Chapter Ten

Section 10.1

1. The general solution of the ODE is $y(x) = c_1 \cos x + c_2 \sin x$. Imposing the first boundary condition, it is necessary that $c_1 = 0$. Therefore $y(x) = c_2 \sin x$. Taking its derivative, $y'(x) = c_2 \cos x$. Imposing the second boundary condition, we require that $c_2 \cos \pi = 1$. The latter equation is satisfied only if $c_2 = -1$. Hence the solution of the boundary value problem is $y(x) = -\sin x$.

4. The general solution of the differential equation is $y(x) = c_1 \cos x + c_2 \sin x$. It follows that $y'(x) = -c_1 \sin x + c_2 \cos x$. Imposing the first boundary condition, we find that $c_2 = 1$. Therefore $y(x) = c_1 \cos x + \sin x$. Imposing the second boundary condition, we require that $c_1 \cos L + \sin L = 0$. If $\cos L \neq 0$, that is, as long as $L \neq (2k-1)\pi/2$, with k an integer, then $c_1 = -\tan L$. The solution of the boundary value problem is

$$y(x) = -\tan L \cos x + \sin x .$$

If $\cos L = 0$, the boundary condition results in $\sin L = 0$. The latter two equations are inconsistent, which implies that the BVP has no solution.

5. The general solution of the *homogeneous* differential equation is

$$y(x) = c_1 \cos x + c_2 \sin x .$$

Using any of a number of methods, including the *method of undetermined coefficients*, it is easy to show that a *particular solution* is $Y(x) = x$. Hence the general solution of the given differential equation is $y(x) = c_1 \cos x + c_2 \sin x + x$. The first boundary condition requires that $c_1 = 0$. Imposing the second boundary condition, it is necessary that $c_2 \sin \pi + \pi = 0$. The resulting equation has *no solution*. We conclude that the boundary value problem has no solution.

6. Using the *method of undetermined coefficients*, it is easy to show that the general solution of the ODE is $y(x) = c_1 \cos\sqrt{2}\,x + c_2 \sin\sqrt{2}\,x + x/2$. Imposing the first boundary condition, we find that $c_1 = 0$. The second boundary condition requires that $c_2 \sin\sqrt{2}\,\pi + \pi/2 = 0$. It follows that $c_2 = -\pi/2\sin\sqrt{2}\,\pi$. Hence the solution of the boundary value problem is

$$y(x) = -\frac{\pi}{2\sin\sqrt{2}\,\pi}\sin\sqrt{2}\,x + \frac{x}{2} .$$

8. The general solution of the *homogeneous* differential equation is

$$y(x) = c_1 \cos 2x + c_2 \sin 2x .$$

Using the method of undetermined coefficients, a *particular solution* is $Y(x) = \sin x/3$.

Hence the general solution of the given differential equation is

$$y(x) = c_1 \, cos \, 2x + c_2 \, sin \, 2x + \frac{1}{3} sin \, x \, .$$

The first boundary condition requires that $c_1 = 0$. The second boundary requires that $c_2 \, sin \, 2\pi + \frac{1}{3} sin \, \pi = 0$. The latter equation is valid for *all* values of c_2. Therefore the solution of the boundary value problem is

$$y(x) = c_2 \, sin \, 2x + \frac{1}{3} sin \, x \, .$$

9. Using the *method of undetermined coefficients*, it is easy to show that the general solution of the ODE is $y(x) = c_1 \, cos \, 2x + c_2 \, sin \, 2x + cos \, x/3$. It follows that $y'(x) = -2c_1 \, sin \, 2x + 2c_2 \, cos \, 2x - sin \, x/3$. Imposing the first boundary condition, we find that $c_2 = 0$. The second boundary condition requires that

$$-2c_1 \, sin \, 2\pi - \frac{1}{3} sin \, \pi = 0 \, .$$

The resulting equation is satisfied for all values of c_1. Hence the solution is the family of functions

$$y(x) = c_1 \, cos \, 2x + \frac{1}{3} cos \, x \, .$$

10. The general solution of the differential equation is

$$y(x) = c_1 \, cos \sqrt{3} \, x + c_2 \, sin \sqrt{3} \, x + \frac{1}{2} cos \, x \, .$$

Its derivative is $y'(x) = -\sqrt{3} \, c_1 \, sin \sqrt{3} \, x + \sqrt{3} \, c_2 \, cos \sqrt{3} \, x - sin \, x/2$. The first boundary condition requires that $c_2 = 0$. Imposing the second boundary condition, we obtain $-\sqrt{3} \, c_1 \, sin \sqrt{3} \, \pi = 0$. It follows that $c_1 = 0$. Hence the solution of the BVP is $y(x) = cos \, x/2$.

12. Assuming that $\lambda > 0$, we can set $\lambda = \mu^2$. The general solution of the differential equation is

$$y(x) = c_1 \, cos \, \mu x + c_2 \, sin \, \mu x \, ,$$

so that $y'(x) = -\mu c_1 \, sin \, \mu x + \mu c_2 \, cos \, \mu x$. Imposing the first boundary condition, it follows that $c_2 = 0$. Therefore $y(x) = c_1 \, cos \, \mu x$. The second boundary condition requires that $c_1 \, cos \, \mu \pi = 0$. For a nontrivial solution, it is necessary that $cos \, \mu \pi = 0$, that is, $\mu \pi = (2n-1)\pi/2$, with $n = 1, 2, \cdots$. Therefore the *eigenvalues* are

$$\lambda_n = \frac{(2n-1)^2}{4} \, , \; n = 1, 2, \cdots .$$

The corresponding *eigenfunctions* are given by

$$y_n = cos\,\frac{(2n-1)x}{2}\,,\; n = 1, 2, \cdots .$$

Assuming that $\lambda < 0$, we can set $\lambda = -\mu^2$. The general solution of the differential equation is

$$y(x) = c_1\, cosh\, \mu x + c_2\, sinh\, \mu x\,,$$

so that $y'(x) = \mu c_1\, sinh\, \mu x + \mu c_2\, cosh\, \mu x$. Imposing the first boundary condition, it follows that $c_2 = 0$. Therefore $y(x) = c_1\, cosh\, \mu x$. The second boundary condition requires that $c_1\, cosh\, \mu\pi = 0$, which results in $c_1 = 0$. Hence the only solution is the trivial solution. Finally, with $\lambda = 0$, the general solution of the ODE is

$$y(x) = c_1 x + c_2\,.$$

It is easy to show that the boundary conditions require that $c_1 = c_2 = 0$. Therefore all of the eigenvalues are *positive*.

