Chapter Eleven

Section 11.1

1. Since the right hand sides of the ODE and the boundary conditions are all *zero*, the boundary value problem is *homogeneous*.

3. The right hand side of the ODE is *nonzero*. Therefore the boundary value problem is *nonhomogeneous*.

6. The ODE can also be written as

$$y'' + \lambda\left(1 + x^2\right)y = 0 .$$

Although the second boundary condition has a more general form, the boundary value problem is *homogeneous*.

7. First assume that $\lambda = 0$. The general solution of the ODE is $y(x) = c_1 x + c_2$. The boundary condition at $x = 0$ requires that $c_2 = 0$. Imposing the second condition,

$$c_1(\pi + 1) + c_2 = 0 .$$

It follows that $c_1 = c_2 = 0$. Hence there are no nontrivial solutions.

Suppose that $\lambda = -\mu^2$. In this case, the general solution of the ODE is

$$y(x) = c_1 cosh\, \mu x + c_2 sinh\, \mu x .$$

The first boundary condition requires that $c_1 = 0$. Imposing the second condition,

$$c_1\,(cosh\, \mu\pi + \mu sinh\, \mu\pi) + c_2(sinh\, \mu\pi + \mu cosh\, \mu\pi) = 0 .$$

The two boundary conditions result in

$$c_2(tanh\, \mu\pi + \mu) = 0 .$$

Since the *only* solution of the equation $tanh\, \mu\pi + \mu = 0$ is $\mu = 0$, we have $c_2 = 0$. Hence there are no nontrivial solutions.

Let $\lambda = \mu^2$, with $\mu > 0$. Then the general solution of the ODE is

$$y(x) = c_1 cos\, \mu x + c_2 sin\, \mu x .$$

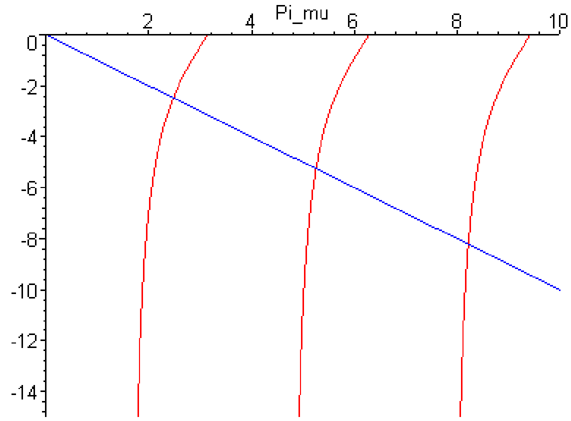
Imposing the boundary conditions, we obtain $c_1 = 0$ and

$$c_1\,(cos\, \mu\pi - \mu sin\, \mu\pi) + c_2(sin\, \mu\pi + \mu cos\, \mu\pi) = 0 .$$

For a *nontrivial* solution of the ODE, we require that $sin\, \mu\pi + \mu cos\, \mu\pi = 0$. Note that

$$cos\, \mu\pi = 0 \Rightarrow sin\, \mu\pi = 0 ,$$

which is false. It follows that $tan\, \mu\pi = -\mu$. From a plot of $\pi\, tan\, \pi\mu$ and $-\pi\mu$,



we find that there is a sequence of solutions, $\mu_1 \approx 0.7876$, $\mu_2 \approx 1.6716$, $\cdots$; For large values of n,

$$\pi\,\mu_n \approx (2n-1)\frac{\pi}{2}\,.$$

Therefore the eigenfunctions are $\phi_n(x) = sin\,\mu_n x$, with corresponding eigenvalues

$$\lambda_1 \approx 0.6204\,,\ \lambda_2 \approx 2.7943\,, \cdots .$$

Asymptotically,

$$\lambda_n \approx \frac{(2n-1)^2}{4}\,.$$

8. With $\lambda = 0$, the general solution of the ODE is $y(x) = c_1 x + c_2$. Imposing the two boundary conditions, $c_1 = 0$ and $2c_1 + c_2 = 0$. It follows that $c_1 = c_2 = 0$. Hence there are no nontrivial solutions.

Setting $\lambda = -\mu^2$, the general solution of the ODE is

$$y(x) = c_1 cosh\,\mu x + c_2 sinh\,\mu x\,.$$

The first boundary condition requires that $c_2 = 0$. Imposing the second condition,

$$c_1\,(cosh\,\mu + \mu sinh\,\mu) + c_2(sinh\,\mu + \mu cosh\,\mu) = 0\,.$$

The two boundary conditions result in

$$c_1(1 + \mu\,tanh\,\mu) = 0\,.$$

Since $\mu\,tanh\,\mu \geq 0$, it follows that $c_1 = 0$, and there are no nontrivial solutions.

Let $\lambda = \mu^2$, with $\mu > 0$. Then the general solution of the ODE is

$$y(x) = c_1 cos\,\mu x + c_2 sin\,\mu x\,.$$

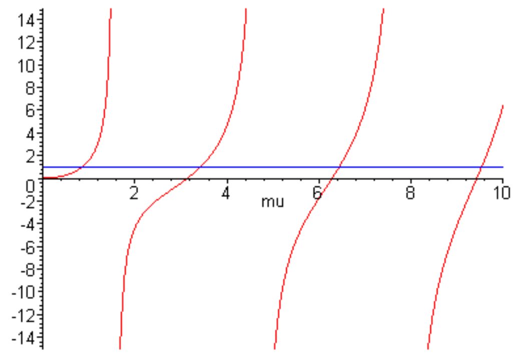
Imposing the boundary conditions, we obtain $c_2 = 0$ and

$$c_1\,(cos\,\mu - \mu sin\,\mu) + c_2(sin\,\mu + \mu cos\,\mu) = 0\,.$$

For a *nontrivial* solution of the ODE, we require that $cos\,\mu - \mu sin\,\mu = 0$. First note that

$$cos\,\mu = 0 \Rightarrow \mu = 0 \text{ or } sin\,\mu = 0\,.$$

Therefore we find that $1 - \mu\,tan\,\mu = 0$. From a plot of $\mu\,tan\,\mu$, there is a sequence of



solutions, $\mu_1 \approx 0.8603\,,\, \mu_2 \approx 3.4256\,,\cdots$; For large n,

$$\mu_n \approx (n-1)\pi\,.$$

Therefore the eigenfunctions are $\phi_n(x) = cos\,\mu_n x$, with corresponding eigenvalues

$$\lambda_1 \approx 0.7402\,,\, \lambda_2 \approx 11.7349\,,\cdots.$$

Asymptotically,

$$\lambda_n \approx (n-1)^2\pi^2\,.$$

12. First note that $P(x) = 1\,,\, Q(x) = -2x$ and $R(x) = \lambda$. Based on Prob. 11, the integrating factor is a solution of the ODE

$$\mu'(x) = -2x\,\mu(x)\,.$$

The differential equation is first order linear, with solution $\mu(x) = c\,exp(-x^2)$. It then follows that the *Hermite equation* can be written as

$$\left[e^{-x^2}y'\right]' + \lambda\,e^{-x^2}y = 0\,.$$

14. For the *Laguerre equation*, $P(x) = x\,,\, Q(x) = 1 - x$ and $R(x) = \lambda$. Using the result of Prob. 11, the integrating factor is a solution of the ODE

$$x\,\mu'(x) = -x\,\mu(x)\,.$$

The general solution of $\mu'(x) = -\mu(x)$ is $\mu(x) = c\,e^{-x}$. Therefore the *Laguerre equation* can be written as

$$[x\,e^{-x}y']' + \lambda\,e^{-x}y = 0\,.$$

15. For the *Chebyshev equation*, $P(x) = 1 - x^2$, $Q(x) = -x$ and $R(x) = \alpha^2$. The integrating factor is a solution of the ODE

$$\big(1 - x^2\big)\mu'(x) = x\,\mu(x)\,.$$

The differential equation is separable, with

$$\frac{d\mu}{\mu} = \frac{x}{1 - x^2}\,.$$

The general solution of the resulting ODE is

$$\mu(x) = \frac{c}{\sqrt{|1 - x^2|}}\,.$$

Recall that the *Chebyshev equation* is typically defined for $|x| \le 1$. Therefore it can also be written as

$$\left[\sqrt{1 - x^2}\,y'\right]' + \frac{\alpha^2}{\sqrt{1 - x^2}}\,y = 0\,.$$

16. We consider solutions of the form $u(x\,,t) = X(x)T(t)$. Substitution into the PDE results in

$$XT'' + c\,XT' + k\,XT = \alpha^2 X''T\,.$$

Dividing both sides of the equation by XT, we obtain

$$\frac{XT''}{XT} + c\,\frac{XT'}{XT} + k = \alpha^2\frac{X''T}{XT}\,,$$

that is,

$$\frac{T''}{T} + c\,\frac{T'}{T} = \alpha^2\frac{X''}{X} - k\,.$$

Since both sides of the resulting equation are functions of different variables, each must be
equal to a constant, say $-\lambda$. Therefore we obtain two ordinary differential equations

$$\alpha^2 X'' + (\lambda - k)X = 0 \quad \text{and} \quad T'' + cT' + \lambda T = 0\,.$$

17(a). Setting $y = s(x)u$, we have $y' = s'u + s\,u'$ and $y'' = s''u + 2s'u' + s\,u''$. Substitution into the given ODE results in

$$s''u + 2s'u' + s\,u'' - 2(s'u + s\,u') + (1 + \lambda)su = 0\,.$$

Collecting the various terms,

$$s\,u'' + (2s' - 2s)u' + [s'' - 2s' + (1+\lambda)s]u = 0\,.$$

The second term on the left vanishes as long as $s' = s$.

(b). With $s(x) = e^x$, the transformed differential equation can be written as

$$u'' + \lambda u = 0\,.$$

Since the boundary conditions are *homogeneous*, we also have $u(0) = u(1) = 0$. It now follows that the eigenfunctions are $u_n = sin\,\sqrt{\lambda_n}\,x$, with corresponding eigenvalues

$$\lambda_n = n^2\pi^2.$$

Therefore the eigenfunctions for the original problem are $\phi_n(x) = e^x sin\,n\pi x$, with corresponding eigenvalues

$$1 + \lambda_n = 1 + n^2\pi^2.$$

(c). The given equation is a second order *constant coefficient* differential equation. The characteristic equation is

$$r^2 - 2r + (1+\lambda) = 0\,,$$

with roots $r_{1,2} = 1 \pm \sqrt{-\lambda}$.

If $\lambda = 0$, then the general solution is $y = c_1 e^x + c_2\,xe^x$. Imposing the two boundary conditions, we find that $c_1 = c_2 = 0$, and hence there are no nontrivial solutions. If $\lambda < 0$, then the general solution is

$$y = c_1 exp\Big(1 + \sqrt{-\lambda}\Big)x + c_2 exp\Big(1 - \sqrt{-\lambda}\Big)x\,.$$

It again follows that $c_1 = c_2 = 0$, and hence there are no nontrivial solutions.

Therefore $\lambda > 0$, and the general solution is

$$y = c_1 e^x cos\,\sqrt{\lambda}\,x + c_2\,e^x sin\,\sqrt{\lambda}\,x\,.$$

Invoking the boundary conditions, we have $c_1 = 0$ and $c_2 e\,sin\sqrt{\lambda} = 0$. For a nontrivial
solution, $\sqrt{\lambda} = n\pi$.

19. First write the differential equation as

$$y'' + (1+\lambda)y' + \lambda y = 0\,,$$

which is a second order *constant coefficient* differential equation. The characteristic equation is

$$r^2 + (1+\lambda)r + \lambda = 0\,,$$

with roots $r_1 = -1$ and $r_2 = -\lambda$. For $\lambda \neq 1$, the general solution is

$$y = c_1 e^{-x} + c_2\, e^{-\lambda x}.$$

Imposing the boundary conditions, we require that $c_1 + c_2 = 0$ and $c_1 e^{-1} + c_2\, e^{-\lambda} = 0$. For a nontrivial solution, it follows that $e^{-1} = e^{-\lambda}$, and hence $\lambda = 1$, which is contrary to the assumption.

If $\lambda = 1$, then the general solution is

$$y = c_1 e^{-x} + c_2\, x e^{-x}.$$

The boundary conditions require that $c_1 = 0$ and $c_1 e^{-1} + c_2\, e^{-1} = 0$. Hence there are no nontrivial solutions.

21. Suppose that $\lambda = 0$. In that case the general solution is $y = c_1 x + c_2$. The boundary
conditions require that $c_1 + 2c_2 = 0$ and $c_1 + c_2 = 0$. We find that $c_1 = c_2 = 0$, and hence there are no nontrivial solutions.

