The Laplace Transform

1 Laplace transform definitions

1.1 Improper integrals

The **Laplace transform** involves an integral from zero to infinity, which is a so-called **improper integral**. Such an integral is defined as

$$\int_a^\infty f(t)\,dt = \lim_{A\to\infty}\int_a^A f(t)\,dt. \tag{1.1}$$

Such an integral can **converge** to a certain value or **diverge**.

1.2 Integral transforms

An **integral transform** is a relation of the form

$$F(s) = \int_\alpha^\beta K(s,t)\,f(t)\,dt, \tag{1.2}$$

where $K(s,t)$ is a given function, called the **kernel** of the transformation. This relation transforms the function f into another function F, which is called the **transform** of f.

1.3 Laplace transform

One such integral transform is the **Laplace transform**, which is often useful for linear differential equations. In this transform, $K(s,t) = e^{-st}$, $\alpha = 0$ and $\beta = \infty$. So the Laplace transform, denoted by $L\{f(t)\}$ (even though the L is often written slightly different), is defined as

$$L\{f(t)\} = F(s) = \int_0^\infty e^{-st}\,f(t)\,dt. \tag{1.3}$$

Now suppose $|f(t)| \leq Ke^{at}$ for $t \geq M$ for certain constants K, a and M, then the Laplace transformation exists for $s > a$. An overview of Laplace transforms can be seen in table 1.

Function $f(t) = L^{-1}\{F(s)\}$	**Laplace Transform** $F(s) = L\{f(t)\}$	**Range**	**Notes**
1	$\frac{1}{s}$	$s > 0$	
t^n	$\frac{n!}{s^{n+1}}$	$s > 0$	$n =$ positive integer
e^{at}	$\frac{1}{s-a}$	$s > a$	
$\sin at$	$\frac{a}{s^2+a^2}$	$s > 0$	
$\cos at$	$\frac{s}{s^2+a^2}$	$s > 0$	
$\sinh at$	$\frac{a}{s^2-a^2}$	$s > \lvert a\rvert$	
$\cosh at$	$\frac{s}{s^2-a^2}$	$s > \lvert a\rvert$	

Table 1: Laplace transforms of basic functions.

1.4 Linear operators

It can also be shown that the Laplace transform is a **linear operator**, meaning that for any constants c_1 and c_2 and functions $f_1(t)$ and $f_2(t)$,

$$L\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 L\{f_1(t)\} + c_2 L\{f_2(t)\}. \tag{1.4}$$

Using this theorem and table 1, it is possible to transform many functions quite easily.

Define $L^{-1}\{F(s)\}$ as the **inverse transform** of $F(s)$, meaning that $f(t) = L^{-1}\{L\{f(t)\}\}$. Then also L^{-1} is a linear operator. So this gives

$$L^{-1}\{c_1 F_1(s) + c_2 F_2(s)\} = c_1 L^{-1}\{F_1(s)\} + c_2 L^{-1}\{F_2(s)\} \tag{1.5}$$

1.5 Laplace transform of derivatives

The Laplace transform of $f'(t)$ is related to the Laplace transform of $f(t)$ (if it exists), by the equation

$$L\{f'(t)\} = sL\{f(t)\} - f(0). \tag{1.6}$$

If $f^{(n)}$ is the n'th derivative of f, then also

$$L\{f^{(n)}(t)\} = s^n L\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - s f^{(n-2)}(0) - f^{(n-1)}(0). \tag{1.7}$$

2 Functions and operators

2.1 Unit step function

The **unit step function**, also called the **Heaviside function**, is denoted by $u_c(t)$. It is defined such that $u_c(t) = 0$ if $t < c$ and $u_c(t) = 1$ for $t \geq c$.

(In other words, in an equation like $u_c(t)f(t)$, the function u_c "activates" the function $f(t)$ only for $t \geq c$, meaning for values of t smaller than c, the function is just 0. To "deactivate" the function $f(t)$, the function $(1 - u_c(t))f(t)$ can be used.)