13. Assuming that $\lambda > 0$, we can set $\lambda = \mu^2$. The general solution of the differential equation is

$$y(x) = c_1\, cos\, \mu x + c_2\, sin\, \mu x\,,$$

so that $y'(x) = -\mu c_1\, sin\, \mu x + \mu c_2\, cos\, \mu x$. Imposing the first boundary condition, it follows that $c_2 = 0$. The second boundary condition requires that $c_1\, sin\, \mu\pi = 0$. For a nontrivial solution, we must have $\mu\pi = n\pi$, $n = 1, 2, \cdots$. It follows that the *eigenvalues* are

$$\lambda_n = n^2,\; n = 1, 2, \cdots,$$

and the corresponding *eigenfunctions* are

$$y_n = cos\, nx\,,\; n = 1, 2, \cdots .$$

Assuming that $\lambda < 0$, we can set $\lambda = -\mu^2$. The general solution of the differential equation is

$$y(x) = c_1\, cosh\, \mu x + c_2\, sinh\, \mu x\,,$$

so that $y'(x) = \mu c_1\, sinh\, \mu x + \mu c_2\, cosh\, \mu x$. Imposing the first boundary condition, it follows that $c_2 = 0$. The second boundary condition requires that $c_1\, sinh\, \mu\pi = 0$. The latter equation is satisfied only for $c_1 = 0$.

Finally, for $\lambda = 0$, the solution is $y(x) = c_1 x + c_2$. Imposing the boundary conditions, we find that $y(x) = c_2$. Therefore $\lambda = 0$ is *also* an eigenvalue, with corresponding eigenfunction $y_0(x) = 1$.

14. It can be shown, as in Prob. 12 , that $\lambda > 0$. Setting $\lambda = \mu^2$, the general solution of the resulting ODE is

$$y(x) = c_1 \cos \mu x + c_2 \sin \mu x \,,$$

with $y'(x) = -\mu c_1 \sin \mu x + \mu c_2 \cos \mu x$. Imposing the first boundary condition, we find that $c_2 = 0$. Therefore $y(x) = c_1 \cos \mu x$. The second boundary condition requires that $c_1 \cos \mu L = 0$. For a nontrivial solution, it is necessary that $\cos \mu L = 0$, that is, $\mu = (2n-1)\pi/(2L)$, with $n = 1, 2, \cdots$. Therefore the *eigenvalues* are

$$\lambda_n = \frac{(2n-1)^2 \pi^2}{4L^2} \,,\; n = 1, 2, \cdots .$$

The corresponding *eigenfunctions* are given by

$$y_n = \cos \frac{(2n-1)\pi x}{2L} \,,\; n = 1, 2, \cdots .$$

16. Assuming that $\lambda > 0$, we can set $\lambda = \mu^2$. The general solution of the differential equation is

$$y(x) = c_1 \cosh \mu x + c_2 \sinh \mu x \,.$$

The first boundary condition requires that $c_1 = 0$. Therefore $y(x) = c_2 \sinh \mu x$ and $y'(x) = c_2 \cosh \mu x$. Imposing the second boundary condition, it is necessary that $c_2 \cosh \mu L = 0$. The latter equation is valid only for $c_2 = 0$. The only solution is the trivial solution.

Assuming that $\lambda > 0$, we set $\lambda = -\mu^2$. The general solution of the resulting ODE is

$$y(x) = c_1 \cos \mu x + c_2 \sin \mu x \,.$$

Imposing the first boundary condition, we find that $c_1 = 0$. Hence $y(x) = c_2 \sin \mu x$ and $y'(x) = c_2 \cos \mu x$. In order to satisfy the second boundary condition, it is necessary that $c_2 \cos \mu L = 0$. For a nontrivial solution, $\mu = (2n-1)\pi/(2L)$, with $n = 1, 2, \cdots$. Therefore the *eigenvalues* are

$$\lambda_n = -\frac{(2n-1)^2 \pi^2}{4L^2} \,,\; n = 1, 2, \cdots .$$

The corresponding *eigenfunctions* are given by

$$y_n = \sin \frac{(2n-1)\pi x}{2L} \,,\; n = 1, 2, \cdots .$$

Finally, for $\lambda = 0$, the general solution is *linear*. Based on the boundary conditions, it follows that $y(x) = 0$. Therefore all of the eigenvalues are negative.

$17(a)$. Setting $\lambda = \mu^2$, write the general solution of the ODE $y'' + \mu^2 y = 0$ as

$$y(x) = k_1 e^{i\mu x} + k_2 e^{-i\mu x}.$$

Imposing the boundary conditions $y(0) = y(\pi) = 0$, we obtain the system of equations

$$k_1 + k_2 = 0$$
$$k_1 e^{i\mu\pi} + k_2 e^{-i\mu\pi} = 0\,.$$

The system has a *nontrivial* solution if and only if the coefficient matrix is *singular*. Set the determinant equal to zero to obtain

$$e^{-i\mu\pi} - e^{i\mu\pi} = 0\,.$$

(b). Let $\mu = \nu + i\sigma$. Then $i\mu\pi = i\nu\pi - \sigma\pi$, and the previous equation can be written as

$$e^{\sigma\pi}e^{-i\nu\pi} - e^{-\sigma\pi}e^{i\nu\pi} = 0\,.$$

Using Euler's relation, $e^{i\nu\pi} = cos\,\nu\pi + i\,sin\,\nu\pi$, we obtain

$$e^{\sigma\pi}(cos\,\nu - i\,sin\,\nu) - e^{-\sigma\pi}(cos\,\nu + i\,sin\,\nu) = 0\,.$$

Equating the real and imaginary parts of the equation,

$$(e^{\sigma\pi} - e^{-\sigma\pi})cos\,\nu\pi = 0$$
$$(e^{\sigma\pi} + e^{-\sigma\pi})sin\,\nu\pi = 0\,.$$

(c). Based on the second equation, $\nu = n\,,\ n \in \mathbb{I}$. Since $cos\,n\pi \neq 0$, it follows that $e^{\sigma\pi} = e^{-\sigma\pi}$, or $e^{2\sigma\pi} = 1$. Hence $\sigma = 0$, and $\mu = n\,, n \in \mathbb{I}$.

