Sturm-Liouville Problems

1 Homogeneous Problems

1.1 Sturm-Liouville problems

In this chapter, we will be examining differential equations of the form

$$(p(x)y')' - q(x)y + \lambda r(x)y = 0, \tag{1.1}$$

where $p(x)$, $q(x)$ and $r(x)$ are given functions. y is a function of x and y' denotes the derivative with respect to x. Let's define the differential operator L to be

$$L[y] = -(p(x)y')' + q(x)y. \tag{1.2}$$

We can now rewrite the differential equation to

$$L[y] = \lambda r(x)y. \tag{1.3}$$

By using $L = 1$ we can also rewrite the boundary conditions to

$$\alpha_1 y(0) + \alpha_2 y'(0) = 0, \qquad \beta_1 y(1) + \beta_2 y'(1) = 0. \tag{1.4}$$

Such types of problems are called **Sturm-Liouville problems**.

1.2 Lagrange's identity

Lagrange's identity is

$$\int_0^1 (L[u]v - uL[v])\, dx = \left[-p(x)\left(u'(x)v(x) - u(x)v'(x)\right)\right]_0^1. \tag{1.5}$$

By using the boundary conditions of equation 1.4 we can show that the right side of this identity is 0, and thus also

$$\int_0^1 (L[u]v - uL[v])\, dx = 0. \tag{1.6}$$

Using the inner product, defined in the previous chapter, we can also write this as $(L[u], v) - (u, L[v]) = 0$.

1.3 Sturm-Liouville problem properties

Several things are known about Sturm-Liouville problems. It can be shown that all eigenvalues λ (for which there are nontrivial solutions to the problem) are real. In fact, if we find two eigenvalues λ_1 and λ_2 (with $\lambda_1 \neq \lambda_2$) and corresponding eigenfunctions ϕ_1 and ϕ_2, then

$$\int_0^1 r(x)\phi_1(x)\phi_2(x)dx = 0. \tag{1.7}$$

Also all eigenvalues are **simple**, meaning that each eigenvalue has only one eigenfunction (if you don't consider multiples of that eigenfunction). Furthermore, the eigenvalues can be ordered according to increasing magnitude, such that $\lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots$, where $\lambda_n \to \infty$ as $n \to \infty$.

1.4 Orthogonality

Equation 1.7 expresses the property of **orthogonality** of the eigenfunctions with respect to the weight function $r(x)$. The eigenfunctions are said to form an **orthogonal set** with respect to $r(x)$.

Every eigenvalue has one corresponding eigenfunction. However, every multiple of this eigenfunction is actually also an eigenfunction. So we can choose our eigenfunctions such that

$$\int_0^1 r(x)\phi_n^2(x)dx = 1. \tag{1.8}$$

Eigenfunctions satisfying this condition are said to be **normalized**. Normalized eigenfunctions are said to form an **orthonormal set** with respect to $r(x)$.

1.5 Expressing a function as a sum of eigenfunctions

Suppose we have found all the normalized eigenfunctions ϕ_n of a Sturm-Liouville problem. Can we now express a given function $f(x)$ as a sum of these eigenfunctions? If so, then a solution $f(x)$ could be written as

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x). \tag{1.9}$$

The only trick is to find the coefficients. To find any coefficient c_m, we can multiply the above equation by $r(x)\phi_m(x)$ and then integrate from 0 to 1, like

$$\int_0^1 = r(x)\phi_m(x)f(x)dx = \sum_{n=1}^{\infty} c_n \int_0^1 r(x)\phi_m(x)\phi_n(x)dx = c_m, \tag{1.10}$$

where we have used equation 1.7 in the last step. It follows that

$$c_m = \int_0^1 r(x)\phi_m(x)f(x)dx = (f(x), r(x)\phi_m(x)). \tag{1.11}$$

2 Nonhomogeneous Problems

2.1 Nonhomogeneous Sturm-Liouville problems

We have spend enough time on homogeneous problems. Now let's turn our attention to the nonhomogeneous problems. These problems have the form

$$L[y] = -\left(p(x)y'\right)' + q(x)y = \mu r(x)y + f(x), \tag{2.1}$$

where μ is a given constant. Note the extra term $f(x)$. Let the boundary conditions once more be

$$\alpha_1 y(0) + \alpha_2 y'(0) = 0, \qquad \beta_1 y(1) + \beta_2 y'(1) = 0. \tag{2.2}$$

To solve this problem, we first look at the homogeneous problem $L[y] = \lambda r(x)y$ with eigenvalues $\lambda_1, \lambda_2, \ldots$ and corresponding eigenfunction $\phi_1, \phi_2, \ldots$. We will assume that the solution $y = \phi(x)$ can be written as

$$\phi(x) = \sum_{n=1}^{\infty} b_n \phi_n(x). \tag{2.3}$$

However, this time we can not find the coefficients b_n in the way we are used to. Instead, we can find them with a small detour. First define the coefficients c_n as

$$c_n = \int_0^1 f(x)\phi_n(x)dx. \tag{2.4}$$

The coefficients b_n can then be found using

$$b_n = \frac{c_n}{\lambda_n - \mu}. \tag{2.5}$$

If $\lambda_n \neq \mu$ for all n, then the solution will simply be equal to

$$y = \phi(x) = \sum_{n=1}^{\infty} \frac{c_n}{\lambda_n - \mu}\phi_n(x). \tag{2.6}$$

However, if $\lambda_n = \mu$ for some n, then there is a problem. If $c_n \neq 0$ (for the same n), then we are dividing by zero. It can then be shown that the nonhomogeneous problem simply doesn't have a solution. However, if also $c_n = 0$, then b_n remains arbitrary. In this case there are infinitely many solutions to the nonhomogeneous problem.

2.2 Nonhomogeneous heat conduction problems

The generalized heat conduction equation is given by

$$r(x)u_t = (p(x)u_x)_x - q(x)u + F(x,t), \tag{2.7}$$

with two boundary conditions and one initial condition, being

$$u_x(0,t) - h_1 u(0,t) = 0, \qquad u_x(1,t) + h_2 u(1,t) = 0, \qquad \text{and} \qquad u(x,0) = f(x). \tag{2.8}$$

We assume any solution will have the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\phi_n(x), \tag{2.9}$$

where $\phi_n(x)$ are the eigenfunctions of the problem. To find the coefficients b_n we need to do several steps. First we need to find two intermediate coefficients B_n and $\gamma_n(t)$, given by

$$B_n = \int_0^1 r(x)f(x)\phi_n(x)dx, \tag{2.10}$$

$$\gamma_n(t) = \int_0^1 F(x,t)\phi_n(x)dx. \tag{2.11}$$

Now the coefficient b_n can be calculated using

$$b_n(t) = B_n + e^{-\lambda_n t}\int_0^t e^{\lambda_n s}\gamma_n(s)ds. \tag{2.12}$$

All the necessary coefficients are now known. The solution can be found by using the sum

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\phi_n(x). \tag{2.13}$$