xy\,dm & -\int xz\,dm \\ -\int xy\,dm & \int (x^2+z^2)\,dm & -\int yz\,dm \\ -\int xz\,dm & -\int yz\,dm & \int (x^2+y^2)\,dm \end{bmatrix}. \tag{2.1}$$

The diagonal terms are called the **mass moments of inertia**. The non-diagonal terms are known as the **products of inertia**. By the way, in the above equation, all distance are measured with respect to point O.

2.2 The Parallel Axis Theorem

Let's suppose we know the value of I_{xx} or I_{xy} with respect to some axis, but want to know it with respect to another axis. In that case, we can use the **Parallel Axis Theorem**. It states that

$$I_{xx} = I_{xx_C} + m\left(d_y^2 + d_z^2\right) \qquad \text{and} \qquad I_{xy} = I_{xy_C} + m d_x d_y. \tag{2.2}$$

In this equation, I_{xx_C} and I_{xy_C} indicate the moment/product of inertia with respect to the CG. Also, the parameters d indicate the distance over which the axis is shifted. (You can apply the same trick for the other moments/products of inertia as well. You then just have to adjust the subscripts.)

2.3 The moment of inertia with respect to a line

Sometimes, we want to find the moment of inertia I_{ll} with respect to a line l. This moment of inertia is now defined as

$$I_{ll} = \int h^2 \, dm, \tag{2.3}$$

where h denotes the distance from the corresponding point to the line l. Let's suppose that the line is denoted by a unit vector $\hat{\mathbf{e}}$. The moment of inertia with respect to the line l is then given by

$$I_{ll} = \hat{\mathbf{e}}^T [I_O] \hat{\mathbf{e}}. \tag{2.4}$$

2.4 Principal axes

What happens to the inertia tensor, when we change the axes? In this case, also the inertia tensor changes. We can change the axes such that the inertia tensor becomes a diagonal matrix. The corresponding axes are called the **principal axes**. (The directions of the principal axes are equal to the directions of the eigenvectors of the original inertia tensor.) We denote these axes by the subscripts 1, 2 and 3. We thus have

$$[I_O] = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}. \tag{2.5}$$

Getting rid of the nondiagonal components of the inertia tensor often comes in handy, as we will soon see.

3 Euler's equations of motion

3.1 Linear and angular momentum

We will now examine the linear and angular momentum of a body. We start with the linear momentum. This is, in fact, quite easy. It can be shown that the linear momentum of the body $\mathbf{p}$ is given by $\mathbf{p} = m\mathbf{v_c}$. So, to find the linear momentum, we only have to consider the velocity of the CG of the body.

Now let's find the angular momentum $\mathbf{L_O}$ with respect to some point O. (Don't confuse the angular momentum $\mathbf{L}$ with the Lagrangian function L.) It can be shown that

$$\mathbf{L_O} = \mathbf{r_C} \times m\mathbf{v_O} + \int \mathbf{r} \times (\omega \times \mathbf{r}) \, dm. \tag{3.1}$$

(In the above equation, all the position vectors $\mathbf{r}$ are with respect to point O.) We usually pick O such that the first term of this relation is zero. To accomplish this, we can do two things. We can pick a point that is fixed in space, such that $\mathbf{v_O} = \mathbf{0}$. We can also pick the CG, implying that $\mathbf{r_C} = \mathbf{0}$. That leaves us only with the right side. Using the inertia tensor, we can also rewrite the right side. We then remain with

$$\mathbf{L_O} = [I_O]\omega. \tag{3.2}$$

Remember that this equation only holds with respect to a fixed point, or with respect to the CG of the body.

3.2 Equations of motion for a rigid body

We know that the resultant force $\mathbf{F_R}$ effects the linear momentum of a body. Similarly, the **resultant moment N** effects the angular momentum. It does this according to

$$\mathbf{N_C} = \sum \mathbf{M_{C_i}} = \frac{d\mathbf{L_C}}{dt}, \qquad \text{or} \qquad \mathbf{N_O} = \sum \mathbf{M_{O_i}} = \frac{d\mathbf{L_O}}{dt}, \tag{3.3}$$

where O is a certain reference point. (Again, the above equation only holds if O is a fixed point, or O is the CG.) Let's examine the time derivative of the angular momentum. It is given by