Section 10.2

1. The period of the function $sin\, \alpha x$ is $T = 2\pi/\alpha$. Therefore the function $sin\, 5x$ has period $T = 2\pi/5$.

2. The period of the function $cos\, \alpha x$ is also $T = 2\pi/\alpha$. Therefore the function cos
$2\pi x$
has period $T = 2\pi/2\pi = 1$.

4. Based on Prob. 1, the period of the function $sin\, \pi x/L$ is $T = 2\pi/(\pi/L) = 2L$.

6. Let $T > 0$ and consider the equation $(x+T)^2 = x^2$. It follows that $2Tx + T^2 = 0$ and $2x + T = 0$. Since the latter equation is *not* an identity, the function x^2 cannot be periodic with finite period.

8. The function is defined on intervals of length $(2n+1) - (2n-1) = 2$. On any two *consecutive* intervals, $f(x)$ is identically equal to 1 on one of the intervals and alternates between 1 and -1 on the other. It follows that the period is $T = 4$.

9. On the interval $L < x < 2L$, a simple *shift to the right* results in

$$f(x) = -(x - 2L) = 2L - x\,.$$

On the interval $-3L < x < -2L$, a simple *shift to the left* results in

$$f(x) = -(x + 2L) = -2L - x\,.$$

11. The next fundamental period *to the left* is on the interval $-2L < x < 0$. Hence the interval $-L < x < 0$ is the second half of a fundamental period. A simple *shift to the left* results in

$$f(x) = L - (x + 2L) = -L - x\,.$$

12. First note that

$$cos\frac{m\pi x}{L} cos\frac{n\pi x}{L} = \frac{1}{2}\left[cos\frac{(m-n)\pi x}{L} + cos\frac{(m+n)\pi x}{L}\right]$$

and

$$cos\frac{m\pi x}{L} sin\frac{n\pi x}{L} = \frac{1}{2}\left[sin\frac{(n-m)\pi x}{L} + sin\frac{(m+n)\pi x}{L}\right].$$

It follows that

$$\begin{aligned}\int_{-L}^{L} cos\frac{m\pi x}{L} cos\frac{n\pi x}{L} dx &= \frac{1}{2}\int_{-L}^{L}\left[cos\frac{(m-n)\pi x}{L} + cos\frac{(m+n)\pi x}{L}\right]dx \\ &= \frac{1}{2}\frac{L}{\pi}\left\{\frac{sin[(m-n)\pi x/L]}{m-n} + \frac{sin[(m+n)\pi x/L]}{m+n}\right\}\Bigg|_{-L}^{L} \\ &= 0\,,\end{aligned}$$

as long as $m+n$ and $m-n$ are not zero. For the case $m = n$,

$$\begin{aligned}\int_{-L}^{L}\left(cos\frac{n\pi x}{L}\right)^2 dx &= \frac{1}{2}\int_{-L}^{L}\left[1 + cos\frac{2n\pi x}{L}\right]dx \\ &= \frac{1}{2}\left\{x + \frac{sin(2n\pi x/L)}{2n\pi/L}\right\}\Bigg|_{-L}^{L} \\ &= L\,.\end{aligned}$$

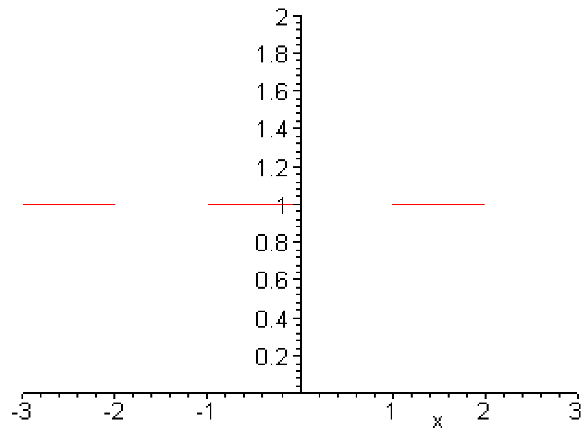
Likewise,

$$\begin{aligned}\int_{-L}^{L} cos\frac{m\pi x}{L} sin\frac{n\pi x}{L} dx &= \frac{1}{2}\int_{-L}^{L}\left[sin\frac{(n-m)\pi x}{L} + sin\frac{(m+n)\pi x}{L}\right]dx \\ &= \frac{1}{2}\frac{L}{\pi}\left\{\frac{cos[(n-m)\pi x/L]}{m-n} - \frac{cos[(m+n)\pi x/L]}{m+n}\right\}\Bigg|_{-L}^{L} \\ &= 0\,,\end{aligned}$$

as long as $m+n$ and $m-n$ are not zero. For the case $m = n$,

$$\begin{aligned}\int_{-L}^{L} cos\frac{m\pi x}{L} sin\frac{n\pi x}{L} dx &= \frac{1}{2}\int_{-L}^{L} sin\frac{2n\pi x}{L} dx \\ &= -\frac{1}{2}\left\{\frac{cos(2n\pi x/L)}{2n\pi/L}\right\}\Bigg|_{-L}^{L} \\ &= 0\,.\end{aligned}$$

$14(a)$. For $L = 1$,



(b). The Fourier coefficients are calculated using the *Euler-Fourier* formulas:

$$\begin{aligned} a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)dx \\ &= \frac{1}{L}\int_{-L}^{0} dx \\ &= 1\,. \end{aligned}$$

For $n > 0$,

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)cos\frac{n\pi x}{L}dx \\ &= \frac{1}{L}\int_{-L}^{0} cos\frac{n\pi x}{L}dx \\ &= 0\,. \end{aligned}$$

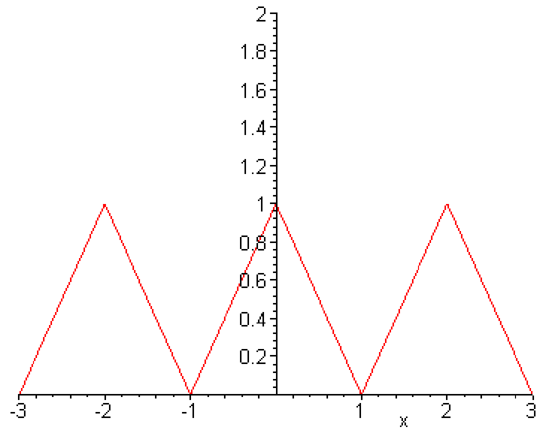
Likewise,

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(x)sin\frac{n\pi x}{L}dx \\ &= \frac{1}{L}\int_{-L}^{0} sin\frac{n\pi x}{L}dx \\ &= \frac{-1+(-1)^n}{n\pi}\,. \end{aligned}$$

It follows that $b_{2k} = 0$ and $b_{2k-1} = -2/[(2k-1)\pi]$, $k = 1, 2, 3, \cdots$. Therefore the Fourier series for the given function is

$$f(x) = \frac{1}{2} - \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{2k-1}sin\frac{(2k-1)\pi x}{L}\,.$$

16(a).