(a). Let $\lambda = \mu^2$, with $\mu > 0$. Then the general solution of the ODE is

$$y(x) = c_1 cos\, \mu x + c_2 sin\, \mu x\,.$$

The boundary conditions require that

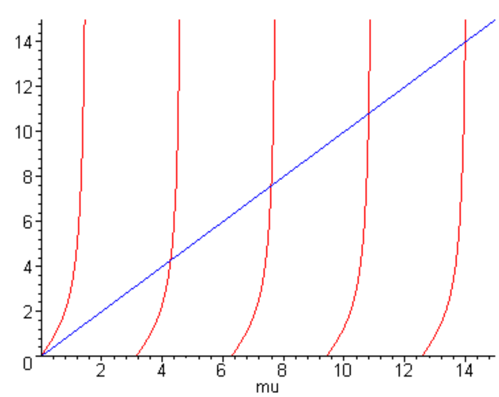
$$2c_1 + \mu\, c_2 = 0 \text{ and } c_1 cos\, \mu + c_2 sin\, \mu = 0\,.$$

These equations have a nonzero solution only if

$$2sin\, \mu - \mu\, cos\, \mu = 0\,,$$

which can also be written as

$$2tan\, \mu - \mu = 0\,.$$



Based on the graph, the positive roots of the determinantal equation are

$$\mu_1 \approx 4.2748\,,\ \mu_2 \approx 7.5965\,,\cdots;\ \text{for large } n,\, \mu_n \approx (2n+1)\frac{\pi}{2}\,.$$

Therefore the eigenvalues are

$$\lambda_1 \approx 18.2738\,,\ \lambda_2 \approx 57.7075\,,\cdots;\ \text{for large } n,\, \lambda_n \approx (2n+1)^2\frac{\pi^2}{4}\,.$$

(b). Setting $\lambda = -\mu^2 < 0$, the general solution of the ODE is

$$y(x) = c_1 cosh\,\mu x + c_2 sinh\,\mu x\,.$$

Imposing the boundary conditions, we obtain the equations

$$2c_1 + \mu\, c_2 = 0 \text{ and } c_1 cosh\,\mu + c_2 sinh\,\mu = 0\,.$$

These equations have a nonzero solution only if

$$2 sinh\,\mu - \mu\, cosh\,\mu = 0\,.$$

The latter equation is satisfied only for $\mu = 0$ and $\mu = \pm 1.9150$. Hence the only *negative* eigenvalue is $\lambda_{-1} = 3.6673$.

24. Based on the physical problem, $\lambda = m\omega^2/EI > 0$. Let $\lambda = \mu^4$. The characteristic equation is $r^4 - \mu^4 = 0$, with roots $r_{1,2} = \pm\,\mu i$, $r_3 = -\mu$ and $r_4 = \mu$. Hence the general solution is

$$y(x) = c_1 cosh\,\mu x + c_2 sinh\,\mu x + c_3 cos\,\mu x + c_4 sin\,\mu x\,.$$

(a). Simply supported on both ends : $y(0) = y''(0) = 0$; $y(L) = y''(L) = 0$. Invoking the boundary conditions, we obtain the system of equations

$$\begin{aligned} c_1 + c_3 &= 0 \\ c_1 - c_3 &= 0 \\ c_1 cosh\,\mu L + c_2 sinh\,\mu L + c_3 cos\,\mu L + c_4 sin\,\mu L &= 0 \\ c_1\mu^2 cosh\,\mu L + c_2\mu^2 sinh\,\mu L - c_3\mu^2 cos\,\mu L - c_4\mu^2 sin\,\mu L &= 0\,. \end{aligned}$$

The determinantal equation is

$$\mu^4 sinh\,\mu L\, sin\,\mu L = 0\,.$$

The nonzero roots are $\mu_n = n\pi/L$, $n = 1, 2, \cdots$. The first two equations result in $c_1 = c_3 = 0$. The last two equations,

$$\begin{aligned} c_2 sinh\,n\pi + c_4 sin\,n\pi &= 0 \\ c_2 sinh\,n\pi - c_4 sin\,n\pi &= 0\,, \end{aligned}$$

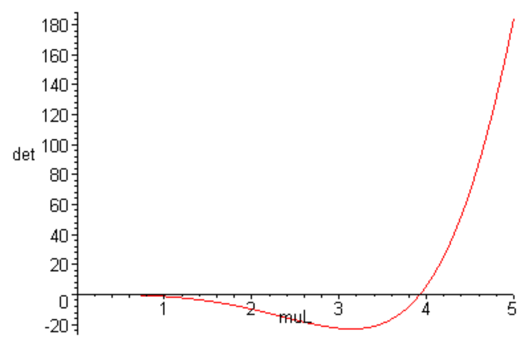
imply that $c_2 = 0$. Therefore the eigenfunctions are $\phi_n = sin\,\mu_n x$, with corresponding eigenvalues $\lambda_n = n^4\pi^4/L^4$.

(b). Simply supported : $y(0) = y''(0) = 0$; clamped : $y(L) = y'(L) = 0$.
Invoking the boundary conditions, we obtain the system of equations

$$\begin{aligned} c_1 + c_3 &= 0 \\ c_1 - c_3 &= 0 \\ c_1 cosh\,\mu L + c_2 sinh\,\mu L + c_3 cos\,\mu L + c_4 sin\,\mu L &= 0 \\ c_1\,\mu sinh\,\mu L + c_2\,\mu cosh\,\mu L - c_3\,\mu sin\,\mu L + c_4\,\mu cos\,\mu L &= 0\,. \end{aligned}$$

The determinantal equation is

$$2\mu^3 sinh\,\mu L\,cos\,\mu L - 2\mu^3 cosh\,\mu L\,sin\,\mu L = 0\,.$$



Based on numerical analysis, $\mu_1 \approx 3.9266/L$ and $\mu_2 \approx 7.0686/L$.

The first two equations result in $c_1 = c_3 = 0$. The last two equations,

$$\begin{aligned} c_2 sinh\,\mu_n L + c_4 sin\,\mu_n L &= 0 \\ c_2\,cosh\,\mu_n L + c_4\,cos\,\mu_n L &= 0\,, \end{aligned}$$

imply that

$$c_2 = -\frac{sin\,\mu_n L}{sinh\,\mu_n L}c_4\,.$$

Therefore the eigenfunctions are

$$\phi_n = -\frac{sin\,\mu_n L}{sinh\,\mu_n L}sinh\,\mu_n x + sin\,\mu_n x\,,$$

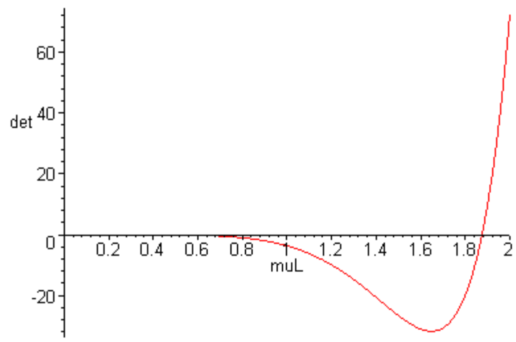
with corresponding eigenvalues $\lambda_n = \mu_n^4$.

(c). Clamped : $y(0) = y'(0) = 0$; free : $y''(L) = y'''(L) = 0$.
Invoking the boundary conditions, we obtain the system of equations

$$c_1 + c_3 = 0$$
$$\mu c_2 + \mu c_4 = 0$$
$$c_1\mu^2 cosh\,\mu L + c_2\mu^2 sinh\,\mu L - c_3\mu^2 cos\,\mu L - c_4\mu^2 sin\,\mu L = 0$$
$$c_1\,\mu^3 sinh\,\mu L + c_2\,\mu^3 cosh\,\mu L + c_3\,\mu^3 sin\,\mu L - c_4\,\mu^3 cos\,\mu L = 0\,.$$

The determinantal equation is

$$1 + cosh\,\mu L\,cos\,\mu L = 0\,.$$



The first two *nonzero* roots are $\mu_1 \approx 1.8751/L$ and $\mu_2 \approx 4.6941/L$. With $c_3 = -c_1$ and $c_4 = -c_2$, the system of equations reduce to

$$c_1(cosh\,\mu_n L + cos\,\mu_n L) + c_2(sinh\,\mu_n L + sin\,\mu_n L) = 0$$
$$c_1(sinh\,\mu_n L - sin\,\mu_n L) + c_2(cosh\,\mu_n L + cos\,\mu_n L) = 0\,.$$

Let $A_n = (cosh\,\mu_n L + cos\,\mu_n L)/(sinh\,\mu_n L + sin\,\mu_n L)$. The eigenfunctions are given by

$$\phi_n(x) = cosh\,\mu_n x - cos\,\mu_n x + A_n(sin\,\mu_n x - sinh\,\mu_n x)\,,$$

with corresponding eigenvalues $\lambda_n = \mu_n^4$.

25(a). Assume that the solution has the form $u(x\,,t) = X(x)T(t)$. Substitution into the
PDE results in

$$\frac{E}{\rho}X''T = XT''\,.$$

Dividing both sides of the equation by XT, we obtain

$$\frac{E}{\rho}\frac{X''T}{XT} = \frac{XT''}{XT}\,,$$

that is,

$$\frac{X''}{X} = \frac{\rho}{E}\frac{T''}{T}\,.$$

Since both sides of the resulting equation are functions of different variables, each must be
equal to a constant, say $-\lambda$. Therefore we obtain two ordinary differential equations

$$X'' + \lambda X = 0 \quad \text{and} \quad T'' + \lambda\frac{E}{\rho}T = 0\,.$$

(b). Given that $u(0\,,t) = X(0)T(t)$ for $t > 0$, it follows that $X(0) = 0$. The second boundary condition can be expressed as

$$EAX'(L)T(t) + mX(L)T''(t) = 0\,, \;\; t > 0\,.$$

From the result in Part (a),

$$EAX'(L)T(t) - \lambda m\frac{E}{\rho}X(L)T(t) = 0\,, \;\; t > 0\,.$$

Since the condition is to be satisfied for all $t > 0$, we arrive at the boundary condition

$$X'(L) - \lambda\frac{m}{\rho A}X(L) = 0\,.$$

(c). If $\lambda = 0$, the general solution of the spatial equation is

$$X(x) = c_1 x + c_2\,.$$

The boundary condition require that $c_1 = c_2 = 0$. Hence there are no nontrivial solutions.
If $\lambda = -\mu^2 < 0$, then the general solution is

$$X(x) = c_1 cosh\,\mu x + c_2 sinh\,\mu x\,.$$

The first boundary condition implies that $c_1 = 0$. The second boundary condition requires
that

$$c_2 cosh\,\mu L + c_2\,\mu\frac{m}{\rho A}sinh\,\mu L = 0\,.$$

The solution is nontrivial only if

$$\mu\,tanh\,\mu L = -\frac{\rho A}{m}\,.$$

Since $\mu\,tanh\,\mu L \geq 0$, there are no nontrivial solutions.

Let $\lambda = \mu^2 > 0$. The general solution of the spatial equation is

$$X(x) = c_1 cos\,\mu x + c_2 sin\,\mu x\,.$$

The first boundary condition implies that $c_1 = 0$. The second boundary condition requires
that

$$c_2\, cos\, \mu L - c_2\, \mu \frac{m}{\rho A} sin\, \mu L = 0\,.$$

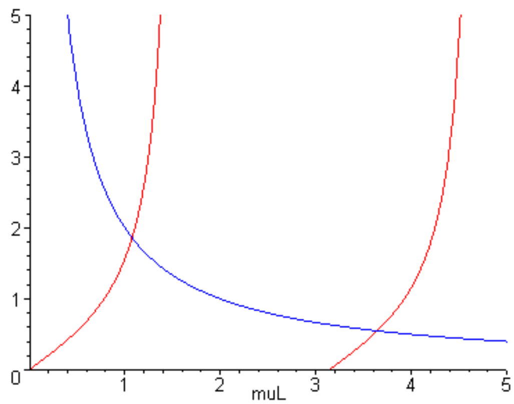
For a nontrivial solution, it is necessary that

$$cos\, \mu L - \mu \frac{m}{\rho A} sin\, \mu L = 0\,,$$

or

$$tan\, \mu L = \frac{\rho A}{m\mu}\,.$$

For the case $(m/\rho AL) = 0.5\,,$



we find that $\mu_1 L \approx 1.0769$ and $\mu_2 L \approx 3.6436$. Therefore the eigenfunctions are given by $\phi_n(x) = sin\, \mu_n x$. The corresponding eigenvalues are solutions of

$$cos\, \sqrt{\lambda_n} L - \frac{L}{2} \sqrt{\lambda_n} sin\, \sqrt{\lambda_n} L = 0\,.$$

The first two eigenvalues are approximated as $\lambda_1 \approx 1.1597/L^2$ and $\lambda_2 \approx 13.276/L^2$.