The Laplace transform of u_c, with range $s > 0$, is

$$L\{u_c(t)\} = \frac{e^{-cs}}{s}. \tag{2.1}$$

If $F(s) = L\{f(t)\}$ and $f(t) = L^{-1}\{F(s)\}$, then

$$L\{u_c(t) f(t-c)\} = e^{-cs} F(s) \qquad \Leftrightarrow \qquad u_c(t) f(t-c) = L^{-1}\{e^{-cs} F(s)\}. \tag{2.2}$$

Analogous, it can be shown that

$$L\{e^{ct} f(t)\} = F(s-c) \qquad \Leftrightarrow \qquad e^{ct} f(t) = L^{-1}\{F(s-c)\}. \tag{2.3}$$

2.2 Delta function

The **Dirac delta function** $\delta(t)$ (also called the **delta function** or the **unit impulse function**) is defined such that

$$\delta(t) = 0 \text{ for } t \neq 0 \qquad \text{and} \qquad \int_{-\infty}^{\infty} \delta(t)\,dt = 1. \tag{2.4}$$

The Laplace transform of this function is

$$L\{\delta(t-t_0)\} = e^{-t_0 s}. \tag{2.5}$$

From this follows that $L\{\delta(t)\} = e^0 = 1$. And finally, the integral of the product of the delta function and any continuous function f is

$$\int_{-\infty}^{\infty} \delta(t-t_0) f(t) dt = f(t_0). \tag{2.6}$$

2.3 The convolution integral

If $F(s) = L\{f(t)\}$ and $G(s) = L\{g(t)\}$ both exist for $s > a \geq 0$, then

$$H(s) = F(s)G(s) = L\{h(t)\}, \qquad s > a, \tag{2.7}$$

where $h(t)$ is

$$h(t) = \int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t f(\tau)g(t-\tau)d\tau. \tag{2.8}$$

Note the difference between t and τ. The function h is known as the **convolution** of f and g and the integrals in the last equation are known as **convolution integrals**. It is conventional to write the above equation as $h(t) = (f * g)(t)$.

The $*$ is more or less similar to a multiplication. The following rules apply.

$$f * g = g * f, \tag{2.9}$$

$$f * (g_1 + g_2) = f * g_1 + f * g_2, \tag{2.10}$$

$$(f * g) * h = f * (g * h), \tag{2.11}$$

$$f * 0 = 0 * f. \tag{2.12}$$

However, in general $f * 1 \neq f$. Keep these rules in mind.

3 Solving differential equations

3.1 Solving a second order initial value problem

Suppose we have a differential equation of the form

$$ay'' + by' + cy = f(t), \tag{3.1}$$

with a, b and c constants. Taking the Laplace transform of both sides, and applying equations 1.4 and 1.7, gives

$$aL\{y''\} + bL\{y'\} + cL\{y\} = a(s^2 L\{y\} - sy(0) - y'(0)) + b(sL\{y\} - y(0)) + cL\{y\} = L\{f(t)\} = F(s). \tag{3.2}$$

Solving this for $L\{y\} = Y(s)$ gives

$$Y(s) = L\{y\} = \frac{(as+b)y(0) + ay'(0)}{as^2 + bs + c} + \frac{F(s)}{as^2 + bs + c}. \tag{3.3}$$

Now $L\{y\}$ is known. To find $f(t)$, we simply need to transform it back: $f(t) = L^{-1}\{L\{y\}\}$. But the inverse Laplace transform is not always easy to find. This problem is known as **the inversion problem for the Laplace transform**.

3.2 Inversion problem for the Laplace transform

To find $L^{-1}\{F(s)\}$ for some function $F(s)$, it's wise to split $F(s)$ up in pieces that occur in table 1, and use equation 1.5 to inversely transform all the pieces. However, this splitting up in pieces often isn't that easy. Especially when fractions are present, this can be difficult. That's why the following example shows a method in which fractions can be split up.