(b). The Fourier coefficients are calculated using the *Euler-Fourier* formulas:

$$\begin{aligned} a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)dx \\ &= \int_{-1}^{0}(x+1)dx + \int_{0}^{1}(1-x)dx \\ &= 1\,. \end{aligned}$$

For $n > 0$,

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)cos\frac{n\pi x}{L}dx \\ &= \int_{-1}^{0}(x+1)cos\, n\pi x\, dx + \int_{0}^{1}(1-x)cos\, n\pi x\, dx \\ &= -2\frac{-1+(-1)^n}{n^2\pi^2}\,. \end{aligned}$$

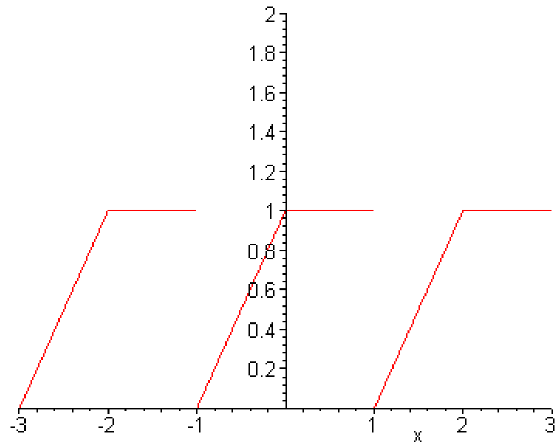
It follows that $a_{2k} = 0$ and $a_{2k-1} = 4/\left[(2k-1)^2\pi^2\right],\ k = 1, 2, 3, \cdots$. Likewise,

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(x)sin\frac{n\pi x}{L}dx \\ &= \int_{-1}^{0}(x+1)sin\, n\pi x\, dx + \int_{0}^{1}(1-x)sin\, n\pi x\, dx \\ &= 0\,. \end{aligned}$$

Therefore the Fourier series for the given function is

$$f(x) = \frac{1}{2} + \frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\,cos(2k-1)\pi x\,.$$

$17(a)$. For $L = 1$,



(b). The Fourier coefficients are calculated using the *Euler-Fourier* formulas:

$$\begin{aligned} a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)dx \\ &= \frac{1}{L}\int_{-L}^{0}(x+L)dx + \frac{1}{L}\int_{0}^{L} L\,dx \\ &= 3L/2\,. \end{aligned}$$

For $n > 0$,

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)cos\frac{n\pi x}{L}dx \\ &= \frac{1}{L}\int_{-L}^{0}(x+L)cos\frac{n\pi x}{L}dx + \frac{1}{L}\int_{0}^{L} L\,cos\frac{n\pi x}{L}dx \\ &= \frac{L(1-cos\,n\pi)}{n^2\pi^2}\,. \end{aligned}$$

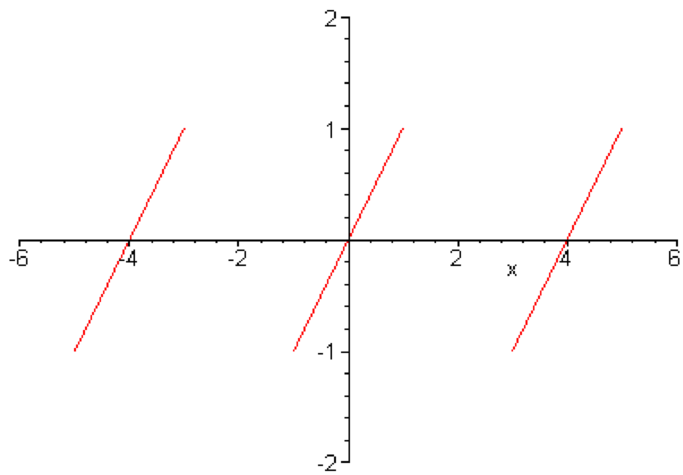
Likewise,

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(x)sin\frac{n\pi x}{L}dx \\ &= \frac{1}{L}\int_{-L}^{0}(x+L)sin\frac{n\pi x}{L}dx + \frac{1}{L}\int_{0}^{L} L\,sin\frac{n\pi x}{L}dx \\ &= -\frac{L\,cos\,n\pi}{n\pi}\,. \end{aligned}$$

Note that $cos\,n\pi = (-1)^n$. It follows that the Fourier series for the given function is

$$f(x) = \frac{3L}{4} + \frac{L}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{2}{(2n-1)^2}cos\frac{(2n-1)\pi x}{L} - \frac{(-1)^n\pi}{n}sin\frac{n\pi x}{L}\right].$$

$18(a)$.



(b). The Fourier coefficients are calculated using the *Euler-Fourier* formulas:

$$\begin{aligned} a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)dx \\ &= \frac{1}{2}\int_{-1}^{1} x\,dx \\ &= 0\,. \end{aligned}$$

For $n > 0$,

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)cos\frac{n\pi x}{L}dx \\ &= \frac{1}{2}\int_{-1}^{1} x\,cos\frac{n\pi x}{L}dx \\ &= 0\,. \end{aligned}$$

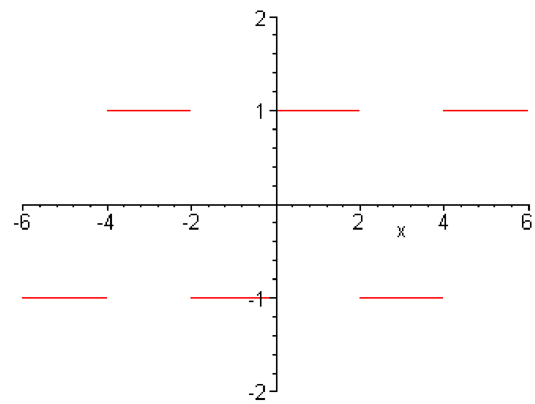
Likewise,

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(x)sin\frac{n\pi x}{L}dx \\ &= \frac{1}{2}\int_{-1}^{1} x\,sin\frac{n\pi x}{L}dx \\ &= \frac{2}{n^2\pi^2}\left(2\,sin\frac{n\pi}{2} - n\pi\,cos\frac{n\pi}{2}\right). \end{aligned}$$

Therefore the Fourier series for the given function is

$$f(x) = \sum_{n=1}^{\infty} \left[\frac{4}{n^2\pi^2} \, sin\frac{n\pi}{2} - \frac{2}{n\pi} \, cos\frac{n\pi}{2} \right] sin\frac{n\pi x}{2} \, .$$

$19(a)$.