Section 11.2

2. Based on the boundary conditions, $\lambda > 0$. The general solution of the ODE is

$$y(x) = c_1 cos\sqrt{\lambda}x + c_2 sin\sqrt{\lambda}x\,.$$

The boundary condition $y'(0) = 0$ requires that $c_2 = 0$. Imposing the second boundary condition, we find that $c_1 cos\sqrt{\lambda} = 0$. So for a nontrivial solution, $\sqrt{\lambda} = (2n-1)\pi/2$, $n = 1, 2, \cdots$. Therefore the eigenfunctions are given by

$$\phi_n(x) = k_n cos\frac{(2n-1)\pi x}{2}\,.$$

In this problem, $r(x) = 1$, and the normalization condition is

$$k_n^2 \int_0^1 \left[cos\frac{(2n-1)\pi x}{2}\right]^2 dx = 1\,.$$

It follows that $k_n^2 = 2$. Therefore the normalized eigenfunctions are

$$\phi_n(x) = \sqrt{2}\,cos\frac{(2n-1)\pi x}{2}\,, \quad n = 1, 2, \cdots.$$

3. Based on the boundary conditions, $\lambda \geq 0$. For $\lambda = 0$, the eigenfunction is
$\phi_0(x) = k_0$.
Set $k_0 = 1$. With $\lambda > 0$, the general solution of the ODE is

$$y(x) = c_1 cos\sqrt{\lambda}x + c_2 sin\sqrt{\lambda}x\,.$$

Invoking the boundary conditions, we require that $c_2 = 0$ and $c_1\sqrt{\lambda}\,sin\sqrt{\lambda} = 0$.
Since
$\lambda > 0$, the eigenvalues are $\lambda_n = n^2\pi^2,\ n = 1, 2, \cdots$, with corresponding eigenfunctions

$$\phi_n(x) = k_n cos\,n\pi x\,.$$

The normalization condition is

$$k_n^2 \int_0^1 cos^2 n\pi x\,dx = 1\,.$$

It follows that $k_n^2 = 2$. Therefore the normalized eigenfunctions are

$$\phi_0(x) = 1\,,\ \text{and}\ \phi_n(x) = \sqrt{2}\,cos\,n\pi x\,, \quad n = 1, 2, \cdots.$$

4. From Prob. 8 in Section 11.1, the eigenfunctions are $\phi_n(x) = k_n cos\,\sqrt{\lambda_n}\,x$, in which
$cos\sqrt{\lambda_n} - \sqrt{\lambda_n}\,sin\sqrt{\lambda_n} = 0$. The normalization condition is

$$k_n^2 \int_0^1 cos^2 \sqrt{\lambda_n}\, x\, dx = 1\,.$$

First note that

$$\int_0^1 cos^2 \sqrt{\lambda_n}\, x\, dx = \frac{cos\,\sqrt{\lambda_n}\ sin\,\sqrt{\lambda_n} + \sqrt{\lambda_n}}{2\sqrt{\lambda_n}}\,.$$

Based on the determinantal equation,

$$\begin{aligned} \frac{cos\,\sqrt{\lambda_n}\ sin\,\sqrt{\lambda_n} + \sqrt{\lambda_n}}{2\sqrt{\lambda_n}} &= \frac{1 + sin^2\,\sqrt{\lambda_n}}{2} \\ &= \frac{3 - cos\,2\sqrt{\lambda_n}}{4}\,. \end{aligned}$$

Therefore

$$k_n^2 = \frac{4}{3 - cos\,2\sqrt{\lambda_n}}$$

and the normalized eigenfunctions are given by

$$\phi_n(x) = \frac{2\,cos\,\sqrt{\lambda_n}\,x}{\sqrt{3 - cos\,2\sqrt{\lambda_n}}}\,.$$

6. As shown in Prob. 1, the normalized eigenfunctions are

$$\phi_n(x) = \sqrt{2}\,sin\frac{(2n-1)\pi x}{2}\,, \quad n = 1, 2, \cdots.$$

Based on Eq. (34), with $r(x) = 1$, the coefficients in the eigenfunction expansion are given by

$$\begin{aligned} c_m &= \int_0^1 f(x)\phi_m(x)dx \\ &= \sqrt{2}\int_0^1 sin\frac{(2m-1)\pi x}{2}dx \\ &= \frac{2\sqrt{2}}{(2m-1)\pi}\,. \end{aligned}$$

Therefore we obtain the formal expansion

$$1 = \frac{2\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}sin\frac{(2n-1)\pi x}{2}\,.$$

8. We consider the normalized eigenfunctions

$$\phi_n(x) = \sqrt{2}\, sin\frac{(2n-1)\pi x}{2}\,, \quad n = 1, 2, \cdots.$$

Based on Eq. (34), with $r(x) = 1$, the coefficients in the eigenfunction expansion are given by

$$\begin{aligned} c_m &= \int_0^1 f(x)\phi_m(x)dx \\ &= \sqrt{2}\int_0^{1/2} sin\frac{(2m-1)\pi x}{2}dx \\ &= \frac{2\sqrt{2}}{(2m-1)\pi}\left[1 - cos\frac{(2m-1)\pi}{4}\right]. \end{aligned}$$

Therefore we obtain the formal expansion

$$f(x) = \frac{2\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\left[1 - cos\frac{(2n-1)\pi}{4}\right] sin\frac{(2n-1)\pi x}{2}\,.$$

9. The normalized eigenfunctions are

$$\phi_n(x) = \sqrt{2}\, sin\frac{(2n-1)\pi x}{2}\,, \quad n = 1, 2, \cdots.$$

Based on Eq. (34), with $r(x) = 1$, the coefficients in the eigenfunction expansion are given by

$$\begin{aligned} c_m &= \int_0^1 f(x)\phi_m(x)dx \\ &= \sqrt{2}\int_0^{1/2} 2x\, sin\frac{(2m-1)\pi x}{2}dx + \sqrt{2}\int_{1/2}^{1} sin\frac{(2m-1)\pi x}{2}dx \\ &= \frac{8}{(2m-1)^2\pi^2}\left[sin\frac{m\pi}{2} - cos\frac{m\pi}{2}\right]. \end{aligned}$$

Therefore the formal expansion of the given function is

$$f(x) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{sin\frac{n\pi}{2} - cos\frac{n\pi}{2}}{(2n-1)^2} sin\frac{(2n-1)\pi x}{2}\,.$$

11. From Prob. 4, the normalized eigenfunctions are given by

$$\phi_n(x) = \frac{2\, cos\sqrt{\lambda_n}\, x}{\sqrt{3 - cos\, 2\sqrt{\lambda_n}}}\,,$$

in which the eigenvalues satisfy $cos\sqrt{\lambda_n} - \sqrt{\lambda_n}\, sin\sqrt{\lambda_n} = 0$. Based on Eq. (34), the coefficients in the eigenfunction expansion are given by

$$\begin{aligned} c_m &= \int_0^1 f(x)\phi_m(x)dx \\ &= \frac{2}{\sqrt{3 - cos\,2\sqrt{\lambda_m}}}\int_0^1 x\, cos\,\sqrt{\lambda_m}\, x dx \\ &= \frac{\sqrt{2}\left(2\,cos\,\sqrt{\lambda_m} - 1\right)}{\lambda_m \alpha_m}\,, \end{aligned}$$

in which $\alpha_m = \sqrt{1 + sin^2\sqrt{\lambda_m}}$.

12. The normalized eigenfunctions are given by

$$\phi_n(x) = \frac{2\,cos\,\sqrt{\lambda_n}\, x}{\sqrt{3 - cos\,2\sqrt{\lambda_n}}}\,,$$

in which the eigenvalues satisfy $cos\sqrt{\lambda_n} - \sqrt{\lambda_n}\, sin\sqrt{\lambda_n} = 0$. Based on Eq. (34), the coefficients in the eigenfunction expansion are given by

$$\begin{aligned} c_m &= \int_0^1 f(x)\phi_m(x)dx \\ &= \frac{2}{\sqrt{3 - cos\,2\sqrt{\lambda_m}}}\int_0^1 (1-x) cos\,\sqrt{\lambda_m}\, x dx \\ &= \frac{\sqrt{2}\left(1 - cos\,\sqrt{\lambda_m}\right)}{\lambda_m \alpha_m}\,, \end{aligned}$$

in which $\alpha_m = \sqrt{1 + sin^2\sqrt{\lambda_m}}$.

13. We consider the normalized eigenfunctions

$$\phi_n(x) = \frac{2\,cos\,\sqrt{\lambda_n}\, x}{\sqrt{3 - cos\,2\sqrt{\lambda_n}}}\,,$$

in which the eigenvalues satisfy $cos\sqrt{\lambda_n} - \sqrt{\lambda_n}\, sin\sqrt{\lambda_n} = 0$. The coefficients in the eigenfunction expansion are given by

$$\begin{aligned} c_n &= \int_0^1 f(x)\phi_n(x)dx \\ &= \frac{2}{\sqrt{3 - cos\, 2\sqrt{\lambda_n}}} \int_0^{1/2} cos\,\sqrt{\lambda_n}\, x dx \\ &= \frac{\sqrt{2}\, sin\big(\sqrt{\lambda_n}/2\big)}{\sqrt{\lambda_n}\,\alpha_n}\,, \end{aligned}$$

in which $\alpha_n = \sqrt{1 + sin^2\sqrt{\lambda_n}}$.

15. The differential equation can be written as

$$\big[\big(1 + x^2\big)y'\big]\,' + y = 0\,,$$

with $p(x) = -1 - x^2$ and $q(x) = 1$. The boundary conditions are homogeneous and *separated*. Hence the BVP is *self-adjoint*.

16. Since the boundary conditions are *not* separated, the inner product is computed: Given u and v, sufficiently smooth and satisfying the boundary conditions,

$$\begin{aligned} (L[u], v) &= \int_0^1 [u''v + uv]dx \\ &= u'v\Big|_0^1 - \int_0^1 [u'v' + uv]dx \\ &= [u'v - uv']\Big|_0^1 + (u, L[v]). \end{aligned}$$

Based on the given boundary conditions,

$$\begin{aligned} u'(1)v(1) - u'(0)v(0) &= u(0)v(1) + 2u(1)v(0) \\ -u(1)v'(1) + u(0)v'(0) &= -u(1)v(0) - 2u(0)v(1)\,. \end{aligned}$$

Since

$$[u'v - uv']\Big|_0^1 = u(1)v(0) - u(0)v(1)\,,$$

the BVP is *not* self-adjoint.

18. The differential equation can be written as

$$-[y']' = \lambda y\,,$$

with $p(x) = 1$, $q(x) = 0$, and $r(x) = 1$. The boundary conditions are homogeneous and *separated*. Hence the BVP is *self-adjoint*.

19. If $a_2 = 0$, then

$$u'(1)v(1) - u(1)v'(1) = -\frac{b_2}{b_1}u'(1)v'(1) + \frac{b_2}{b_1}u'(1)v'(1) = 0\,,$$

and since $u(0) = v(0) = 0$,

$$u'(0)v(0) - u(0)v'(0) = 0\,.$$

If $b_2 = 0$, then $u(1) = v(1) = 0$ implies that

$$u'(1)v(1) - u(1)v'(1) = 0\,.$$

Furthermore,

$$u'(0)v(0) - u(0)v'(0) = -\frac{a_2}{a_1}u'(0)v'(0) + \frac{a_2}{a_1}u'(0)v'(0) = 0\,.$$

Clearly, the results are also true if $a_2 = b_2 = 0$.

20. Suppose that $\phi_1(x)$ and $\phi_2(x)$ are linearly independent eigenfunctions associated with an eigenvalue λ. The Wronskian is given by

$$W(\phi_1\,,\phi_2)(x) = \phi_1(x)\phi_2'(x) - \phi_2(x)\phi_1'(x).$$

Each of the eigenfunctions satisfies the boundary condition $a_1 y(0) + a_2 y'(0) = 0$. If either $a_1 = 0$ or $a_2 = 0$, then clearly $W(\phi_1\,,\phi_2)(0) = 0$. On the other hand, if a_2 is *not* equal to zero, then

$$\begin{aligned} W(\phi_1\,,\phi_2)(0) &= \phi_1(0)\phi_2'(0) - \phi_2(0)\phi_1'(0) \\ &= -\frac{a_1}{a_2}\phi_1(0)\phi_2(0) + \frac{a_1}{a_2}\phi_2(0)\phi_1(0) \\ &= 0\,. \end{aligned}$$

By Theorem 3.3.2, $W(\phi_1\,,\phi_2)(x) = 0$ for all $0 \le x \le 1$. Based on Theorem 3.3.3, $\phi_1(x)$ and $\phi_2(x)$ must be linearly *dependent*. Hence λ must be a simple eigenvalue.