$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt}\left(L_x\mathbf{i} + L_y\mathbf{j} + L_z\mathbf{k}\right) = \dot{L}_x\mathbf{i} + \dot{L}_y\mathbf{j} + \dot{L}_z\mathbf{k} + \omega \times \mathbf{L}. \tag{3.4}$$

The last term in the above equation comes from the fact that the unit vectors may change too. We can combine the above two equations with the knowledge that $\mathbf{L_O} = [I_O]\omega$. This will then give us the equations of motion for a rotating rigid body. These equations will, however, be rather big. But we can use a trick to make a lot of terms disappear out of these equations. We can use the principal axes of the object. If we do that, it follows that

$$N_1 = I_1\dot{\omega}_1 - (I_2 - I_3)\,\omega_2\omega_3, \tag{3.5}$$
$$N_2 = I_2\dot{\omega}_2 - (I_3 - I_1)\,\omega_3\omega_1, \tag{3.6}$$
$$N_3 = I_3\dot{\omega}_3 - (I_1 - I_2)\,\omega_1\omega_2. \tag{3.7}$$

The above equations are known as **Euler's equations of motion**. They only hold with respect to the principal axes of the body.

3.3 Analyzing Euler's equations of motion

We can derive some interesting facts from Euler's equations of motion. First, we can see that the angular acceleration $\dot{\omega}$ doesn't only depend on the moment applied. It also depends on the rotation rates about the other two principal axes. This is sometimes important to keep in mind.

We can also ask ourselves, when is a **steady rotation** possible, without any outside moments acting on the system? (So when is $\dot{\omega}_1 = \dot{\omega}_2 = \dot{\omega}_3 = 0$ possible, while also $N_1 = N_2 = N_3 = 0$?) We can now see that, to ensure a steady rotation, 2 out of the 3 terms ω_1, ω_2 and ω_3 must be zero. In other words, a steady rotation is only possible about one of the principal axes.

Now let's assume that we're trying to rotate our body about one of the principal axes. For example, we try to rotate it about the third principal axis. In the real world, small perturbations are always present. So although ω_1 and ω_2 are much smaller than ω_3, they are not exactly zero. Sometimes, these perturbations damp out. (The rotation is stable.) In other cases, these perturbations grow. (The rotation is unstable.) It can be derived that the **condition for instability** is

$$I_1 > I_3 > I_2, \qquad \text{or} \qquad I_2 > I_3 > I_1. \tag{3.8}$$

A steady rotation is thus **stable** if it is performed about the principal axis with either the smallest or the largest moment of inertia. It is **unstable** if it is performed about the principal axis with the moment of inertia which is between the values of the other two.

4 Lagrange's equations of motion

4.1 Kinetic energy

There is another way to analyze rotating bodies. We then have to make use of the kinetic energy of the body. But how do we find the kinetic energy? We can do that, using the general equation

$$T = \frac{1}{2} m v_O^2 + \mathbf{v_O} \cdot \omega \times \int \mathbf{r}\, dm + \frac{1}{2} \int (\omega \times \mathbf{r}) \cdot (\omega \times \mathbf{r})\, dm. \tag{4.1}$$

This equation works for any point O on the body. But it's kind of a horrific equation. To simplify it, we usually pick O such that the middle term disappears. Again, we can do this in two ways. Either O is fixed ($\mathbf{v_O} = \mathbf{0}$) or O is the CG (the integral is zero). If we do the first (and thus have O fixed), then we can rewrite the above equation to

$$T = \frac{1}{2} \omega^T [I_O] \omega. \tag{4.2}$$

If, on the other hand, we choose point O to be the CG, then we have

$$T = \frac{1}{2} m v_C^2 + \frac{1}{2} \omega^T [I_C] \omega. \tag{4.3}$$

4.2 Euler angles

We have previously only looked at rotations, as seen from the rotating object itself. We haven't examined rotations from an inertial reference frame. When doing this, we need to know something about the orientation of the body, with respect to the inertial reference frame. For this, **Euler angles** come in handy. They allow us to express orientations.

Let's suppose we start at an inertial coordinate system, having axes X, Y and Z. We want to transform this to a coordinate system xyz, connected to a body. First, we rotate the inertial coordinate system by an angle ϕ about the Z axis. The new X axis is known as the **line of nodes** ξ. Second, we rotate the coordinate system about the line of nodes (the current X axis) by an angle θ. We do this to change the Z axis into the z axis. Finally, we rotate the system by an angle ψ about the z axis. We have now arrived at the xyz coordinate system.