(b). The Fourier *cosine* coefficients are given by

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x) cos\frac{n\pi x}{L} dx \\ &= \frac{1}{2}\int_{-2}^{0} - cos\frac{n\pi x}{2} dx + \frac{1}{2}\int_{0}^{2} cos\frac{n\pi x}{2} dx \\ &= 0\,. \end{aligned}$$

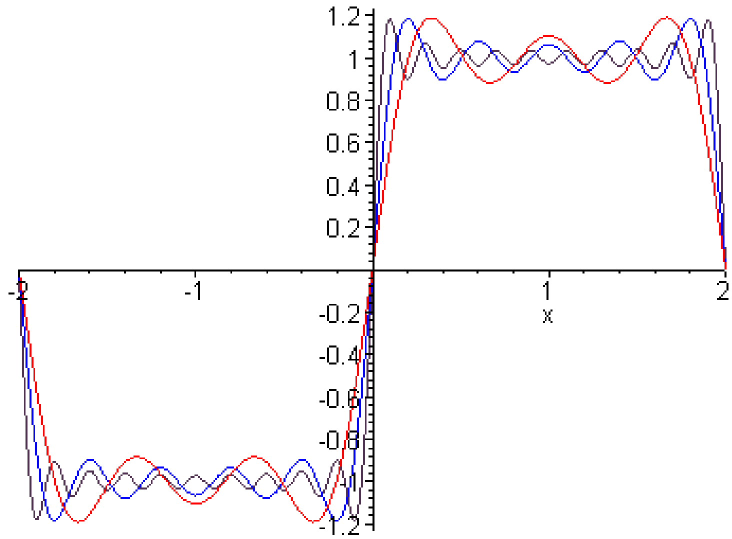
The Fourier *sine* coefficients are given by

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(x) sin\frac{n\pi x}{L} dx \\ &= \frac{1}{2}\int_{-2}^{0} - sin\frac{n\pi x}{2} dx + \frac{1}{2}\int_{0}^{2} sin\frac{n\pi x}{2} dx \\ &= 2\frac{1 - cos\, n\pi}{n\pi}\,. \end{aligned}$$

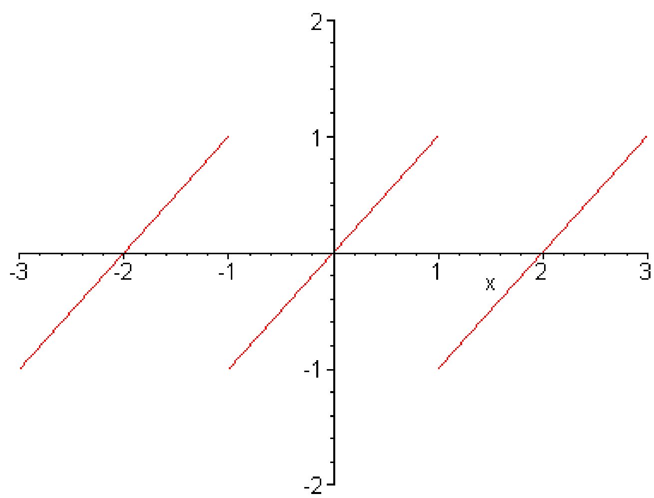
Therefore the Fourier series for the given function is

$$f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} sin\frac{(2n-1)\pi x}{2} \, .$$

(c).



$20(a)$.



(b). The Fourier *cosine* coefficients are given by

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x) cos \frac{n\pi x}{L} dx \\ &= \int_{-1}^{1} x \cos n\pi x dx \\ &= 0 \,. \end{aligned}$$

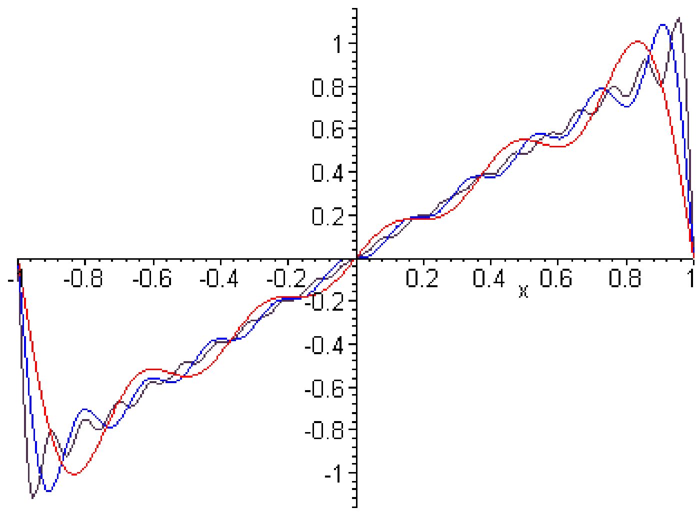
The Fourier *sine* coefficients are given by

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(x) sin\frac{n\pi x}{L} dx \\ &= \int_{-1}^{1} x\, sin\, n\pi x dx \\ &= -2\frac{cos\, n\pi}{n\pi}\,. \end{aligned}$$

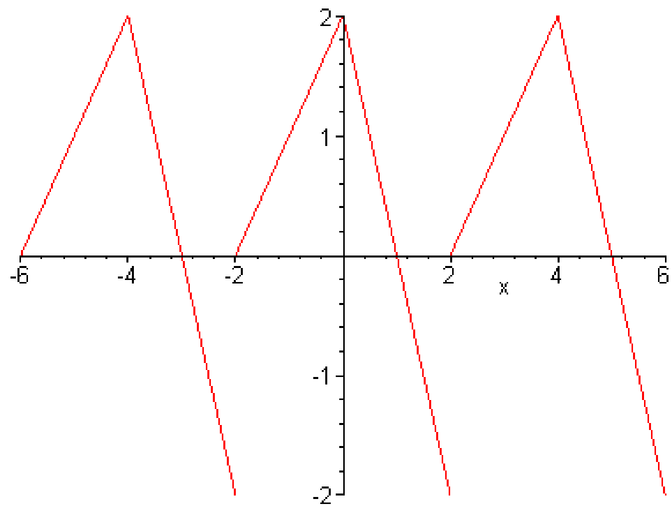
Therefore the Fourier series for the given function is

$$f(x) = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n} sin\, n\pi x\,.$$

(c).