22. We consider the operator

$$L[y] = -[p(x)y']' + q(x)y$$

on the interval $0 < x < 1$, together with the boundary conditions

$$a_1 y(0) + a_2 y'(0) = 0\ ,\quad b_1 y(1) + b_2 y'(1) = 0\,.$$

Let $u = \phi + i\psi$ and $v = \xi + i\eta$. If u and v both satisfy the boundary conditions, then the real and imaginary parts also satisfy the same boundary conditions. Using the inner product

$$(u\,,v) = \int_0^1 u(x)\overline{v}(x)dx\,,$$

$$\begin{aligned}(L[u]\,,v) &= \int_0^1 \big[-[p(x)u']'\overline{v}+q(x)u\overline{v}\big]dx \\ &= \int_0^1 \big\{-[p(x)(\phi'+i\psi')]'\overline{v}+q(x)u\overline{v}\big\}dx \\ &= -\,p(x)(\phi'+i\psi')\overline{v}\Big|_0^1+\int_0^1 \{p(x)(\phi'+i\psi')\,\overline{v}'+q(x)u\overline{v}\}dx\,.\end{aligned}$$

Integrating by parts, again,

$$\int_0^1 \{p(x)(\phi'+i\psi')\overline{v}'\}dx=(\phi+i\psi)p(x)\overline{v}'\Big|_0^1-\int_0^1 \big\{[p(x)\overline{v}']'u\big\}dx\,.$$

Collecting the boundary terms,

$$p(x)[(\phi'+i\psi')\overline{v}-(\phi+i\psi)\overline{v}']\Big|_0^1=p(x)[(\phi'+i\psi')(\xi-i\eta)-(\phi+i\psi)(\xi'-i\eta')]\Big|_0^1.$$

The *real* part is given by

$$\begin{aligned}p(x)[(\phi'\xi+\psi'\eta)-(\phi\xi'+\psi\eta')]\Big|_0^1 &= p(x)[(\phi'\xi-\phi\xi')+(\psi'\eta-\psi\eta')]\Big|_0^1 \\ &= p(x)[\phi'\xi-\phi\xi']\Big|_0^1+p(x)[\psi'\eta-\psi\eta']\Big|_0^1.\end{aligned}$$

Since ϕ , ψ , ξ and η satisfy the boundary conditions, it follows that

$$p(x)[(\phi'\xi+\psi'\eta)-(\phi\xi'+\psi\eta')]\Big|_0^1=0\,.$$

Similarly, the *imaginary* part also vanishes. That is,

$$p(x)[(\psi'\xi-\psi\xi')-(\phi'\eta-\phi\eta')]\Big|_0^1=0\,.$$

Therefore

$$\begin{aligned}(L[u]\,,v) &= \int_0^1 \big\{-[p(x)\overline{v}']'u+q(x)u\overline{v}\big\}dx \\ &= (L[\overline{v}]\,,\overline{u}) \\ &= \overline{(\overline{u}\,,L[\overline{v}])}\,.\end{aligned}$$

The result follows from the fact that $\overline{(\overline{u}\,,L[\overline{v}])}=(u\,,L[v])$.

24. Based on the physical problem, $\lambda=P/EI>0$. Let $\lambda=\mu^2$. The characteristic equation is $r^4+\mu^2r^2=0$, with roots $r_{1,2}=0$, $r_3=-\mu i$ and $r_4=\mu i$. Hence the general solution is

$$y(x)=c_1+c_2\,x+c_3cos\,\mu x+c_4sin\,\mu x\,.$$

(a). Simply supported on both ends : $y(0) = y''(0) = 0\,;\; y(L) = y''(L) = 0\,.$
Invoking the boundary conditions, we obtain the system of equations

$$
\begin{aligned}
c_1 + c_3 &= 0 \\
c_3 &= 0 \\
c_3 cos\,\mu L + c_4 sin\,\mu L &= 0 \\
c_1 + c_2 L + c_3 cos\,\mu L + c_4 sin\,\mu L &= 0\,.
\end{aligned}
$$

The determinantal equation is

$$sin\,\mu L = 0\,.$$

The nonzero roots are $\mu_n = n\pi/L\,,\; n = 1, 2, \cdots$. Therefore the eigenfunctions are $\phi_n = sin\,\mu_n x$, with corresponding eigenvalues $\lambda_n = n^2\pi^2/L^2$. Hence the smallest eigenvalue is $\lambda_1 = \pi^2/L^2$.

(b). Simply supported : $y(0) = y''(0) = 0\,;$ clamped : $y(L) = y'(L) = 0\,.$
Invoking the boundary conditions, we obtain the system of equations

$$
\begin{aligned}
c_1 + c_3 &= 0 \\
c_3 &= 0 \\
c_2 - c_3 \mu sin\,\mu L + c_4 \mu cos\,\mu L &= 0 \\
c_1 + c_2 L + c_3\, cos\,\mu L + c_4\, sin\,\mu L &= 0\,.
\end{aligned}
$$

The determinantal equation is

$$\mu L\, cos\,\mu L - sin\,\mu L = 0\,.$$

It follows that the eigenfunctions are given by

$$\phi_n(x) = sin\sqrt{\lambda_n}\,x - \left(\sqrt{\lambda_n} cos\sqrt{\lambda_n} L\right) x\,,$$

and the eigenvalues satisfy the equation $L\sqrt{\lambda_n}\, cos\,\sqrt{\lambda_n}\,L - sin\,\sqrt{\lambda_n}\,L = 0\,.$
The smallest eigenvalue is estimated as $\lambda_1 \approx (4.4934)^2/L^2$.

(c). Clamped : $y(0) = y'(0) = 0\,;$ clamped : $y(L) = y'(L) = 0\,.$
Invoking the boundary conditions, we obtain the system of equations

$$
\begin{aligned}
c_1 + c_3 &= 0 \\
c_2 + \mu c_4 &= 0 \\
c_1 + c_2 L + c_3 cos\,\mu L + c_4 sin\,\mu L &= 0 \\
c_2 - c_3\, \mu sin\,\mu L + c_4\, \mu cos\,\mu L &= 0\,.
\end{aligned}
$$

The determinantal equation is

$$2 - 2cos\,\mu L = \mu L\, sin\,\mu L\,.$$

It follows that the eigenfunctions are given by

$$\phi_n(x) = 1 - cos\sqrt{\lambda_n}\, x\,,$$

and the eigenvalues satisfy the equation $2 - 2cos\,\sqrt{\lambda_n}L = \sqrt{\lambda_n}L\,sin\,\sqrt{\lambda_n}L$.
The smallest eigenvalue is $\lambda_1 = (2\pi)^2/L^2$.

26. As shown is Prob. 25 , the general solution is

$$y(x) = c_1 + c_2\,x + c_3cos\,\mu x + c_4sin\,\mu x\,.$$

Imposing the boundary conditions, we obtain the system of equations

$$\begin{aligned} c_2 &= 0 \\ c_1 + c_3 &= 0 \\ c_2 + \mu c_4 &= 0 \\ c_3cos\,\mu L + c_4sin\,\mu L &= 0\,. \end{aligned}$$

For a nontrivial solution, it is necessary that

$$cos\,\mu L = 0\,.$$

We find that $c_2 = c_4 = 0$, and hence the eigenfunctions are given by

$$\phi_n(x) = 1 - cos\sqrt{\lambda_n}\, x\,.$$

The corresponding eigenvalues are $\lambda_n = (2n-1)^2\pi^2/4L^2,\, n = 1, 2, \cdots$. The smallest eigenvalue is $\lambda_1 = \pi^2/4L^2$.

Section 11.3

4. The eigensystem of the associated homogeneous problem is given in Prob. 11 of Section 11.2 . The normalized eigenfunctions are

$$\phi_n(x) = \frac{\sqrt{2}\,cos\,\sqrt{\lambda_n}\,x}{\sqrt{1 + sin^2\sqrt{\lambda_n}}}\,,$$

in which the eigenvalues satisfy $cos\sqrt{\lambda_n} - \sqrt{\lambda_n}\,sin\sqrt{\lambda_n} = 0$. Rewrite the given differential equation as $-y'' = 2y + x$. Since $\mu = 2 \neq \lambda_n$, the formal solution of the nonhomogeneous problem is

$$y(x) = \sum_{n=1}^{\infty} \frac{c_n}{\lambda_n - 2}\phi_n(x),$$

in which

$$\begin{aligned} c_n &= \int_0^1 f(x)\phi_n(x)dx \\ &= \frac{\sqrt{2}}{\sqrt{1 + sin^2\sqrt{\lambda_n}}}\int_0^1 x\,cos\,\sqrt{\lambda_n}\,x\,dx \\ &= \frac{\sqrt{2}\left(2\,cos\,\sqrt{\lambda_n} - 1\right)}{\lambda_n\sqrt{1 + sin^2\sqrt{\lambda_n}}}\,. \end{aligned}$$

Therefore we obtain the formal expansion

$$y(x) = 2\sum_{n=1}^{\infty} \frac{\sqrt{2}\left(2\,cos\,\sqrt{\lambda_n} - 1\right)\,cos\,\sqrt{\lambda_n}\,x}{\lambda_n(\lambda_n - 2)\left(1 + sin^2\sqrt{\lambda_n}\right)}\,.$$

5. The solution follows that in Prob. 1 , except that the coefficients are given by

$$\begin{aligned} c_n &= \int_0^1 f(x)\phi_n(x)dx \\ &= \sqrt{2}\int_0^{1/2} 2x\,sin\,n\pi x\,dx + \sqrt{2}\int_{1/2}^1 (2 - 2x)\,sin\,n\pi x\,dx \\ &= 4\frac{\sqrt{2}\,sin(n\pi/2)}{n^2\pi^2}\,. \end{aligned}$$

Therefore the formal solution is

$$y(x) = 8\sum_{n=1}^{\infty} \frac{sin(n\pi/2)\ sin\,n\pi x}{n^2\pi^2(n^2\pi^2 - 2)}\,.$$

6. The differential equation can be written as $-y'' = \mu y + f(x)$. Note that $q(x) = 0$ and $r(x) = 1$. As shown in Prob. 1 in Section 11.2, the normalized eigenfunctions are

$$\phi_n(x) = \sqrt{2}\, sin\frac{(2n-1)x}{2},$$

with associated eigenvalues $\lambda_n = (2n-1)^2\pi^2/4$. Based on Theorem 11.3.1, the formal solution is given by

$$y(x) = \sqrt{2}\sum_{n=1}^{\infty}\frac{c_n}{(\lambda_n - \mu)}\, sin\frac{(2n-1)x}{2},$$

as long as $\mu \neq \lambda_n$. The coefficients in the series expansion are computed as

$$c_n = \sqrt{2}\int_0^1 f(x) sin\frac{(2n-1)x}{2}\, dx.$$

7. As shown in Prob. 1 in Section 11.2, the normalized eigenfunctions are

$$\phi_n(x) = \sqrt{2}\, cos\frac{(2n-1)x}{2},$$

with associated eigenvalues $\lambda_n = (2n-1)^2\pi^2/4$. Based on Theorem 11.3.1, the formal solution is given by

$$y(x) = \sqrt{2}\sum_{n=1}^{\infty}\frac{c_n}{(\lambda_n - \mu)}\, cos\frac{(2n-1)x}{2},$$

as long as $\mu \neq \lambda_n$. The coefficients in the series expansion are computed as

$$c_n = \sqrt{2}\int_0^1 f(x) cos\frac{(2n-1)x}{2}\, dx.$$

9. The normalized eigenfunctions are

$$\phi_n(x) = \frac{\sqrt{2}\, cos\sqrt{\lambda_n}\, x}{\sqrt{1 + sin^2\sqrt{\lambda_n}}}.$$

The eigenvalues satisfy $cos\sqrt{\lambda_n} - \sqrt{\lambda_n}\, sin\sqrt{\lambda_n} = 0$. Based on Theorem 11.3.1, the formal solution is given by

$$y(x) = \sqrt{2}\sum_{n=1}^{\infty}\frac{c_n\, cos\sqrt{\lambda_n}\, x}{(\lambda_n - \mu)\sqrt{1 + sin^2\sqrt{\lambda_n}}},$$

as long as $\mu \neq \lambda_n$. The coefficients in the series expansion are computed as

$$c_n = \frac{\sqrt{2}}{\sqrt{1+sin^2\sqrt{\lambda_n}}}\int_0^1 f(x)cos\sqrt{\lambda_n}\,x\,dx\,.$$

