Lagrangian dynamics

Newton examined forces. From this, he derived accelerations and such. Lagrange did something different. He examined energy, by using generalized coordinates. And that's what we'll do in this chapter as well.

1 System configurations and coordinates

1.1 Degrees of freedom

Let's consider a particle in a three-dimensional space. We need three numbers to fully describe this particle. We thus say that the particle has three **degrees of freedom**. On the other hand, a rigid body has six degrees of freedom. (Three due to its position, and three due to its orientation.) For a general system, the number of degrees of freedom is denoted by NDOF. We usually thus need NDOF numbers (called **coordinates**) to describe the system.

1.2 System configurations

Describing the **configuration** of a system can be done in many ways. (We could use many kinds of coordinate systems.) However, we want to be able to work with any description of the system. To accomplish this, we define **generalized coordinates** q_i. The collection of all possible sets of coordinates $(q_1, q_2, \ldots, q_n)$ is called the **configuration space**. By the way, the formulation of dynamics problems in terms of generalized coordinates is known as **Lagrangian dynamics**.

1.3 Transforming coordinates

Once a problem is described in certain generalized coordiantes, it can also be described in other coordinate systems. For this, we use **coordinate transformations**, like

$$q_i = q_i(x_1, x_2, \ldots, x_n, t) \qquad \text{and similarly} \qquad x_i = x_i(q_1, q_2, \ldots, q_n, t). \tag{1.1}$$

The latter part of the above equation is known as the **inverse transformation**.

2 Generalizing energy, momentum and forces

We have generalized coordinates. It would be nice if we could generalize other parameters as well. That's what we'll do in this part.

2.1 Kinetic energy

Let's examine a system with generalized coordinates $\mathbf{q} = (q_1, q_2, \ldots, q_n)$. The **generalized velocities** $\dot{q}_i$ of the system are the time derivatives of the coordinates. In other words,

$$\dot{q}_i = \frac{dq_i}{dt}. \tag{2.1}$$

From the generalized velocities, the **kinetic energy**, in terms of the generalized coordinates, can be derived. It can be shown that

$$T = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_{ij}\dot{q}_i\dot{q}_j + \sum_{i=1}^{n}\beta_i\dot{q}_i + \gamma = T_2 + T_1 + T_0. \tag{2.2}$$

In this equation, T_2 is the collection of terms with the α_{ij} coefficients, T_1 with the β_i coefficients and $T_0 = \gamma$. If the transformations from the generalized coordinates to the actual coordinates do not depend on time, then $T_1 = T_0 = 0$. (This is the case if $x_i = x_i(q_1, q_2, \ldots, q_n)$, or, equivalently, $\partial x_i/\partial t = 0$ for all i.) In this case, we call the system a **natural system**.

Sadly, the above relation isn't a very easy one. The coefficients α_{ij}, β_j and γ depend on the system and the way in which the generalized coordinates are defined. Finding them requires some skill and work.

2.2 The generalized momentum

The **generalized momentum** in q_i direction p_i can be derived from the kinetic energy. It is defined as

$$p_i = \frac{\partial T}{\partial \dot{q}_i}. \tag{2.3}$$

When calculating this partial derivative, all other variables (including the time t) remain fixed.

2.3 Generalized forces

Now let's examine **generalized forces Q**. To do this, we examine work. We can remember that work is force times displacement in the corresponding direction. We thus have

$$\delta W = \sum_{i=1}^{n} F_i \, \delta x_i. \tag{2.4}$$

By using the transformations to generalized coordinates, we can rewrite this to

$$\delta W = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} F_j \frac{\partial x_j}{\partial q_i} \right) \delta q_i = \sum_{i=1}^{n} Q_i \, \delta q_i. \tag{2.5}$$

In this equation, Q_i is the generalized force in the direction of the coordinate q_i. It is given by

$$Q_i = \sum_{j=1}^{n} F_j \frac{\partial x_j}{\partial q_i}. \tag{2.6}$$

It may be important to note that the amount of work performed does not depend on changes in time.

Let's examine a force that is conservative in the real physical word. The corresponding potential function is V. (We thus have $\mathbf{F} = -\nabla V$.) In this case, it can be shown that the corresponding generalized force is also conservative. In fact, we have

$$Q_i = -\frac{\partial V}{\partial q_i} \qquad \text{or, equivalently,} \qquad \mathbf{Q} = -\nabla V. \tag{2.7}$$

Note that, to use this function, we first have to transform V. First, V was a function of the physical coordinates. (So $V = V(x_1, x_2, \ldots, x_n)$.) But, to use the above equation, V has to be a function of the generalized coordinates. (Thus $V = V(q_1, q_2, \ldots, q_n)$.)

3 The generalized equations of motion

3.1 Finding the generalized equations of motion

Newton once stated that $\mathbf{F} = m\ddot{\mathbf{r}}$ or, equivalently, $\mathbf{F} = \dot{\mathbf{p}}$. This relation doesn't always work with generalized forces. This is because, in generalized coordinates, force is not the time derivative of momentum.