$22(a)$.



(b). The Fourier *cosine* coefficients are given by

$$\begin{aligned} a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)dx \\ &= \frac{1}{2}\int_{-2}^{0}(x+2)dx + \frac{1}{2}\int_{0}^{2}(2-2x)dx \\ &= 1\,, \end{aligned}$$

and for $n > 0$,

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)cos\frac{n\pi x}{L}dx \\ &= \frac{1}{2}\int_{-2}^{0}(x+2)cos\frac{n\pi x}{2}dx + \frac{1}{2}\int_{0}^{2}(2-2x)cos\,\frac{n\pi x}{2}dx \\ &= 6\frac{(1-cos\,n\pi)}{n^2\pi^2}\,. \end{aligned}$$

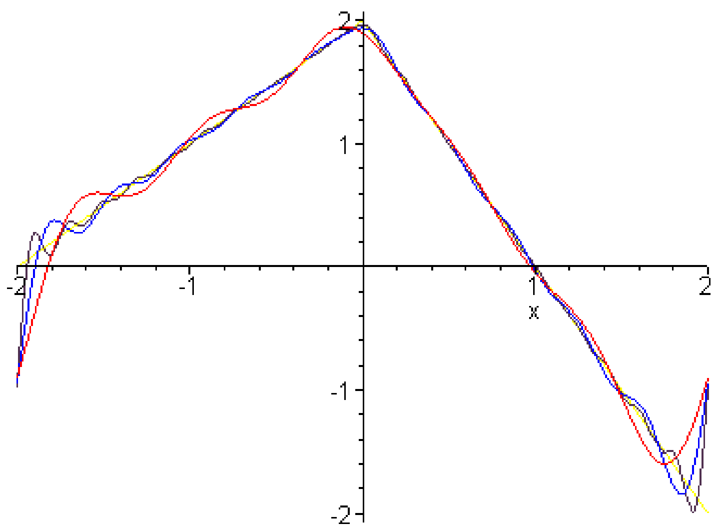
The Fourier *sine* coefficients are given by

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(x)sin\frac{n\pi x}{L}dx \\ &= \frac{1}{2}\int_{-2}^{0}(x+2)sin\frac{n\pi x}{2}dx + \frac{1}{2}\int_{0}^{2}(2-2x)sin\,\frac{n\pi x}{2}dx \\ &= 2\frac{cos\,n\pi}{n\pi}\,. \end{aligned}$$

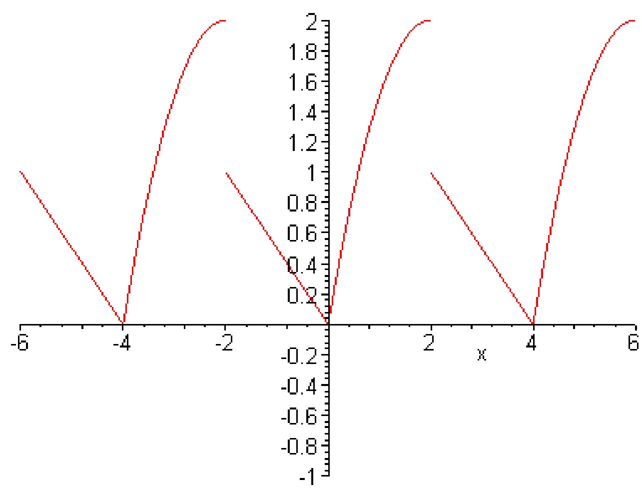
Therefore the Fourier series for the given function is

$$f(x) = \frac{1}{2} + \frac{12}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\,cos\frac{(2n-1)\pi x}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}sin\frac{n\pi x}{2}\,.$$

(c).



$23(a)$.



(b). The Fourier *cosine* coefficients are given by

$$\begin{aligned} a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)dx \\ &= \frac{1}{2}\int_{-2}^{0}\Big(-\frac{x}{2}\Big)dx + \frac{1}{2}\int_{0}^{2}\left(2x - \frac{1}{2}x^2\right)dx \\ &= 11/6\,, \end{aligned}$$

and for $n > 0$,

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)cos\frac{n\pi x}{L}dx \\ &= \frac{1}{2}\int_{-2}^{0}\Big(-\frac{x}{2}\Big)cos\frac{n\pi x}{2}dx + \frac{1}{2}\int_{0}^{2}\left(2x - \frac{1}{2}x^2\right)cos\,\frac{n\pi x}{2}dx \\ &= -\,\frac{(5 - cos\,n\pi)}{n^2\pi^2}\,. \end{aligned}$$

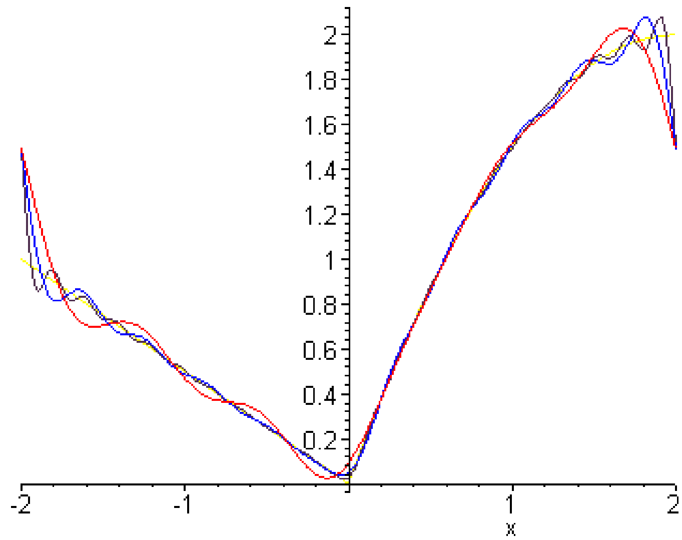
The Fourier *sine* coefficients are given by