13. The differential equation can be written as $-y'' = \pi^2 y + cos\,\pi x - a$. Note that $\mu = \pi^2$ and $f(x) = cos\,\pi x - a$. Furthermore, $\mu = \pi^2$ is an eigenvalue corresponding to the eigenfunction $\phi_1(x) = \sqrt{2}\,sin\,\pi x$. A solution exists only if $f(x)$ and $\phi_1(x)$ are *orthogonal*. Since

$$\int_0^1 (cos\,\pi x - a)sin\,\pi x\,dx = -2a/\pi\,,$$

there exists a solution as long as $a = 0$. In that case, the ODE is

$$y'' + \pi^2 y = -cos\,\pi x\,.$$

The complementary solution is $y_c(x) = c_1 cos\,\pi x + c_2 sin\,\pi x$. A particular solution is $Y(x) = Ax\,cos\,\pi x + Bx\,sin\,\pi x$. Using the *method of undetermined coefficients*, we find that $A = 0$ and $B = -1/2\pi$. Therefore the general solution is

$$y(x) = c_1 cos\,\pi x + c_2 sin\,\pi x - \frac{x}{2\pi}sin\,\pi x\,.$$

The boundary conditions require that $c_1 = 0$. Hence the solution of the boundary value problem is

$$y(x) = c_2 sin\,\pi x - \frac{x}{2\pi}sin\,\pi x\,.$$

15. Let $y(x) = \phi_1(x) + \phi_2(x)$. It follows that $L[y] = L[\phi_1] + L[\phi_2] = f(x)$. Also,

$$\begin{aligned} a_1 y(0) + a_2 y'(0) &= a_1\phi_1(0) + a_1\phi_2(0) + a_2\phi_1'(0) + a_2\phi_2'(0) \\ &= a_1\phi_1(0) + a_2\phi_1'(0) + a_1\phi_2(0) + a_2\phi_2'(0) \\ &= \alpha\,. \end{aligned}$$

Similarly, the boundary condition at $x = 1$ is satisfied as well.

16. The complementary solution is $y_c(x) = c_1 cos\,\pi x + c_2 sin\,\pi x$. A particular solution is $Y(x) = A + Bx$. Using the *method of undetermined coefficients*, we find that $A = 0$ and $B = 1$. Therefore the general solution is

$$y(x) = c_1 cos\,\pi x + c_2 sin\,\pi x + x\,.$$

Imposing the boundary conditions, we find that $c_1 = 1$. Therefore the solution of the BVP is

$$y(x) = cos\,\pi x + c_2 sin\,\pi x + x\,.$$

Now attempt to solve the problem as shown in Prob. 15. Let BVP-1 be given by

$$u'' + \pi^2 u = \pi^2 x\,,$$
$$u(0) = 0\,, \quad u(1) = 0\,.$$

The general solution of the ODE is

$$u(x) = c_1 cos\,\pi x + c_2 sin\,\pi x + x\,.$$

The boundary conditions require that $c_1 = 0$ and $-c_1 + 1 = 0$. We find that BVP-1 has no solution. Let BVP-2 be given by

$$v'' + \pi^2 v = 0\,,$$
$$v(0) = 1\,, \quad v(1) = 0\,.$$

The general solution of the ODE is $v(x) = c_1 cos\,\pi x + c_2 sin\,\pi x$. Imposing the boundary conditions, we obtain $c_1 = 1$ and $-c_1 = 0$. Thus BVP-2 has no solution.

17. Setting $y(x) = u(x) + v(x)$, substitution results in

$$u'' + v'' + p(x)[u' + v'] + q(x)[u + v] = u'' + p(x)u' + q(x)u + \\ + v'' + p(x)v' + q(x)v\,.$$

Since the left hand side of the equation is *zero*,

$$u'' + p(x)u' + q(x)u = -[v'' + p(x)v' + q(x)v].$$

Furthermore, $u(0) = y(0) - v(0) = 0$ and $u(1) = y(1) - v(1) = 0$. The simplest function having the assumed properties is $v(x) = (b - a)x + a$. In this case,

$$g(x) = (a - b)p(x) + (a - b)x\,q(x) - a\,q(x)\,.$$

20. The associated homogeneous PDE is $u_t = u_{xx}$, $0 < x < 1$, with

$$u_x(0\,,t) = 0,\ u_x(1,t) + u(1,t) = 0 \text{ and } u(x\,,0) = 1 - x.$$

Applying the method of *separation of variables*, we obtain the eigenvalue problem $X'' + \lambda X = 0$, with boundary conditions $X'(0) = 0$ and $X'(1) + X(1) = 0$. It was shown in Prob. 4, in Section 11.2, that the normalized eigenfunctions are

$$\phi_n(x) = \frac{\sqrt{2}\,cos\,\sqrt{\lambda_n}\,x}{\sqrt{1 + sin^2\sqrt{\lambda_n}}}\,,$$

where $cos\sqrt{\lambda_n} - \sqrt{\lambda_n}\,sin\sqrt{\lambda_n} = 0.$

We assume a solution of the form

$$u(x\,,t) = \sum_{n=1}^{\infty} b_n(t)\phi_n(x)\,.$$

Substitution into the given PDE results in

$$\begin{aligned}\sum_{n=1}^{\infty} b_n'(t)\phi_n(x) &= \sum_{n=1}^{\infty} b_n(t)\phi_n''(x) + e^{-t} \\ &= -\sum_{n=1}^{\infty} \lambda_n b_n(t)\phi_n(x) + e^{-t},\end{aligned}$$

that is,

$$\sum_{n=1}^{\infty} [b_n'(t) + \lambda_n b_n(t)]\phi_n(x) = e^{-t}.$$

We now note that

$$1 = \sum_{n=1}^{\infty} \frac{\sqrt{2}\, sin\sqrt{\lambda_n}}{\sqrt{\lambda_n}\sqrt{1 + sin^2\sqrt{\lambda_n}}}\,\phi_n(x)\,.$$

Therefore

$$e^{-t} = \sum_{n=1}^{\infty} \beta_n\, e^{-t}\phi_n(x)\,,$$

in which $\beta_n = \sqrt{2}\, sin\sqrt{\lambda_n}/\left[\sqrt{\lambda_n}\sqrt{1 + sin^2\sqrt{\lambda_n}}\right]$. Combining these results,

$$\sum_{n=1}^{\infty} \left[b_n'(t) + \lambda_n b_n(t) - \beta_n e^{-t}\right]\phi_n(x) = 0\,.$$

Since the resulting equation is valid for $0 < x < 1$, it follows that

$$b_n'(t) + \lambda_n b_n(t) = \beta_n e^{-t},\ n = 1, 2, \cdots.$$

Prior to solving the sequence of ODEs, we establish the initial conditions. These are obtained from the expansion

$$u(x\,,0) = 1 - x = \sum_{n=1}^{\infty} \alpha_n\phi_n(x)\,,$$

in which $\alpha_n = \sqrt{2}\left(1 - cos\sqrt{\lambda_n}\right)/\left[\lambda_n\sqrt{1 + sin^2\sqrt{\lambda_n}}\right]$. That is, $b_n(0) = \alpha_n$. Therefore the solutions of the first order ODEs are

$$b_n(t) = \frac{\beta_n\left(e^{-t} - e^{-\lambda_n t}\right)}{(\lambda_n - 1)} + \alpha_n e^{-\lambda_n t},\ n = 1, 2, \cdots.$$

Hence the solution of the boundary value problem is

$$u(x\,,t)=\sum_{n=1}^{\infty}\left[\frac{\beta_n\left(e^{-t}-e^{-\lambda_n t}\right)}{(\lambda_n-1)}+\alpha_n e^{-\lambda_n t}\right]\phi_n(x)\,.$$

21. Based on the boundary conditions, the normalized eigenfunctions are given by

$$\phi_n(x)=\sqrt{2}\,sin\,n\pi x\,,$$

with associated eigenvalues $\lambda_n=n^2\pi^2$. We now assume a solution of the form

$$u(x\,,t)=\sum_{n=1}^{\infty}b_n(t)\phi_n(x)\,.$$

Substitution into the given PDE results in

$$\begin{aligned}\sum_{n=1}^{\infty}b_n'(t)\phi_n(x)\ &=\sum_{n=1}^{\infty}b_n(t)\phi_n''(x)+1-|1-2x|\\ &=\ -\sum_{n=1}^{\infty}\lambda_n b_n(t)\phi_n(x)+1-|1-2x|,\end{aligned}$$

that is,

$$\sum_{n=1}^{\infty}[b_n'(t)+\lambda_n b_n(t)]\phi_n(x)=1-|1-2x|.$$

It was shown in Prob. 5 that

$$1-|1-2x|=\sum_{n=1}^{\infty}4\frac{\sqrt{2}\,sin(n\pi/2)}{n^2\pi^2}\phi_n(x)\,.$$

Substituting on the right hand side and collecting terms, we obtain

$$\sum_{n=1}^{\infty}\left[b_n'(t)+\lambda_n b_n(t)-4\frac{\sqrt{2}\,sin(n\pi/2)}{n^2\pi^2}\right]\phi_n(x)=0.$$

Since the resulting equation is valid for $0<x<1$, it follows that

$$b_n'(t)+n^2\pi^2 b_n(t)=4\frac{\sqrt{2}\,sin(n\pi/2)}{n^2\pi^2},\ \ n=1,2,\cdots.$$

Based on the given initial condition, we also have $b_n(0)=0$, for $n=1,2,\cdots$. The solutions of the first order ODEs are

$$b_n(t)=4\frac{\sqrt{2}\,sin(n\pi/2)}{n^4\pi^4}\left(1-e^{-n^2\pi^2 t}\right),\ \ n=1,2,\cdots.$$

Hence the solution of the boundary value problem is

$$u(x\,,t) = \frac{8}{\pi^4}\sum_{n=1}^{\infty}\frac{sin(n\pi/2)}{n^4}\left(1 - e^{-n^2\pi^2 t}\right)sin\, n\pi x\,.$$

$23(a)$. Let $u(x\,,t)$ be a solution of the boundary value problem and $v(x)$ be a solution of the related BVP. Substituting for $u(x\,,t) = w(x\,,t) + v(x)$, we have

$$r(x)u_t = r(x)w_t$$

and

$$\begin{aligned}[p(x)u_x]_x - q(x)u + F(x) &= [p(x)w_x]_x - q(x)w + [p(x)v']' - q(x)v + F(x)\\ &= [p(x)w_x]_x - q(x)w - F(x) + F(x)\\ &= [p(x)w_x]_x - q(x)w\,.\end{aligned}$$

Hence $w(x\,,t)$ is a solution of the *homogeneous* PDE

$$r(x)w_t = [p(x)w_x]_x - q(x)w\,.$$

The required *boundary conditions* are

$$\begin{aligned}w(0\,,t) = u(0\,,t) - v(0) = 0\,,\\ w(1\,,t) = u(1\,,t) - v(1) = 0\,.\end{aligned}$$

The associated *initial condition* is $w(x\,,0) = u(x\,,0) - v(x) = f(x) - v(x)$.

(b). Let $v(x)$ be a solution of the ODE

$$[p(x)v']' - q(x)v = \,-F(x)\,,$$

and satisfying the boundary conditions $v'(0) - h_1 v(0) = T_1$, $v'(1) + h_2 v(1) = T_2$. If $w(x\,,t) = u(x\,,t) - v(x)$, then it is easy to show the w satisfies the PDE and initial condition given in Part (a). Furthermore,

$$\begin{aligned}w_x(0\,,t) - h_1 w(0\,,t) &= u_x(0\,,t) - v'(0) - h_1 u(0\,,t) + h_1 v(0)\\ &= u_x(0\,,t) - h_1 u(0\,,t) - v'(0) + h_1 v(0)\\ &= 0\,.\end{aligned}$$

Similarly, the other boundary condition is also homogeneous.

25. In this problem, $F(x) = -\pi^2 cos\, \pi x$. First find a solution of the boundary value problem

$$v'' = \pi^2 cos\, \pi x \ , \ v'(0) = 0 \ , \ v(1) = 1 \, .$$

The general solution is $v(x) = Ax + B - cos\, \pi x$. Imposing the initial conditions, the solution of the related BVP is $v(x) = -cos\, \pi x$. Now let $w(x\,,t) = u(x\,,t) + cos\, \pi x$. It follows that $w(x\,,t)$ satisfies the *homogeneous* boundary value problem, and the initial condition $w(x\,,0) = cos(3\pi x/2) - cos\, \pi x - (-cos\, \pi x) = cos(3\pi x/2)$.

