Calculus of variations

The calculus of variations has turned out to be a handy tool for solving dynamic problems. What is the calculus of variations? And how can we apply it to dynamic problems? That's what we'll look at now.

1 The basic principles of calculus of variations

1.1 The definition of a functional

A **function** (like $y(x)$) takes one or more numbers as input. The output is also a number. A generalization of a function is a functional. A **functional** (like $I[y(x)]$) takes one or more functions as input. The output is again a number. A specific kind of functional is the **integral functional**. The general form of such a functional is

$$I[y(x)] = \int_{x_1}^{x_2} F\left(x, y(x), y'(x)\right) dx. \tag{1.1}$$

The integrand F is called the **Lagrangian** of the integral function. An example of such an integral functional (for the so-called **brachistochrone problem**) is

$$I[y(x)] = \int_{x_1}^{x_2} \frac{\sqrt{1 + [y'(x)]}^2}{\sqrt{2gy(x)}} dx. \tag{1.2}$$

An **admissible function** is a function $y(x)$ that satisfies certain conditions. These conditions are usually differentiability conditions (we, for example, want continuously differentiable functions) and boundary conditions (we, for example, want to have $y(x_1) = y_1$ and $y(x_2) = y_2$). The full set of all admissible functions is called the **domain** of the functional.

1.2 The variational problem

Let's suppose we have a certain functional $I[y(x)]$. It then often occurs that we want to find the function $y(x)$ which minimizes or maximizes the functional $I[y(x)]$. This kind of problem is called a **variational problem**. The resulting function $y(x)$ is called an **extremal**.

Finding an extremum of a function is easy: you simply set the derivative to zero. But functionals don't really have a derivative. For that reason, we examine

$$I[y^*(x) + \varepsilon\eta(x)] = \int_{x_1}^{x_2} F(x, y^*(x) + \varepsilon\eta(x), y^{*\prime}(x) + \varepsilon\eta'(x))dx, \tag{1.3}$$

where $y^*(x)$ is a certain admissible function. Also, the **perturbation** $\eta(x)$ is an arbitrary function, such that $y^*(x) + \varepsilon\eta(x)$ is still an admissible function. The term $\varepsilon\eta(x)$ is known as the **variation**. It is often also written as $\delta y^*(x)$, with δ being the **variational operator**. We then also write $I[\varepsilon\eta] = I[\delta y^*] = \delta I$.

The above function now only depends on one variable, being ε. So, to find the extremum, we simply have to find the derivative with respect to ε and set it to zero. This will give us

$$\frac{dI}{d\varepsilon}[y^* + \varepsilon\eta] = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta'\right) dx = 0. \tag{1.4}$$

We can now apply integration by parts to rewrite the term $\frac{\partial F}{\partial y'}\frac{d\eta}{dx}$. The result will be

$$\frac{dI}{d\varepsilon}[y^* + \varepsilon\eta] = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right) \eta\, dx + \left.\frac{\partial F}{\partial y'}\eta\right|_{x_1}^{x_2} = 0. \tag{1.5}$$

This is a rather interesting equation. From it, we can derive conditions which any extremal $y(x)$ must satisfy. We will examine these conditions in the upcoming paragraph.

1.3 Extremum conditions

Let's examine the last equation of the previous paragraph. It must hold for any function $\eta(x)$. For this reason, the part within brackets has to be zero for every x. In other words, we must have

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right). \tag{1.6}$$

This equation is known as the **Euler-Lagrange equation** for the functional $I[y(x)]$. It is the equation we have to use to find the extremal $y(x)$.

Now let's look at the term $\frac{\partial F}{\partial y'}\eta\Big|_{x_1}^{x_2}$. This term also has to be zero. However, it is a bit difficult to analyze. This is because some problems have so-called **essential boundary conditions** (EBCs) $y(x_1) = y_1$ and $y(x_2) = y_2$. We know that both $y^*(x)$ and $y^*(x) + \varepsilon\eta(x)$ have to satisfy these conditions. (They are both admissible functions.) This implies that $\eta(x_1) = \eta(x_2) = 0$. So, in this case, the term $\frac{\partial F}{\partial y'}\eta\Big|_{x_1}^{x_2}$ is automatically zero. We can't derive any conditions from it. (That is, other than the EBCs that were already present.)

However, other problems only have one or zero EBCs. For example, let's examine a problem having only the EBC $y(x_1) = y_1$. We thus have $\eta(x_1) = 0$. But we don't necessarily have $\eta(x_2) = 0$. In this case, we therefore must have

$$\left.\frac{\partial F}{\partial y'}\right|_{x=x_2} = 0. \tag{1.7}$$

If we don't have the EBC $y(x_1) = y_1$ as well, then we should also have

$$\left.\frac{\partial F}{\partial y'}\right|_{x=x_1} = 0. \tag{1.8}$$

The above two boundary conditions are called **natural boundary conditions** (NBCs). So we see that every end always has a boundary condition. Sometimes an EBC has been specified. If this is not the case, an NBC will automatically be present.

1.4 Generalizing the variational problem

We can make the variational problem a bit more general. We can, for example, examine a Lagrangian function with several more input functions $F(x, y_1, \ldots, y_n, \dot{y}_1, \ldots, \dot{y}_n)$. In this case, the resulting Euler-Lagrange equation will be

$$\frac{\partial F}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_i'}\right) = 0, \tag{1.9}$$

for every i from 1 to n. We thus have n conditions, resulting in n (possibly coupled) differential equations. Solving this set of equations is usually quite difficult.