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(x)sin\frac{n\pi x}{L}dx \\ &= \frac{1}{2}\int_{-2}^{0}\Big(-\frac{x}{2}\Big)sin\frac{n\pi x}{2}dx + \frac{1}{2}\int_{0}^{2}\left(2x - \frac{1}{2}x^2\right)sin\,\frac{n\pi x}{2}dx \\ &= \frac{4 - (4 + n^2\pi^2)cos\,n\pi}{n^3\pi^3}\,. \end{aligned}$$

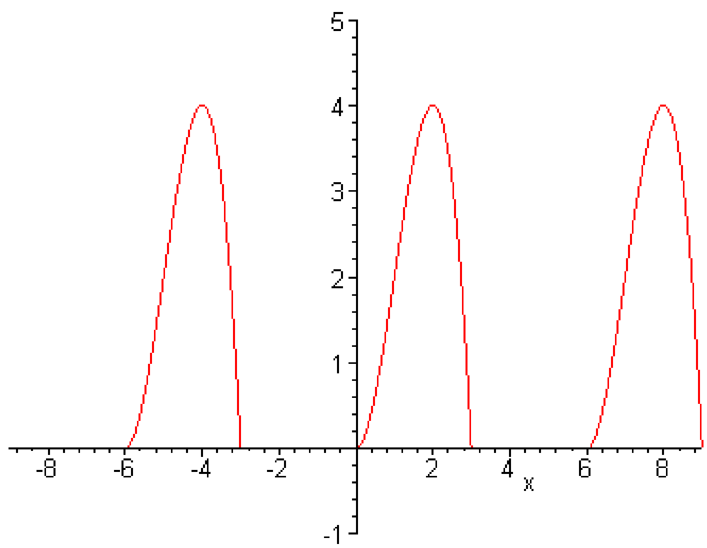
Therefore the Fourier series for the given function is

$$\begin{aligned} f(x) &= \frac{11}{12} + \frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{[(-1)^n - 5]}{n^2}\,cos\frac{n\pi x}{2}\; + \\ &\quad + \frac{1}{\pi^3}\sum_{n=1}^{\infty}\frac{[4 - (4 + n^2\pi^2)(-1)^n]}{n^3}sin\frac{n\pi x}{2}\,. \end{aligned}$$

(c).



24(a).



(b). The Fourier *cosine* coefficients are given by

$$\begin{aligned} a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)dx \\ &= \frac{1}{3}\int_0^3 x^2(3-x)dx \\ &= 9/4\,, \end{aligned}$$

and for $n > 0$,

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x) cos \frac{n\pi x}{L} dx \\ &= \frac{1}{3}\int_{0}^{3} x^2(3-x) cos\,\frac{n\pi x}{3} dx \\ &= -27\frac{(6-6\,cos\,n\pi + n^2\pi^2 cos\,n\pi)}{n^4\pi^4}\,. \end{aligned}$$

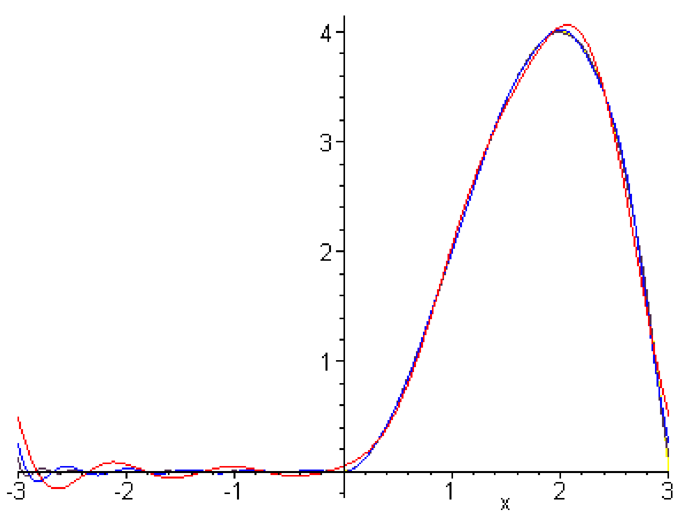
The Fourier *sine* coefficients are given by

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(x) sin \frac{n\pi x}{L} dx \\ &= \frac{1}{3}\int_{0}^{3} x^2(3-x) sin\,\frac{n\pi x}{3} dx \\ &= -54\frac{1+2\,cos\,n\pi}{n^3\pi^3}\,. \end{aligned}$$

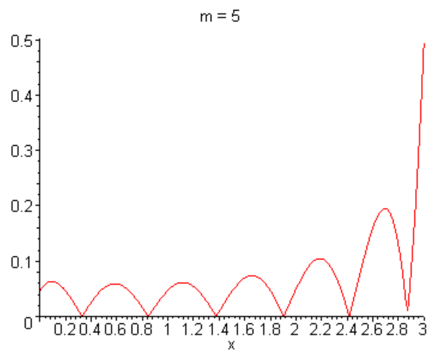
Therefore the Fourier series for the given function is

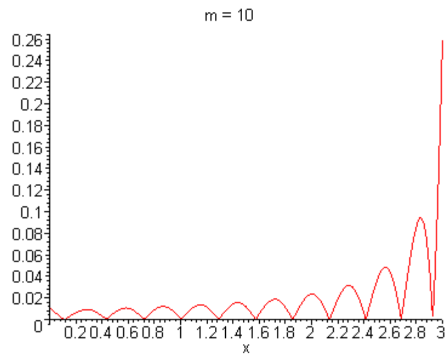
$$\begin{aligned} f(x) &= \frac{9}{8} - 27\sum_{n=1}^{\infty}\left[\frac{6[1-(-1)^n]}{n^4\pi^4} + \frac{(-1)^n}{n^2\pi^2}\right] cos\frac{n\pi x}{3} - \\ &- \frac{54}{\pi^3}\sum_{n=1}^{\infty}\frac{[1+2(-1)^n]}{n^3} sin\frac{n\pi x}{3}\,. \end{aligned}$$

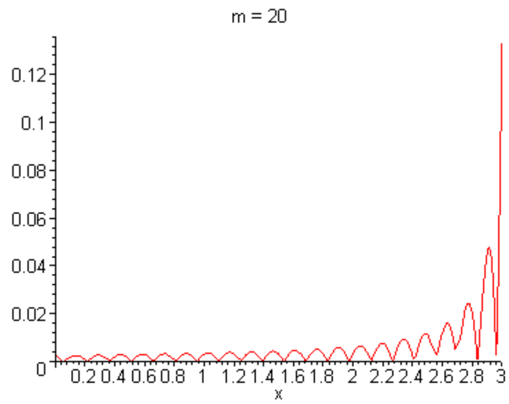
(c).