We now seek solutions of the homogeneous problem of the form

$$w(x\,,t) = \sum_{n=1}^{\infty} b_n(t)\phi_n(x)\,,$$

in which $\phi_n(x) = \sqrt{2}\; cos\,(2n-1)\pi x/2$ are the *normalized* eigenfunctions of the *homogeneous* problem and $\lambda_n = (2n-1)^2\pi^2/4$, with $n = 1, 2, \cdots$. Substitution into the PDE for w, we have

$$\begin{aligned} \sum_{n=1}^{\infty} b_n'(t)\phi_n(x) &= \sum_{n=1}^{\infty} b_n(t)\phi_n''(x) \\ &= -\sum_{n=1}^{\infty} \lambda_n b_n(t)\phi_n(x)\,. \end{aligned}$$

Since the latter equation is valid for $0 < x < 1$, it follows that

$$b_n'(t) + \lambda_n b_n(t) = 0\,,\ n = 1, 2, \cdots,$$

with $b_n(t) = b_n(0)exp(-\lambda_n t)$. Hence

$$w(x\,,t) = \sum_{n=1}^{\infty} b_n(0)exp(-\lambda_n t)\phi_n(x)\,.$$

Imposing the initial condition, we require that

$$\sqrt{2}\sum_{n=1}^{\infty} b_n(0)cos\,\frac{(2n-1)\pi x}{2} = cos\frac{3\pi x}{2}\,.$$

It is evident that all of the coefficients are *zero*, except for $b_2(0) = 1/\sqrt{2}$. Therefore

$$w(x\,,t) = exp\big(-9\pi^2 t/4\big)cos\frac{3\pi x}{2}\,,$$

and the solution of the original BVP is

$$u(x\,,t) = exp\big(-9\pi^2 t/4\big)cos\frac{3\pi x}{2} - cos\,\pi x\,.$$

26(a). Let $u(x\,,t) = X(x)T(t)$. Substituting into the homogeneous form of (i),

$$r(x)XT'' = [p(x)X']'T - q(x)XT\,.$$

Now divide both sides of the resulting equation by XT to obtain

$$\frac{T''}{T} = \frac{[p(x)X']'}{r(x)X} - \frac{q(x)}{r(x)} = -\lambda\,.$$

It follows that

$$-[p(x)X']' + q(x)X = \lambda r(x)X\,.$$

Since the boundary conditions (ii) are valid for all $t > 0$, we also have

$$X'(0) - h_1X(0) = 0\;,\quad X'(1) + h_2X(1) = 0\,.$$

(b). Let λ_n and $\phi_n(x)$ denote the eigenvalues and eigenfunctions of the BVP in Part (a). Assume a solution, of the PDE (i), of the form

$$u(x\,,t) = \sum_{n=1}^{\infty} b_n(t)\phi_n(x)\,.$$

Substituting into (i),

$$\begin{aligned} r(x)\sum_{n=1}^{\infty} b_n''(t)\phi_n &= \sum_{n=1}^{\infty} b_n(t)\big\{[p(x)\phi_n']' - q(x)\phi_n\big\} + F(x\,,t) \\ &= \sum_{n=1}^{\infty} b_n(t)[-\lambda_n r(x)\phi_n] + F(x\,,t)\,. \end{aligned}$$

Rearranging the terms,

$$r(x)\sum_{n=1}^{\infty}[b_n''(t) + \lambda_n b_n(t)]\phi_n = F(x\,,t)\,,$$

or

$$\sum_{n=1}^{\infty}[b_n''(t) + \lambda_n b_n(t)]\phi_n = \frac{F(x\,,t)}{r(x)}\,.$$

Now expand the right hand side in terms of the eigenfunctions. That is, write

$$\frac{F(x\,,t)}{r(x)} = \sum_{n=1}^{\infty}\gamma_n(t)\phi_n(x)\,,$$

in which

$$\gamma_n(t) = \int_0^1 r(x)\frac{F(x\,,t)}{r(x)}\phi_n(x)dx$$
$$= \int_0^1 F(x\,,t)\phi_n(x)dx\,, \quad n = 1, 2, \cdots.$$

Combining these results, we have

$$\sum_{n=1}^{\infty}[b_n''(t) + \lambda_n b_n(t) - \gamma_n(t)]\phi_n = 0\,.$$

It follows that

$$b_n''(t) + \lambda_n b_n(t) = \gamma_n(t) \;\;,\; n = 1, 2, \cdots.$$

In order to solve this sequence of ODEs, we require initial conditions $b_n(0)$ and $b_n'(0)$. Note that

$$u(x\,,0) = \sum_{n=1}^{\infty} b_n(0)\phi_n(x) \;\text{ and }\; u_t(x\,,0) = \sum_{n=1}^{\infty} b_n'(0)\phi_n(x)\,.$$

Based on the given initial conditions,

$$f(x) = \sum_{n=1}^{\infty} b_n(0)\phi_n(x) \;\text{ and }\; g(x) = \sum_{n=1}^{\infty} b_n'(0)\phi_n(x)\,.$$

Hence $b_n(0) = \alpha_n$ and $b_n'(0) = \beta_n$, the expansion coefficients for $f(x)$ and $g(x)$ in terms of the eigenfunctions, $\phi_n(x)$.

$27(a)$. Since the eigenvectors are *orthogonal*, they form a basis. Given any vector **b**,

$$\mathbf{b} = \sum_{i=1}^{n} b_i \boldsymbol{\xi}^{(i)}.$$

Taking the inner product, with $\boldsymbol{\xi}^{(j)}$, of both sides of the equation, we have

$$\left(\mathbf{b}\,, \boldsymbol{\xi}^{(j)}\right) = b_j\left(\boldsymbol{\xi}^{(j)}\,, \boldsymbol{\xi}^{(j)}\right).$$

(b). Consider solutions of the form

$$\mathbf{x} = \sum_{i=1}^{n} a_i \boldsymbol{\xi}^{(i)}.$$

Substituting into Eq. (i), and using the above form of **b**,

$$\sum_{i=1}^{n} a_i \mathbf{A}\boldsymbol{\xi}^{(i)} - \sum_{i=1}^{n} \mu\, a_i \boldsymbol{\xi}^{(i)} = \sum_{i=1}^{n} b_i \boldsymbol{\xi}^{(i)}.$$

It follows that

$$\sum_{i=1}^{n} [a_i\lambda_i - \mu\, a_i - b_i]\boldsymbol{\xi}^{(i)} = \mathbf{0}\,.$$

Since the eigenvectors are linearly independent,

$$a_i\lambda_i - \mu\, a_i - b_i = 0\,, \text{ for } i = 1, 2, \cdots, n\,.$$

That is,

$$a_i = b_i/(\lambda_i - \mu)\,, \quad i = 1, 2, \cdots, n\,.$$

Assuming that the eigenvectors are *normalized*, the solution is given by

$$\mathbf{x} = \sum_{i=1}^{n} \frac{(\mathbf{b}\,, \boldsymbol{\xi}^{(i)})}{\lambda_i - \mu} \boldsymbol{\xi}^{(i)},$$

as long as μ is *not* equal to one of the eigenvalues.

29. First write the ODE as $y'' + y = -f(x)$. A fundamental set of solutions of the homogeneous equation is given by

$$y_1 = cos\, x \text{ and } y_2 = sin\, x\,.$$

The Wronskian is equal to $W[cos\, x\,, sin\, x] = 1$. Applying the method of *variation of parameters*, a particular solution is

$$Y(x) = y_1(x)u_1(x) + y_2(x)u_2(x)\,,$$

in which

$$u_1(x) = \int_0^x sin(s)f(s)ds \text{ and } u_2(x) = -\int_0^x cos(s)f(s)ds\,.$$

Therefore the general solution is

$$y = \phi(x) = c_1 cos\, x + c_2 sin\, x + cos\, x\int_0^x sin(s)f(s)ds - sin\, x\int_0^x cos(s)f(s)ds.$$

Imposing the boundary conditions, we must have $c_1 = 0$ and

$$c_2\, sin\, 1 + cos\, 1\int_0^1 sin(s)f(s)ds - sin\, 1\int_0^1 cos(s)f(s)ds = 0\,.$$

It follows that

$$c_2 = \frac{1}{sin\, 1}\int_0^1 sin(1-s)f(s)ds\,,$$

and

$$\phi(x) = \frac{sin\,x}{sin\,1}\int_0^1 sin(1-s)f(s)ds - \int_0^x sin(x-s)f(s)ds.$$

Using standard identities,

$$sin\,x \cdot sin(1-s) - sin\,1 \cdot sin(x-s) = sin\,s \cdot sin(1-x)\,.$$

Therefore

$$\frac{sin\,x \cdot sin(1-s)}{sin\,1} - sin(x-s) = \frac{sin\,s \cdot sin(1-x)}{sin\,1}\,.$$

Splitting up the *first* integral, we obtain

$$\begin{aligned}\phi(x) &= \int_0^x \frac{sin\,s \cdot sin(1-x)}{sin\,1} f(s)ds + \int_x^1 \frac{sin\,x \cdot sin(1-s)}{sin\,1} f(s)ds \\ &= \int_0^1 G(x\,,s)f(s)ds\,,\end{aligned}$$

in which

$$G(x\,,s) = \begin{cases} \frac{sin\,s \cdot sin(1-x)}{sin\,1}\,, & 0 \le s \le x \\ \frac{sin\,x \cdot sin(1-s)}{sin\,1}\,, & x \le s \le 1\,. \end{cases}$$

31. The general solution of the homogeneous problem is

$$y = c_1 + c_2\,x\,.$$

By inspection, it is easy to see that $y_1(x) = 1$ satisfies the BC $y'(0) = 0$ and that the function $y_2(x) = 1 - x$ satisfies the BC $y(1) = 0$. The Wronskian of these solutions is $W[y_1\,, y_2] = -1$. Based on Prob. 30 , with $p(x) = 1$, the Green's function is given by

$$G(x\,,s) = \begin{cases} (1-x)\,, & 0 \le s \le x \\ (1-s)\,, & x \le s \le 1\,. \end{cases}$$

Therefore the solution of the given BVP is

$$\phi(x) = \int_0^x (1-x)f(s)ds + \int_x^1 (1-s)f(s)ds\,.$$

32. The general solution of the homogeneous problem is

$$y = c_1 + c_2\,x\,.$$

We find that $y_1(x) = x$ satisfies the BC $y(0) = 0$. Imposing the boundary condition

$y(1) + y'(1) = 0$, we must have $c_1 + 2c_2 = 0$. Hence choose $y_2(x) = -2 + x$. The Wronskian of these solutions is $W[y_1, y_2] = 2$. Based on Prob. 30, with $p(x) = 1$, the Green's function is given by

$$G(x, s) = \begin{cases} s(x-2)/2, & 0 \le s \le x \\ x(s-2)/2, & x \le s \le 1. \end{cases}$$

Therefore the solution of the given BVP is

$$\phi(x) = \frac{1}{2}\int_0^x s(x-2)f(s)ds + \frac{1}{2}\int_x^1 x(s-2)f(s)ds.$$

34. The general solution of the homogeneous problem is

$$y = c_1 + c_2 x.$$

By inspection, it is easy to see that $y_1(x) = x$ satisfies the BC $y(0) = 0$ and that the function $y_2(x) = 1$ satisfies the BC $y'(1) = 0$. The Wronskian of these solutions is $W[y_1, y_2] = -1$. Based on Prob. 30, with $p(x) = 1$, the Green's function is given by

$$G(x, s) = \begin{cases} s, & 0 \le s \le x \\ x, & x \le s \le 1. \end{cases}$$

Therefore the solution of the given BVP is

$$\phi(x) = \int_0^x sf(s)ds + \int_x^1 xf(s)ds.$$

35(a). We proceed to show that if the expression given by Eq. (iv) is substituted into the
integral of Eq. (iii), then the result is the solution of the nonhomogeneous problem. As long as we can interchange the summation and integration,

$$\begin{aligned} y = \phi(x) &= \int_0^1 G(x, s, \mu)f(s)ds \\ &= \sum_{n=1}^{\infty}\frac{\phi_i(x)}{\lambda_i - \mu}\int_0^1 f(s)\phi_i(s)ds. \end{aligned}$$

Note that

$$\int_0^1 f(s)\phi_i(s)ds = c_i.$$

Therefore

$$y = \phi(x) = \sum_{n=1}^{\infty}\frac{c_i\,\phi_i(x)}{\lambda_i - \mu},$$

as given by Eq. (13) in the text. It is assumed that the eigenfunctions are *normalized* and $\lambda_i \neq \mu$.