(So $\dot{\mathbf{p}} \neq \mathbf{Q}$.) Instead, we have

$$\dot{\mathbf{p}} = \mathbf{Q} + \nabla T, \qquad \text{or, in components,} \qquad \dot{p}_i = Q_i + \frac{\partial T}{\partial q_i}. \tag{3.1}$$

We can also recall that $\dot{p}_i = \frac{dp_i}{dt} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)$. Putting things together gives

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i. \tag{3.2}$$

This very important relation is known as **Lagrange's Equations of Motions**. (It holds for every coordinate q_i.)

3.2 The Lagrangian function

In some cases, we can simplify Lagrange's equations of motion. Let's suppose that the force Q_i acting on the system is conservative. We thus have a **conservative system**. So there is a function $V(\mathbf{q})$ such that $Q_i = -\partial V/\partial q_k$. If this is the case, then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0, \tag{3.3}$$

where $L(\mathbf{q}, \dot{\mathbf{q}}, t) = T(\mathbf{q}, \dot{\mathbf{q}}, t) - V(\mathbf{q})$ is known as the **Lagrangian function**.

Sometimes, however, we can't find a potential function $V(\mathbf{q})$ such that $Q_i = -\partial V/\partial q_i$. But, we may be able to find a function $V(\mathbf{q}, \dot{\mathbf{q}}, t)$ such that

$$Q_i = \frac{d}{dt}\left(\frac{\partial V}{d\dot{q}_i}\right) - \frac{\partial V}{\partial q_i}. \tag{3.4}$$

The function $V(\mathbf{q}, \dot{\mathbf{q}}, t)$ is then known as the **generalized potential**. If there is such a generalized potential, then the system is called a **Lagrangian system**. And, if we again define $L(\mathbf{q}, \dot{\mathbf{q}}, t) = T(\mathbf{q}, \dot{\mathbf{q}}, t) - V(\mathbf{q}, \dot{\mathbf{q}}, t)$, then equation (3.3) still holds.

3.3 Lagrangian systems

Energy is not always conserved in a Lagrangian system. However, there may be another quantity that is conserved. We define the **Jacobi energy function** h as

$$h(\mathbf{q}, \dot{\mathbf{q}}, t) = -L + \sum_{i=1}^{n} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i}. \tag{3.5}$$

It can now be shown that

$$\frac{dh}{dt} = -\frac{\partial L}{\partial t}. \tag{3.6}$$

So, if L does not explicitly depend on the time t, then h is constant. It is thus conserved. If this is indeed the case, then we have

$$h = T_2 - T_0 + V. \tag{3.7}$$

For natural systems, we have $T_1 = T_0 = 0$ and thus $h = T + V$. In this case, the mechanical energy $T + V$ is thus conserved as well.

3.4 Ignorable coordinates

Let's consider a Lagrangian system with n degrees of freedom. We suppose that there are m generalized coordinates $q_{n-m+1}, \ldots, q_n$ that do not appear in the Lagrangian L. These coordinates are called **inactive** or **ignorable coordinates**. (We will soon see why.) For these coordinates, we have $\partial L/\partial q_i = 0$, where $n - m < i \leq n$. This implies that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0, \qquad \text{or, equivalently,} \qquad \frac{\partial L}{\partial \dot{q}_i} = C_i, \tag{3.8}$$

where again $n - m < i \leq n$. The above equation is known as an **integral of motion** (a quantity that stays constant during the motion). The constant C_i is related to the momentum corresponding to coordinate q_i.

We use the above relation, when defining the Routhian function. The **Routhian function** R is defined as

$$R = -L + \sum_{i=n-m+1}^{n} C_i \dot{q}_i. \tag{3.9}$$

The ignorable coordinates do not appear in the Routhian function. Instead, the Routhian function contains the constants C_i. By using the Routhian function, we can rewrite the equations of motion for the nonignorable coordinates to

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_i}\right) - \frac{\partial R}{\partial q_i} = 0, \tag{3.10}$$

where this time $1 \leq i \leq n - m$. We now only have $n - m$ equations of motion left. We therefore 'ignore' the equations of motion corresponding to the ignorable coordinates. Once the equations of motion have been solved, the ignorable coordinates can be found using

$$\dot{q}_i = \frac{\partial R}{\partial C_i} \qquad \text{or, equivalently,} \qquad q_i(t) = \int_{t_0}^{t} \frac{\partial R}{\partial C_i} d\tau, \tag{3.11}$$

where $n - m < i \leq n$. By the way, we can also express the Jacobi energy function h as a function of R. We then find that

$$h = R - \sum_{i=1}^{n-m} \dot{q}_i \frac{\partial R}{\partial \dot{q}_k}. \tag{3.12}$$

3.5 Steady motion

A special type of motion is steady motion. In a steady motion, all the nonignorable coordinates are constant. We thus have $\dot{q}_i = \ddot{q}_i = 0$ for $1 \leq i \leq n-m$. Because of this, the equations of motion reduce to

$$\frac{\partial R}{\partial q_i} = 0, \tag{3.13}$$

for $1 \leq i \leq n - m$. On the other hand, the ignorable coordinates have a constant velocity. We thus have $\dot{q}_i = v_i = \text{constant}$ for $n - m < i \leq n$.