Sometimes, we might also have to deal with a function $y(x_1, \ldots, x_n)$ of multiple variables. In this case, the Lagrangian is $F(x_1, \ldots, x_n, y, y_{x_1}, \ldots, y_{x_n})$, where y_{x_i} means the derivative $\frac{\partial y}{\partial x_i}$. If we want to find the extremal of the functional $I[y]$, then we now have to use the condition

$$\frac{\partial F}{\partial y} - \frac{\partial}{\partial x_1}\left(\frac{\partial F}{\partial y_{x_1}}\right) - \frac{\partial}{\partial x_2}\left(\frac{\partial F}{\partial y_{x_2}}\right) - \ldots - \frac{\partial}{\partial x_n}\left(\frac{\partial F}{\partial y_{x_n}}\right) = 0. \tag{1.10}$$

This time, we wind up with a partial differential equation. It's usually rather hard to solve as well.

We could also have to deal with higher derivatives of a function $y(x)$. In this case, the Lagrangian is given by $F(x, y, y', y'', \ldots, y^{(n)})$, where $y^{(i)}$ stands for the i^{th} derivative of the function y. The Euler-Lagrange equation now becomes

$$\frac{\partial F}{\partial y} + \sum_{i=1}^{n} (-1)^i \frac{d^i}{dx^i} \left(\frac{\partial F}{\partial y^{(i)}} \right) = 0. \tag{1.11}$$

We now have a higher-order differential equation. Again, solving it can be rather difficult.

Finally, it can also occur that multiple of the above cases occur simultaneously. In that case, you would have to combine the above conditions. But we won't go into detail on this.

2 Applying the calculus of variations

2.1 Hamilton's principle

It is time to apply the calculus of variations to dynamic systems. For this, we use **Hamilton's principle**. It states that

$$\int_{t_1}^{t_2} (\delta T + \delta W)\, dt = 0. \tag{2.1}$$

For conservative systems, we have $\delta W = -\delta V$. It now follows that the integral

$$I = \int_{t_1}^{t_2} (T - V)\, dt. \tag{2.2}$$

should be at an extremum. In fact, it has to be at a minimum. The above integral is called the **action** of the system. We can rewrite $T - V$ as $L(\mathbf{q}, \dot{\mathbf{q}}, t)$. In this case, it follows that we should have

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = 0, \tag{2.3}$$

for every i from 1 to n. So we see that, from Hamilton's principle, and from the calculus of variations, we have derived Lagrange's equations of motion.

Sometimes, we don't have a conservative system. In this case, the nonconservative forces Q_i^{nc} also have to be taken into account. Rewriting equation (2.1) gives

$$\int_{t_1}^{t_2} \left(\delta T - \delta V + \sum_{i=1}^{n} Q_i^{nc}\, \delta q_i \right) dt = \int_{t_1}^{t_2} \left(\delta L + \sum_{i=1}^{n} Q_i^{nc}\, \delta q_i \right) dt = 0. \tag{2.4}$$

Using the calculus of variations in dynamic problems has a big advantage. This method also works when there are an infinite amount of degrees of freedom. So, when examining **continuous mechanical systems**, like a vibrating string or a bending bar, this method is most likely to be used.

2.2 The Ritz method

Let's consider the variational problem

$$I[y(x)] = \int_{x_1}^{x_2} F\left(x, y(x), y'(x)\right) dx, \qquad \text{with } y(x_1) = y_1 \text{ and } y(x_2) = y_2. \tag{2.5}$$

The **Ritz method** is a method of finding an approximate solution $\bar{y}(x)$ to this problem. To start, we take n linearly independent known functions $h_i(x)$, called **shape functions**. (Deciding which functions to take can be rather difficult. It is something that requires quite some experience. But often $h_1(x) =$

$1, h_2(x) = x, h_3(x) = x^2, \ldots$ works quite well.) We then assume that our approximate solution can be written as

$$\bar{y}(x) = \sum_{i=1}^{n} a_i h_i(x). \tag{2.6}$$

The unknown coefficients a_i are called **degrees of freedom**. It's our job to find them. To do this, we insert the above relation into equation (2.5). This reduces the functional $I[y(x)]$ to a function of n variables, being

$$\Phi(a_1, a_2, \ldots, a_n) = \int_{x_1}^{x_2} F\left(x, \sum_{i=1}^{n} a_i h_i(x), \sum_{i=1}^{n} a_i h_i'(x)\right) dx. \tag{2.7}$$

We can now also apply the boundary conditions $\bar{y}(x_1) = y_1$ and $\bar{y}(x_2) = y_2$. This further reduces the function Φ to a function of $n - 2$ variables.

We want to find the extremums of Φ. To do this, we simply set $\frac{\partial \Phi}{\partial a_i} = 0$ for all remaining a_i. This gives us $n - 2$ equations and $n - 2$ unknowns. It can thus be solved, giving us all the coefficients a_i. Once these coefficients are known, also our approximate solution $\bar{y}(x)$ is known. And it has turned out that $\bar{y}(x)$ is often a rather good approximation of the real solution.