(b). For any fixed value of x, $G(x\,,s\,,\mu)$ is a function of s and the parameter μ. With appropriate assumptions on G, we can write the eigenfunction expansion

$$G(x\,,s\,,\mu) = \sum_{i=1}^{\infty} a_i(x\,,\mu)\phi_i(s)\,.$$

Since the eigenfunctions are *orthonormal* with respect to $r(x)$,

$$\int_0^1 G(x\,,s\,,\mu)r(s)\phi_i(s)ds = a_i(x\,,\mu)\,.$$

Now let

$$y_i(x) = \int_0^1 G(x\,,s\,,\mu)r(s)\phi_i(s)ds\,.$$

Based on the association $f(x) = r(x)\phi_i(x)$, it is evident that

$$L[y_i] = \mu\, r(x)y_i(x) + r(x)\phi_i(x)\,.$$

In order to evaluate the left hand side, we consider the eigenfunction expansion

$$y_i(x) = \sum_{k=1}^{\infty} b_{ik}\phi_k(x)\,.$$

It follows that

$$\begin{aligned} L[y_i] &= \sum_{k=1}^{\infty} b_{ik}L[\phi_k] \\ &= \sum_{k=1}^{\infty} b_{ik}\lambda_k\, r(x)\phi_k(x)\,. \end{aligned}$$

Therefore

$$r(x)\sum_{k=1}^{\infty} b_{ik}\lambda_k\phi_k(x) = \mu\, r(x)\sum_{k=1}^{\infty} b_{ik}\phi_k(x) + r(x)\phi_i(x)\,,$$

and since $r(x) \neq 0$,

$$\sum_{k=1}^{\infty} b_{ik}\lambda_k\phi_k(x) = \mu\sum_{k=1}^{\infty} b_{ik}\phi_k(x) + \phi_i(x)\,.$$

Rearranging the terms, we find that

$$\phi_i(x) = \sum_{k=1}^{\infty} b_{ik}(\lambda_k - \mu)\phi_k(x)\,.$$

Since the eigenfunctions are linearly independent, $b_{ik}(\lambda_k - \mu) = \delta_{ik}$, and thus

$$y_i(x) = \sum_{k=1}^{\infty} \frac{\delta_{ik}}{\lambda_k - \mu}\phi_k(x) = \frac{1}{\lambda_i - \mu}\phi_i(x)\,.$$

We conclude that

$$a_i(x\,,\mu) = \frac{1}{\lambda_i - \mu}\phi_i(x)\,,$$

which verifies that

$$G(x\,,s\,,\mu) = \sum_{i=1}^{\infty} \frac{\phi_i(x)\phi_i(s)}{\lambda_i - \mu}\,.$$

36. First note that $-\,d^2y/ds^2 = 0$ for $s \neq x$. On the interval $0 < s < x$, the solution of the ODE is $y_1(s) = c_1 + c_2 s$. Given that $y(0) = 0$, we have $y_1(s) = c_2 s$. On the interval $x < s < 1$, the solution is $y_2(s) = d_1 + d_2 s$. Imposing the condition $y(1) = 0$, we have $y_2(s) = d_1(1 - s)$. Assuming continuity of the solution, at $s = x$,

$$c_2 x = d_1(1 - x)\,,$$

which gives $c_2 = d_1(1 - x)/x$. Next, integrate both sides of the given ODE over an *infinitesimal* interval containing $s = x$:

$$-\int_{x^-}^{x^+} \frac{d^2y}{ds^2}\,ds = \int_{x^-}^{x^+} \delta(s - x)ds = 1\,.$$

It follows that

$$y'(x^-) - y'(x^+) = 1\,,$$

and hence $c_2 - (\,-\,d_1) = 1$. Solving for the two coefficients, we obtain $c_2 = 1 - x$ and $d_1 = x$. Therefore the solution of the BVP is given by

$$y(s) = \begin{cases} s(1 - x)\,, & 0 \le s \le x \\ x(1 - s)\,, & x \le s \le 1\,, \end{cases}$$

which is identical to the Green's function in Prob. 28.

Section 11.4

1. Let $\phi_n(x) = J_0\big(\sqrt{\lambda_n}\,x\big)$ be the eigenfunctions of the singular problem

$$-\,(x\,y')' = \lambda x y\,, \quad 0 < x < 1\,,$$
$$y\,, y' \text{ bounded as } x \to 0\,,\; y(1) = 0\,.$$

Let $\phi(x)$ be a solution of the given BVP, and set

$$\phi(x) = \sum_{n=0}^{\infty} b_n \phi_n(x)\,. \qquad (*)$$

Then

$$\begin{aligned} -\,(x\,\phi')' &= \mu\, x\phi + f(x) \\ &= \mu\, x\phi + x\,\frac{f(x)}{x}\,. \end{aligned}$$

Substituting $(*)$, we obtain

$$\sum_{n=0}^{\infty} b_n \lambda_n x\,\phi_n(x) = \mu x \sum_{n=0}^{\infty} b_n \phi_n(x) + x \sum_{n=0}^{\infty} c_n \phi_n(x)\,,$$

in which the c_n are the expansion coefficients of $f(x)/x$ for $x > 0$. That is,

$$\begin{aligned} c_n &= \frac{1}{\|\phi_n(x)\|^2} \int_0^1 x\,\frac{f(x)}{x}\phi_n(x)dx \\ &= \frac{1}{\|\phi_n(x)\|^2} \int_0^1 f(x)\phi_n(x)dx\,. \end{aligned}$$

It follows that if $x \neq 0$,

$$\sum_{n=0}^{\infty} [c_n - b_n(\lambda_n - \mu)]\phi_n(x) = 0\,.$$

As long as $\mu \neq \lambda_n$, linear independence of the eigenfunctions implies that

$$b_n = \frac{c_n}{\lambda_n - \mu}\,, \quad n = 1, 2, \cdots.$$

Therefore a formal solution is given by

$$\phi(x) = \sum_{n=0}^{\infty} \frac{c_n}{\lambda_n - \mu}\, J_0\Big(\sqrt{\lambda_n}\,x\Big),$$

in which $\sqrt{\lambda_n}$ are the positive roots of $J_0(x) = 0$.

$3(a)$. Setting $t = \sqrt{\lambda}\, x$, it follows that

$$\frac{dy}{dx} = \sqrt{\lambda}\,\frac{dy}{dt} \text{ and } \frac{d^2y}{dx^2} = \lambda\,\frac{d^2y}{dt^2}\,.$$

The given ODE can be expressed as

$$-\sqrt{\lambda}\,\frac{d}{dt}\left(\frac{t}{\sqrt{\lambda}}\sqrt{\lambda}\,\frac{dy}{dt}\right) + \frac{k^2\sqrt{\lambda}}{t} = \sqrt{\lambda}\,t\,y\,,$$

or

$$-\frac{d}{dt}\left(t\,\frac{dy}{dt}\right) + \frac{k^2}{t} = t\,y\,.$$

An equivalent form is given by

$$t^2\frac{dy}{dt} + t\,\frac{dy}{dt} + \left(t^2 - k^2\right)y = 0\,,$$

which is known as a Bessel equation of order k. A *bounded* solution is $J_k(t)$.

(b). $J_k\left(\sqrt{\lambda}\,x\right)$ satisfies the boundary condition at $x = 0$. Imposing the other boundary
condition, it is necessary that $J_k\left(\sqrt{\lambda}\right) = 0$. Therefore the eigenvalues are given by λ_n, $n = 1, 2\cdots$, where $\sqrt{\lambda_n}$ are the positive zeroes of $J_k(x)$. The eigenfunctions of the BVP are $\phi_n(x) = J_k\left(\sqrt{\lambda_n}\,x\right)$.

(c). The BVP is a *singular Sturm-Liouville* problem with

$$L[y] = -\left(x\,y'\right)' + \frac{k^2}{x}\,y \text{ and } r(x) = 1\,.$$

We note that

$$\begin{aligned}\lambda_n\int_0^1 x\,\phi_n(x)\phi_m(x)dx &= \int_0^1 L[\phi_n]\,\phi_m(x)dx\\ &= \int_0^1 \phi_n(x)L[\phi_m]dx\\ &= \lambda_m\int_0^1 x\,\phi_n(x)\phi_m(x)dx\,.\end{aligned}$$

Therefore

$$(\lambda_n - \lambda_m)\int_0^1 x\,\phi_n(x)\phi_m(x)dx = 0\,.$$

So for $n \neq m$, we have $\lambda_n \neq \lambda_m$ and

$$\int_0^1 x\,\phi_n(x)\phi_m(x)dx = 0\,.$$

(d). Consider the expansion

$$f(x) = \sum_{n=0}^{\infty} a_n \phi_n(x)\,.$$

Multiplying both sides of equation by $x\,\phi_j(x)$ and integrating from 0 to 1, and using the orthogonality of the eigenfunction,

$$\begin{aligned}\int_0^1 x\,f(x)\phi_j(x)dx &= \sum_{n=0}^{\infty} a_n \int_0^1 x\,\phi_j(x)\phi_n(x)dx \\ &= a_j \int_0^1 x\,\phi_j(x)\phi_j(x)dx\,.\end{aligned}$$

Therefore

$$a_j = \int_0^1 x\,f(x)\phi_j(x)dx \Big/ \int_0^1 x[\phi_j(x)]^2 dx\,,\ j = 1, 2, \cdots.$$

(e). Let $\phi(x)$ be a solution of the given BVP, and set

$$\phi(x) = \sum_{n=0}^{\infty} b_n \phi_n(x)\,, \qquad (*)$$

where $\phi_n(x) = J_k\big(\sqrt{\lambda_n}\,x\big)$. Then

$$\begin{aligned}L[\phi] &= \mu\,x\phi + f(x) \\ &= \mu\,x\phi + x\,\frac{f(x)}{x}\,.\end{aligned}$$

Substituting $(*)$, we obtain

$$\sum_{n=0}^{\infty} b_n \lambda_n x\,\phi_n(x) = \mu x \sum_{n=0}^{\infty} b_n \phi_n(x) + x \sum_{n=0}^{\infty} c_n \phi_n(x)\,,$$

in which the c_n are the expansion coefficients of $f(x)/x$ for $x > 0$. That is,

$$\begin{aligned}c_n &= \frac{1}{\|\phi_n(x)\|^2} \int_0^1 x\,\frac{f(x)}{x}\phi_n(x)dx \\ &= \frac{1}{\|J_k\big(\sqrt{\lambda_n}\,x\big)\|^2} \int_0^1 f(x) J_k\Big(\sqrt{\lambda_n}\,x\Big)dx\,.\end{aligned}$$

It follows that if $x \neq 0$,

$$\sum_{n=0}^{\infty}[c_n - b_n(\lambda_n - \mu)]J_k\left(\sqrt{\lambda_n}\, x\right) = 0\,.$$

As long as $\mu \neq \lambda_n$, linear independence of the eigenfunctions implies that

$$b_n = \frac{c_n}{\lambda_n - \mu}\,, \quad n = 1, 2, \cdots.$$

Therefore a formal solution is given by

$$\phi(x) = \sum_{n=0}^{\infty}\frac{c_n}{\lambda_n - \mu}\, J_k\left(\sqrt{\lambda_n}\, x\right).$$

5(a). Setting $\lambda = \alpha^2$ in Prob. 15 of Section 11.1, the *Chebyshev equation* can also be written as

$$-\left[\sqrt{1-x^2}\, y'\right]' = \frac{\lambda}{\sqrt{1-x^2}}\, y\,.$$

Note that

$$p(x) = \sqrt{1-x^2}\ ,\ q(x) = 0\,, \text{and } r(x) = 1/\sqrt{1-x^2}\,,$$

hence both boundary points are singular.

(b). Observe that $p(1-\varepsilon) = \sqrt{2\varepsilon - \varepsilon^2}$ and $p(-1+\varepsilon) = \sqrt{2\varepsilon - \varepsilon^2}$. It follows that if $u(x)$ and $v(x)$ satisfy the boundary conditions (iii), then

$$\lim_{\varepsilon \to 0^+} p(1-\varepsilon)[u'(1-\varepsilon)v(1-\varepsilon) - u(1-\varepsilon)v'(1-\varepsilon)] = 0$$

and

$$\lim_{\varepsilon \to 0^+} p(-1+\varepsilon)[u'(-1+\varepsilon)v(-1+\varepsilon) - u(-1+\varepsilon)v'(-1+\varepsilon)] = 0\,.$$

Therefore Eq. (17) is satisfied and the boundary value problem is *self-adjoint*.