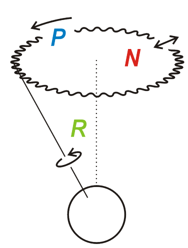

Figure 1: Clarification of the meanings of precession, nutation and rotation.

The angular velocity ω of the body can be expressed in the Euler angles. To be more specific, it can be expressed as a function of the **precession** $\dot{\phi}$, the **nutation** $\dot{\theta}$ and the **spin/rotation** $\dot{\psi}$, according to

$$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \sin\theta\sin\psi & \cos\psi & 0 \\ \sin\theta\cos\psi & -\sin\psi & 0 \\ \cos\theta & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}. \tag{4.4}$$

The precise meaning of the precession, the nutation and the rotation can be seen in figure 1

4.3 The equations of motion for a rotating object

We can also express the kinetic energy due to rotation as a function of the Euler angles. To do this, we use $T_{rot} = \frac{1}{2}\omega^T[I]\omega$. Doing so, while using the principal axes, will give

$$T_{rot} = \frac{1}{2}\left(I_1\left(\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi\right)^2 + I_2\left(\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi\right)^2 + I_3\left(\dot{\phi}\cos\theta + \dot{\psi}\right)^2\right). \tag{4.5}$$

We can now find the moments Q_ϕ, Q_θ and Q_ψ, by using Lagrange's equations of motion. These equations are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i. \tag{4.6}$$

In this equation, q_i can be replaced by ϕ, θ or ψ. Q_i then has to be replaced by the moment Q_ϕ, Q_θ or Q_ψ, respectively.

4.4 Gyrodynamics

The equations of motion that we just derived are usually quite difficult to solve. But for some bodies, like a gyroscope or a top, we can simplify them. We will now examine such a body. Let's suppose that we have a body with the I_3 axis as an axis of symmetry. We thus have $I_1 = I_2 = I$ and $I_3 = I_s$. It follows that the kinetic energy, due to rotation, is given by

$$T_{rot} = \frac{1}{2}\left(I\left(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2\right) + I_s\left(\dot{\phi}\cos\theta + \dot{\psi}\right)^2\right). \tag{4.7}$$

It often also occurs that the potential function V does not depend on ϕ or ψ. Since $L = T - V$, also the Lagrangian function L then does not depend on ϕ or ψ. This means that these two coordinates are ignorable coordinates. We thus also have two integrals of motion, being

$$C_\phi = \frac{dL}{d\dot{\phi}} = I\dot{\phi}\sin^2\theta + I_s\left(\dot{\phi}\cos\theta + \dot{\psi}\right)\cos\theta \qquad \text{and} \qquad C_\psi = \frac{dL}{d\dot{\psi}} = I_s\left(\dot{\phi}\cos\theta + \dot{\psi}\right). \tag{4.8}$$

The second of these equations can also be written as $C_\psi = I_s\omega_3$. In other words, the spin rate ω_3 is constant too. By using the above equations, we can derive the Routhian R and the energy E. They are

$$R = -\frac{1}{2}I\dot{\theta}^2 + \frac{(C_\phi - C_\psi\cos\theta)^2}{2I\sin^2\theta} + \frac{C_\psi^2}{2I_s} + V(\theta), \tag{4.9}$$

$$E = \frac{1}{2}I\dot{\theta}^2 + V_{eff}(\theta), \qquad \text{where} \qquad V_{eff}(\theta) = V(\theta) + \frac{(C_\phi - C_\psi\cos\theta)^2}{2I\sin^2\theta} + \frac{C_\psi^2}{2I_s}. \tag{4.10}$$

4.5 Steady precession

Now let's suppose we want to have a steady rotation, without nutation. We thus want to have θ constant. This is equivalent to $\dot{\theta} = 0$ or $\frac{dV_{eff}}{d\theta} = 0$. If we apply this, and perform a lot of mathematical rewriting, we can find that

$$-I\dot{\phi}^2 \sin\theta \cos\theta + I_s \dot{\phi} \omega_3 \sin\theta + \frac{dV}{d\theta} = 0. \tag{4.11}$$

From this equation, the precession rate $\dot{\phi}$ necessary to maintain equilibrium can be found. In fact, there are two solutions. They are known as the **slow** and the **fast precession**.