Basic concepts

There are quite some advanced techniques in the Dynamics and Stability course. Before we're going to examine those, we need to make sure we have a good foundation. We will examine some basic concepts. You will have seen most of them before. So we'll go through them quickly.

1 Using coordinate systems

1.1 Coordinate systems

Kinematics is the study concerned with describing motion. To describe motion, we need to have **coordinate systems**. Examples are **normal**, **cylindrical** and **spherical** coordinate systems. In these coordinates systems, we denote the **position** of a particle by the vector $\mathbf{r}$.

Every vector has components. We can, for example, write $\mathbf{r}$ in the normal coordinate system as

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}. \tag{1.1}$$

Here, $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ are **unit vectors**. (They per definition have length 1. This is also indicated by the hats.) x, y and z are the components. They are different in different coordinate systems. To transform vectors between coordinate systems, we use **transformation matrices** $[T]$. To transform a vector $\mathbf{r_1}$ in coordinate system 1 to the same vector $\mathbf{r_2}$ in coordinate system 2, we can use

$$\mathbf{r_2} = [T]_1^2 \mathbf{r_1}. \tag{1.2}$$

Transformation matrices are **orthogonal**. This implies that $[T]^{-1} = [T]^T$. The inverse equals the transpose.

1.2 Changing vectors

Next to the velocity, also the **velocity** $\mathbf{v}$ and the **acceleration** $\mathbf{a}$ are vectors. They satisfy

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}}. \tag{1.3}$$

We see that the time rate of change is important. Now let's examine a vector $\mathbf{r} = r\hat{\mathbf{e}}$, where $\hat{\mathbf{e}}$ is a unit vector in some coordinate system 1. We now examine $\mathbf{r}$ from another coordinate system 2. Coordinate system 1 has a **rotation vector** ω with respect to system 2. The time derivative of $\mathbf{r}$ is now given by

$$\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{e}} + r\dot{\hat{\mathbf{e}}}. \tag{1.4}$$

It can be shown that $\dot{\hat{\mathbf{e}}} = \omega \times \hat{\mathbf{e}}$. This implies that

$$\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{e}} + r\left(\omega \times \hat{\mathbf{e}}\right). \tag{1.5}$$

1.3 Non-inertial reference frames

The basis of dynamics and stability are Newton's laws. But sadly, Newton's laws are not valid in a non-inertial reference frame. To find the differences, we examine a non-inertial reference frame F_B with origin B. We do this with respect to an inertial reference frame F_O with origin O. The position vector $\mathbf{r}$ of a particle P can be expressed as

$$\mathbf{r_O} = \mathbf{r_B} + \mathbf{r_{rel}}. \tag{1.6}$$

In this equation, $\mathbf{r_B}$ is the position vector of B, with respect to O. Also, $\mathbf{r_{rel}}$ is the position vector of P with respect to B. It is important to see that $\mathbf{r_O}$ and $\mathbf{r_B}$ are given with respect to F_O, while $\mathbf{r_{rel}}$ is given with respect to F_B.

We can differentiate the above equation, with respect to time. However, as time changes, also the coordinate system of F_B changes. To take that into account, we have to use equation (1.5). It then follows that

$$\mathbf{v_O} = \mathbf{v_B} + (\omega \times \mathbf{r_{rel}}) + \mathbf{v_{rel}}, \tag{1.7}$$

where ω is the rotational velocity of F_B with respect to F_O. Also, $\mathbf{v_{rel}}$ is the velocity of P, as seen from F_B.

We have now expressed the velocity of point P in F_O as a function of the velocity of point P in F_B. We can go even further. Differentiating again will give us the acceleration. Doing this, and working out the results, will give

$$\mathbf{a_O} = \mathbf{a_B} + \alpha \times \mathbf{r_{rel}} + \omega \times (\omega \times \mathbf{r_{rel}}) + 2\,(\omega \times \mathbf{v_{rel}}) + \mathbf{a_{rel}}. \tag{1.8}$$

The vector $\mathbf{a_{rel}}$ is the acceleration of P as seen from F_B. Also, $\alpha = \dot{\omega}$ is the **angular acceleration** of F_B, with respect to F_O. The term $\omega \times (\omega \times \mathbf{r_{rel}})$ is known as the **centrifugal** acceleration. The term $2\,(\omega \times \mathbf{v_{rel}})$ is called the **Coriolis acceleration**.

Now let's suppose a force $\mathbf{F}$ is acting on particle P. Newton's second law implies that $\mathbf{F} = m\mathbf{a_O}$. This implies that

$$m\mathbf{a_{rel}} = \mathbf{F} - m\mathbf{a_B} - m\alpha \times \mathbf{r_{rel}} - m\omega \times (\omega \times \mathbf{r_{rel}}) - 2m\,(\omega \times \mathbf{v_{rel}}) = \mathbf{F} + \mathbf{F_{fict}}. \tag{1.9}$$

Note that, in F_B, we do not have $\mathbf{F} = m\mathbf{a_{rel}}$. Newton's second law therefore does not hold in F_B. Instead, it appears as if some **fictitious force** $\mathbf{F_{fict}}$ is present. This fictitious force strangely effects the motion of the particle in point P. It is important to remember that $\mathbf{F_{fict}}$ is not a real force. It's only a force that appears to be present, if an observer forgets that he's in a non-inertial reference frame.