3.6 Disippative systems

Let's examine the forces Q_i acting on a non-Lagrangian system. Part of these forces Q_i^{lagr} can be derived from a generalized potential function $V(\mathbf{q}, \dot{\mathbf{q}}, t)$, according to equation (3.4). However, another part can not be derived from such a potential function.

Examples of forces that can't be derived from a potential are frictional forces. These forces can not be connected to a potential, since energy is dissipated. In real (physical) coordinates, such forces are usually described by functions like $F_i^{fr} = -c_i \dot{x}_i$. (This relation holds for all i.) From this, we can derive that the frictional generalized forces Q_i^{fr} are given by

$$Q_i^{fr} = -\frac{1}{2}\frac{\partial}{\partial \dot{q}_i}\sum_{i=1}^{n} c_i \dot{x}_i^2 = -\frac{\partial D}{\partial \dot{q}_i}. \tag{3.14}$$

The parameter D is known as **Rayleigh's Dissipation Function**. It is defined as

$$D = \frac{1}{2}\sum_{i=1}^{n} c_i \dot{x}_i^2. \tag{3.15}$$

By using this function, we can rewrite the equations of motion to

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = Q_i^*. \tag{3.16}$$

In this equation, Q_i^* denotes the part of the force Q_i that is not derivable from a potential function or a dissipation function.

4 Constraints

4.1 Holonomic constraints

Systems can have certain **constraints**. Constraints reduce the number of degrees of freedom. Let's examine a system normally having n degrees of freedom. If this system is given c constraints, then the remaining number of degrees of freedom is $\text{NDOF} = n - c$.

There are many types of constraints. So-called **holonomic constraints** can be written as

$$f(q_1, q_2, \ldots, q_n) = \text{constant} \qquad \text{or} \qquad f(q_1, q_2, \ldots, q_n, t) = \text{constant}. \tag{4.1}$$

If the constraint does not depend on time (as in the first relation), then the constraint is **scleronomic**. Otherwise, it is **rheonomic**. We can put holonomic constraints in the so-called **differential form**. To do this, we have to use the chain rule. We then find that

$$df = \frac{\partial f}{\partial q_1}dq_1 + \frac{\partial f}{\partial q_2}dq_2 + \ldots + \frac{\partial f}{\partial q_n}dq_n + \frac{\partial f}{\partial t}dt = 0. \tag{4.2}$$

4.2 Nonholonomic constraints

Some constraints can not be written in the form shown in equation (4.1). These constraints are known as **nonholonomic constraints**. There are two important types of nonhomolic constraints. These are **inequalities** and **nonintegrable differential expressions**. Their forms are, respectively,

$$g(q_1, q_2, \ldots, q_n, t) \geq 0 \qquad \text{and} \qquad A_1 dq_1 + A_2 dq_2 + \ldots + A_n dq_n + A_0 dt = 0, \tag{4.3}$$

where the differential expression (the one on the right) is nonintegrable. By the way, the coefficients A_i don't have to be constant. They can depend on time. Nonholonomic constraints are usually a bit more difficult to deal with than holonomic constraints. Luckily, they appear less frequently too.

4.3 Forces caused by constraints

Let's examine a system. This system has a set of J constraints that can be written as

$$\mathbf{A_j} \cdot \delta \mathbf{q} = A_{j1}\, \delta q_1 + A_{j2}\, \delta q_2 + \ldots + A_{jn} \delta q_n = 0, \tag{4.4}$$

where the index j is between 1 and J. Let's consider one of these constraints. This constraint demands that the position of the system (in the configuration space) moves along a certain $n-1$-dimensional plane. To keep the position of the system in this plane, a reacting **constraint force** R_j acts on the system. The only effect/goal of this force is to keep the configuration of the system in the plane. It thus acts perpendicular to the plane.

Now let's examine a movement $\delta \mathbf{q}$ of the system. Due to the constraint, this movement $\delta \mathbf{q}$ must be performed in the $n-1$-dimensional plane. Since $\mathbf{A_j} \cdot \delta \mathbf{q} = 0$, the vector $\mathbf{A_j}$ must be perpendicular to the plane. This implies that $\mathbf{R_j}$ and $\mathbf{A_j}$ have the same direction. We can thus write $\mathbf{R_j} = \lambda_j \mathbf{A_j}$, where the **Lagrange multiplier** λ_j is (at the moment) an unknown number.

In the equations of motion, we need to take the reaction forces R_j into account. We thus rewrite these equations to

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i + \sum_{j=1}^{J} R_{ji} = Q_i + \sum_{j=1}^{J} \lambda_j A_{ji}. \tag{4.5}$$

We now have $n+J$ equations, being n equations of motion, and J constraint equations. We also have $n+J$ unknowns, being the n coordinates q_i and the J Lagrange multipliers λ_j. To find the unknowns, all the equations have to be solved simultaneously.