(c). For $n \neq 0$,

$$\begin{aligned} n^2\int_{-1}^{1}\frac{T_0(x)\, T_n(x)}{\sqrt{1-x^2}}dx &= \int_{-1}^{1} T_0(x)\, L[T_n]dx \\ &= \int_{-1}^{1} L[T_0]\, T_n(x)dx \\ &= 0\,, \end{aligned}$$

since $L[T_0] = 0 \cdot T_0 = 0$. Otherwise,

$$n^2\int_{-1}^{1}\frac{T_n(x)\,T_m(x)}{\sqrt{1-x^2}}dx \;= \int_{-1}^{1}L[T_n]\,T_m(x)dx$$
$$= \int_{-1}^{1}T_n(x)L[T_m]dx$$
$$= m^2\int_{-1}^{1}\frac{T_n(x)\,T_m(x)}{\sqrt{1-x^2}}dx\,.$$

Therefore

$$\left(n^2-m^2\right)\int_{-1}^{1}\frac{T_n(x)\,T_m(x)}{\sqrt{1-x^2}}dx=0\,.$$

So for $n \neq m$,

$$\int_{-1}^{1}\frac{T_n(x)\,T_m(x)}{\sqrt{1-x^2}}dx=0\,.$$

Section 11.5

3. The equations relating to this problem are given by Eqs. (2) to (17) in the text. Based on the boundary conditions, the eigenfunctions are $\phi_n(x) = J_0(\lambda_n r)$ and the associated eigenvalues $\lambda_1, \lambda_2, \cdots$ are the positive zeroes of $J_0(\lambda)$. The general solution has the form

$$u(r\,,t) = \sum_{n=1}^{\infty}[c_n J_0(\lambda_n r)\,cos\,\lambda_n at + k_n J_0(\lambda_n r)\,sin\,\lambda_n at]\,.$$

The initial conditions require that

$$u(r\,,0) = \sum_{n=1}^{\infty} c_n J_0(\lambda_n r) = f(r)$$

and

$$u_t(r\,,0) = \sum_{n=1}^{\infty} a\lambda_n k_n J_0(\lambda_n r) = g(r)\,.$$

The coefficients c_n and k_n are obtained from the respective eigenfunction expansions. That is,

$$c_n = \frac{1}{\|J_0(\lambda_n r)\|^2}\int_0^1 rf(r)J_0(\lambda_n r)dr$$

and

$$k_n = \frac{1}{a\lambda_n\,\|J_0(\lambda_n r)\|^2}\int_0^1 rg(r)J_0(\lambda_n r)dr\,,$$

in which

$$\|J_0(\lambda_n r)\|^2 = \int_0^1 r[J_0(\lambda_n r)]^2 dr$$

for $n = 1, 2, \cdots$.

8. A more general equation was considered in Prob. 23 of Section 10.5 . Assuming a solution of the form $u(r\,,t) = R(r)T(t)$, substitution into the PDE results in

$$\alpha^2\left[R''T + \frac{1}{r}\,R'T\right] = RT'\,.$$

Dividing both sides of the equation by the factor RT, we obtain

$$\frac{R''}{R} + \frac{1}{r}\,\frac{R'}{R} = \frac{T'}{\alpha^2 T}\,.$$

Since both sides of the resulting differential equation depend on *different* variables, each side must be equal to a constant, say $-\lambda^2$. That is,

$$\frac{R''}{R} + \frac{1}{r}\,\frac{R'}{R} = \frac{T'}{\alpha^2 T} = -\lambda^2\,.$$

It follows that $T' + \alpha^2\lambda^2 T = 0$, and

$$\frac{R''}{R} + \frac{1}{r}\,\frac{R'}{R} = -\lambda^2\,,$$

which can be written as $r^2R'' + r\,R' + \lambda^2 r^2 R = 0$. Introducing the variable $\xi = \lambda r$, the last equation can be expressed as $\xi^2 R'' + \xi\,R' + \xi^2 R = 0$, which is the Bessel equation of order zero.

The temporal equation has solutions which are multiples of $T(t) = exp(-\alpha^2\lambda^2 t)$. The general solution of the Bessel equation is

$$R(r) = b_1 J_0(\lambda_n r) + b_2 Y_0(\lambda_n r)\,.$$

Since the steady state temperature will be *zero*, all solutions must be bounded, and hence we set $b_2 = 0$. Furthermore, the boundary condition $u(1\,,t) = 0$ requires that $R(1) = 0$ and hence $J_0(\lambda) = 0$. It follows that the eigenfunctions are $\phi_n(x) = J_0(\lambda_n r)$, with the associated eigenvalues $\lambda_1, \lambda_2, \cdots$, which are the positive zeroes of $J_0(\lambda)$. Therefore the fundamental solutions of the PDE are $u_n(r\,,t) = J_0(\lambda_n r) exp(-\alpha^2\lambda_n^2 t)$, and the general solution has the form

$$u(r\,,t) = \sum_{n=1}^{\infty} c_n J_0(\lambda_n r) exp\big(-\alpha^2\lambda_n^2 t\big)\,.$$

The initial condition requires that

$$u(r\,,0) = \sum_{n=1}^{\infty} c_n J_0(\lambda_n r) = f(r).$$

The coefficients in the general solution are obtained from the eigenfunction expansion of $f(r)$. That is,

$$c_n = \frac{1}{\|J_0(\lambda_n r)\|^2}\int_0^1 r f(r) J_0(\lambda_n r)dr\,,$$

in which

$$\|J_0(\lambda_n r)\|^2 = \int_0^1 r[J_0(\lambda_n r)]^2 dr \quad (n = 1, 2, \cdots).$$

Section 11.6

1. The *sine expansion* of $f(x) = 1$, on $0 < x < 1$, is given by

$$f(x) = 2\sum_{m=1}^{\infty} \frac{1 - cos\, m\pi}{m\pi}\, sin\, m\pi x\,,$$

with partial sums

$$S_n(x) = 2\sum_{m=1}^{n} \frac{1 - cos\, m\pi}{m\pi}\, sin\, m\pi x\,.$$

The *mean square error* in this problem is

$$R_n = \int_0^1 |1 - S_n(x)|^2 dx\,.$$

Several values are shown in the Table :

n	5	10	15	20
R_n	0.067	0.04	0.026	0.02

Further numerical calculation shows that $R_n < 0.02$ for $n \geq 21$.

3(a). The *sine expansion* of $f(x) = x(1-x)$, on $0 < x < 1$, is given by

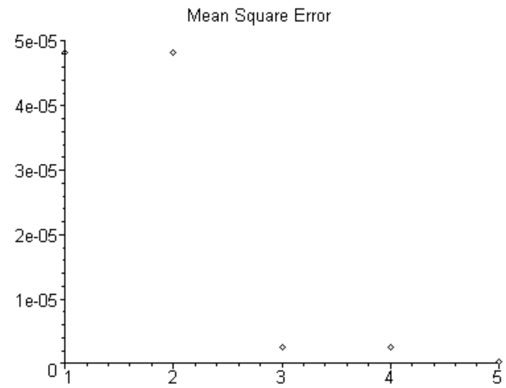
$$f(x) = 2\sum_{m=1}^{\infty} \frac{1 - cos\, m\pi}{m\pi}\, sin\, m\pi x\,,$$

with partial sums

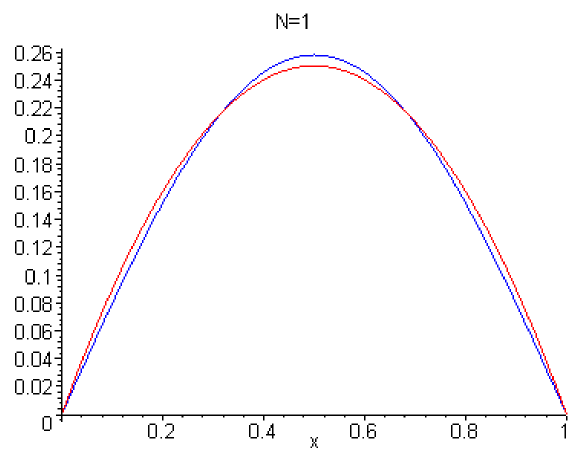
$$S_n(x) = 4\sum_{m=1}^{n} \frac{1 - cos\, m\pi}{m^3\pi^3}\, sin\, m\pi x\,.$$

(b, c). The *mean square error* in this problem is

$$R_n = \int_0^1 |\, x(1-x) - S_n(x)|^2 dx\,.$$



We find that $R_1 = 0.000048$. The graphs of $f(x)$ and $S_1(x)$ are plotted below :



$6(a)$. The function is bounded on intervals not containing $x = 0$, so for $\varepsilon > 0$,

$$\int_\varepsilon^1 f(x)dx = \int_\varepsilon^1 x^{-1/2}\,dx = 2 - 2\sqrt{\varepsilon}\,.$$

Hence the improper integral is evaluated as

$$\int_0^1 f(x)dx = \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 x^{-1/2}\,dx = 2\,.$$

On the other hand, $f^2(x) = 1/x$ for $x \neq 0$, and

$$\int_\varepsilon^1 f^2(x)dx = \int_\varepsilon^1 x^{-1}dx = -\,ln\sqrt{\varepsilon}\,.$$

Therefore the improper integral does not exist.

(b). Since $f^2(x) \equiv 1$, it is evident that the *Riemann integral* of $f^2(x)$ exists. Let

$$P_N = \{0 = x_1\,, x_2, \cdots, x_{N+1} = 1\}$$

be a *partition* of $[0\,,1]$ into equal subintervals. We can always choose a *rational* point, ξ_i , in each of the subintervals so that the Riemann sum

$$R(\xi_1, \xi_2, \cdots, \xi_N) = \sum_{n=1}^{N} f(\xi_n)\frac{1}{N} = 1\,.$$

Likewise, can always choose an *irrational* point, η_i , in each of the subintervals so that the Riemann sum

$$R(\eta_1, \eta_2, \cdots, \eta_N) = \sum_{n=1}^{N} f(\eta_n)\frac{1}{N} = \,-\,1\,.$$

It follows that $f(x)$ is *not* Riemann integrable.

8. With $P_0(x) = 1$ and $P_1(x) = x$, the normalization conditions are satisfied. Using the usual inner product on $[\,-\,1\,,1]$,

$$\int_{-1}^{1} P_0(x)P_1(x)dx = 0$$

and hence the polynomials are also orthogonal. Let $P_2(x) = a_2x^2 + a_1x + a_0$. The normalization condition requires that $a_2 + a_1 + a_0 = 1$. For orthogonality, we need

$$\int_{-1}^{1} \left(a_2x^2 + a_1x + a_0\right)dx = 0 \text{ and } \int_{-1}^{1} x\left(a_2x^2 + a_1x + a_0\right)dx = 0\,.$$

It follows that $a_2 = 3/2\,, a_1 = 0$ and $a_0 = \,-\,1/2$. Hence $P_2(x) = (3x^2 - 1)/2$. Now let $P_3(x) = a_3x^3 + a_2x^2 + a_1x + a_0$. The coefficients must be chosen so that $a_3 + a_2 + a_1 + a_0 = 1$ and the orthogonality conditions

$$\int_{-1}^{1} P_i(x)P_j(x)dx = 0 \quad (i \neq j)$$

are satisfied. Solution of the resulting algebraic equations leads to $a_3 = 5/2\,, a_2 = 0\,,$ $a_1 = \,-\,3/2$ and $a_0 = 0$. Therefore $P_3(x) = (5x^3 - 3x)/2$.

11. The implied sequence of coefficients is $a_n = 1\,, n \geq 1$. Since the limit of these coefficients is *not* zero, the series cannot be an eigenfunction expansion.

13. Consider the eigenfunction expansion

$$f(x) = \sum_{i=1}^{\infty} a_i\phi_i(x).$$

Formally,

$$f^2(x) = \sum_{i=1}^{\infty} a_i^2 \phi_i^2(x) + 2\sum_{i\neq j} a_i a_j \, \phi_i(x)\phi_j(x) \, .$$

Integrating term-by-term,

$$\begin{aligned}\int_0^1 r(x) f^2(x) dx \, &= \, \sum_{i=1}^{\infty} \int_0^1 a_i^2 r(x) \phi_i^2(x) dx + 2\sum_{i\neq j} \int_0^1 a_i a_j \, r(x)\phi_i(x)\phi_j(x) dx \\ &= \sum_{i=1}^{\infty} a_i^2 \int_0^1 \phi_i^2(x) dx \, ,\end{aligned}$$

since the eigenfunctions are orthogonal. Assuming that they are also normalized,

$$\int_0^1 r(x) f^2(x) dx = \sum_{i=1}^{\infty} a_i^2 \, .$$