From the differential equation $y'' + y = \sin 2t$ follows

$$Y(s) = \frac{2s^3 + s^2 + 8s + 6}{(s^2+4)(s^2+1)}. \tag{3.4}$$

We want to split this fraction up in separate fractions, one with denominator $s^2 + 4$ and the other with denominator $s^2 + 1$, like

$$Y(s) = \frac{a}{s^2+4} + \frac{b}{s^2+1} = \frac{a(s^2+1) + b(s^2+4)}{(s^2+4)(s^2+1)}, \tag{3.5}$$

for certain a and b. From this we see that $a(s^2+1) + b(s^2+4) = 2s^3 + s^2 + 8s + 6$. But, if a and b are just constants, there are no third powers of s on the left side of the equation. So let's just suppose that $b = cs + d$. Now it's easy to see that $c = 2$. Working out the rest of the equation gives $a = -2/3$ and $d = 5/3$. So finally we have split up the fraction to

$$Y(s) = \frac{-2/3}{s^2+4} + \frac{2s}{s^2+1} + \frac{5/3}{s^2+1}. \tag{3.6}$$

Using table 1 we can find

$$y = -\frac{1}{3}\sin 2t + 2\cos t + \frac{5}{3}\sin t. \tag{3.7}$$

3.3 Discontinuous forcing functions

If the nonhomogeneous term of the differential equation, also called the **forcing term**, is discontinuous, solving the differential equation can be difficult. To illustrate how to solve such equations, we handle an example. To solve the differential equation

$$y'' + 4y = u_4(t)(t-4) - u_8(t)(t-8) \qquad y(0) = 0 \qquad y'(0) = 0, \tag{3.8}$$

we take the Laplace transform to find

$$Y(s) = \frac{e^{-4s} - e^{-8s}}{s^2(s^2+4)}. \tag{3.9}$$

It is now often wise to define $H = \frac{1}{s^2(s^2+4)}$, such that $Y(s) = (e^{-4s} - e^{-8s})H(s)$. If we define $h(t) = L^{-1}\{H(s)\}$, then taking the inverse Laplace transform, and using equation 2.2, gives

$$y(t) = u_4(t)h(t-4) - u_8(t)h(t-8). \tag{3.10}$$

We only need to find $h(t)$. Rewriting $H(s)$ differently gives

$$H(s) = \frac{1}{4}\frac{1}{s^2} - \frac{1}{8}\frac{2}{s^2+2^2} \qquad \Rightarrow \qquad h(t) = L^{-1}\{H(s)\} = \frac{1}{4}t - \frac{1}{8}\sin 2t, \tag{3.11}$$

which can be inserted in equation 3.10 to get the solution of our differential equation.

3.4 Using the convolution integral

Consider the differential equation

$$ay'' + by' + cy = g(t), \tag{3.12}$$

where a, b and c are constants. Let's define $L\{g(t)\} = G(s)$ and define $\Phi(s)$ and $\Psi(s)$ as

$$\Phi(s) = \frac{(as+b)y_0 + ay_0'}{as^2+bs+c}, \qquad \Psi(s) = \frac{G(s)}{as^2+bs+c}. \tag{3.13}$$

By taking the Laplace transform of the differential equation, we find

$$Y(s) = \Phi(s) + \Psi(s) \qquad \Leftrightarrow \qquad y(t) = \phi(t) + \psi(t), \tag{3.14}$$

where $\phi(t) = L^{-1}\{\Phi(s)\}$ and $\psi(t) = L^{-1}\{\Psi(s)\}$. It is convenient to write $\Psi(s)$ as

$$\Psi(s) = H(s)G(s), \tag{3.15}$$

where $H(s) = \frac{1}{as^2+bs+c}$. The function $H(s)$ is known as the **transfer function**. Using the convolution integral, we can solve for $\psi(t)$:

$$\psi(t) = L^{-1}\{H(s)G(s)\} = \int_0^t h(t-\tau)g(\tau)d\tau, \tag{3.16}$$

where $h(t) = L^{-1}\{H(s)\}$. The function $h(t)$ is called the **impulse response** of the system.