2 Momentum, work, energy and potential functions

2.1 Momentum

The **linear momentum p** of a particle is defined as

$$\mathbf{p} = m\mathbf{v}, \tag{2.1}$$

where m is the mass of the particle. The time rate of change $\dot{\mathbf{p}}$ equals the **resultant force** $\sum \mathbf{F}$ acting on the particle. From this follows that

$$\mathbf{p_1} + \int_{t_1}^{t_2} \sum \mathbf{F}\, dt = \mathbf{p_2}. \tag{2.2}$$

The change in linear momentum $\mathbf{p_2} - \mathbf{p_1}$ is called the **impulse**. The above equation is also known as the **Principle of Impulse and Momentum**.

In a similar way, we can define the **angular momentum** $\mathbf{L_O}$ of a particle about some point O as

$$\mathbf{L_O} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m\mathbf{v}. \tag{2.3}$$

The time rate of change $\dot{\mathbf{L}}_\mathbf{O}$ now equals the **resultant moment** $\sum \mathbf{r} \times \mathbf{F}$. This knowledge gives us the rotational version of the Principle of Impulse and Momentum. It is given by

$$\mathbf{L_1} + \int_{t_1}^{t_2} \sum (\mathbf{r} \times \mathbf{F})\; dt = \mathbf{L_2}. \tag{2.4}$$

2.2 Work and energy

Let's examine a particle moving from position $\mathbf{r_1}$ to position $\mathbf{r_2}$. The **work** done on the particle, by a force $\mathbf{F}$, is given by

$$W_{1\to 2} = \int_{\mathbf{r_1}}^{\mathbf{r_2}} \mathbf{F} \cdot d\mathbf{r}. \tag{2.5}$$

It can be shown that $W_{1\to 2}$ also equals

$$W_{1\to 2} = T_2 - T_1 = \Delta T, \tag{2.6}$$

where $T = \frac{1}{2}mv^2$ is the **kinetic energy** of the particle. The above equation is also known as the **Principle of Work and Energy**.

A force $\mathbf{F}$ is **conservative** if $\nabla \times \mathbf{F} = \mathbf{0}$. In this case, there is a certain **potential function V** satisfying $\mathbf{F} = -\nabla V$. (The minus sign is present by convention.) V is also known as the **potential energy**. It can be determined up to a constant. The point where $V = 0$ is called the **datum**. Usually, first a datum is set. Then, from the datum, the arbitrary constant is derived.

Now let's again examine the work done by force $\mathbf{F}$. It can be shown that

$$W_{1\to 2} = -(V_2 - V_1) = -\Delta V. \tag{2.7}$$

So the work done by a conservative force is independent of the path of the particle. It only depends on the initial and final position. Now let's examine all the forces acting on a particle. The total work done $W_{1\to 2}$ can be split up into two parts: The work done by conservative forces W_{cons} and the work done by non-conservative forces W_{nc}. It follows that

$$W_{nc} = \Delta T + \Delta V, \tag{2.8}$$

where V is the potential function of the conservative forces. If there are no non-conservative forces, then $W_{nc} = 0$. In this case, we have

$$T_1 + V_1 = T_2 + V_2 = E, \tag{2.9}$$

where E is the **total (mechanical) energy** of the particle. It is constant. The above statement is known as the **Conservation of Energy** relation.

2.3 Basic ideas of stability

For simplicity, let's reduce our problem to a one-dimensional problem. We can then rewrite the conservation of energy relation as

$$\frac{1}{2}mv^2 + V(x) = E. \tag{2.10}$$

So, given the total energy E, the magnitude of v only depends on the position x. This means that we can make a plot of v versus x. The resulting curves are called **phase curves**. If we plot multiple phase curves (for different values of E), then we get a **phase plane**, also known as a **phase portrait**.

We can examine a few interesting points in the phase portrait. A **turning point** is a point where the particle changes direction. At such points, we must have $v = 0$ for a brief moment. Similarly, an **equilibrium point** is a point where the resultant force is zero. In other words, $\mathbf{F} = -dV/dx = 0$. This means that, in an equilibrium point, the potential function V must be either at a minimum or at a maximum.

Whether an equilibrium point is a minimum or a maximum of V is, in fact, rather important. It determines whether the equilibrium point is stable. Let's suppose that we give the particle a small deviation from an equilibrium position. If the point is a minimum of V (and thus $d^2V/dx^2 > 0$), then the resulting force will point towards the equilibrium point. (The force is attractive.) The point will thus be **stable**. If, however, the point is a maximum of V (and thus $d^2V/dx^2 < 0$), then the resulting force will point away from the equilibrium point. (The force is repulsive.) The point is therefore **unstable**. We will later go more into depth on the stability of equilibrium points.