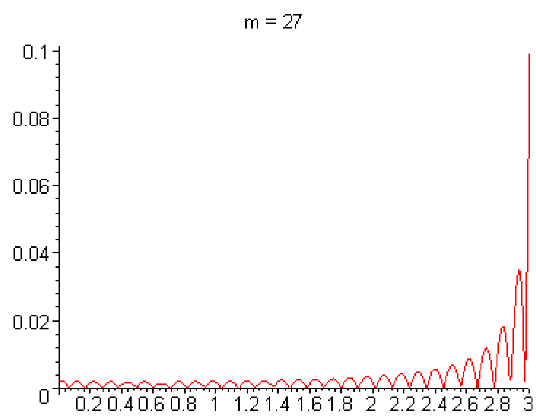
26.



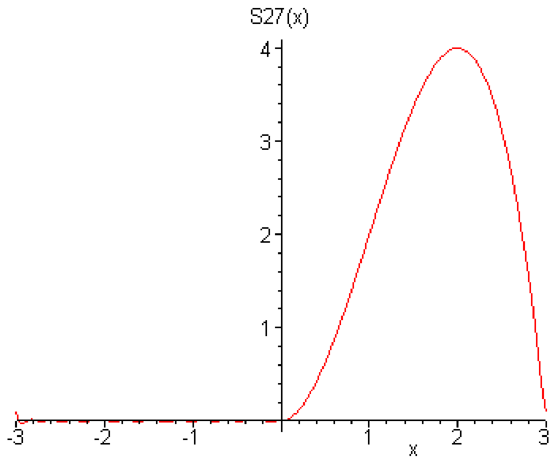




It is evident that $|e_m(x)|$ is greatest at $x = \pm 3$. Increasing the number of terms in the partials sums, we find that if $m \geq 27$, then $|e_m(x)| \leq 0.1$, for all $x \in [-3, 3]$.



Graphing the partial sum $s_{27}(x)$, the convergence is as predicted:



28. Let $x = T + a$, for some $a \in [0, T]$. First note that for any value of h,

$$\begin{aligned} f(x+h) - f(x) &= f(T+a+h) - f(T+a) \\ &= f(a+h) - f(a)\,. \end{aligned}$$

Since f is *differentiable*,

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \\ &= f'(a)\,. \end{aligned}$$

That is, $f'(a+T) = f'(a)$. By induction, it follows that $f'(a+T) = f'(a)$ for every value of a.

On the other hand, if $f(x) = 1 + cos\,x$, then the function

$$\begin{aligned} F(x) &= \int_0^x [1 + cos\,t]dt \\ &= x + sin\,x \end{aligned}$$

is *not* periodic, unless its definition is restricted to a specific interval.

29(a). Based on the hypothesis, the vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are a basis for $\mathbb{R}^3$. Given any vector $\mathbf{u} \in \mathbb{R}^3$, it can be expressed as a linear combination $\mathbf{u} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3$. Taking the inner product of both sides of this equation with $\mathbf{v}_i$, we have

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v}_i &= (a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3) \cdot \mathbf{v}_i \\ &= a_i\,\mathbf{v}_i \cdot \mathbf{v}_i\,, \end{aligned}$$

since the basis vectors are mutually orthogonal. Hence

$$a_i = \frac{\mathbf{u}\cdot\mathbf{v}_i}{\mathbf{v}_i\cdot\mathbf{v}_i}\,,\ i = 1, 2, 3\,.$$

Recall that $\mathbf{u}\cdot\mathbf{v}_i = u\,v_i\,cos\,\theta$, in which θ is the angle between $\mathbf{u}$ and $\mathbf{v}_i$. Therefore

$$a_i = \frac{u\,cos\,\theta}{v_i}\,.$$

Here $u\,cos\,\theta$ is interpreted as the magnitude of the projection of $\mathbf{u}$ in the direction of $\mathbf{v}_i$.

(b). Assuming that a Fourier series converges to a periodic function, $f(x)$,

$$f(x) = \frac{a_0}{2}\phi_0(x) + \sum_{m=1}^{\infty} a_m\phi_m(x) + \sum_{m=1}^{\infty} b_m\psi_m(x)\,.$$

Taking the inner product, defined by

$$(u\,,v) = \int_{-L}^{L} u(x)v(x)dx\,,$$

of both sides of the series expansion with the specified trigonometric functions, we have

$$(f\,,\phi_n) = \frac{a_0}{2}(\phi_0\,,\phi_n) + \sum_{m=1}^{\infty} a_m(\phi_m\,,\phi_n) + \sum_{m=1}^{\infty} b_m(\psi_m\,,\phi_n)$$

for $n = 0, 1, 2, \cdots$.

(c). It also follows that

$$(f\,,\psi_n) = \frac{a_0}{2}(\phi_0\,,\psi_n) + \sum_{m=1}^{\infty} a_m(\phi_m\,,\psi_n) + \sum_{m=1}^{\infty} b_m(\psi_m\,,\psi_n)$$

for $n = 1, 2, \cdots$. Based on the orthogonality conditions,

$$(\phi_m\,,\phi_n) = L\,\delta_{mn}\ ,\ (\psi_m\,,\psi_n) = L\,\delta_{mn}\ ,$$

and $(\psi_m\,,\phi_n) = L\,\delta_{mn}$. Note that $(\phi_0\,,\phi_0) = 2L$. Therefore

$$a_0 = \frac{2(f\,,\phi_0)}{(\phi_0\,,\phi_0)} = \frac{1}{L}\int_{-L}^{L} f(x)\phi_0(x)dx$$

and

$$a_n = \frac{(f\,,\phi_n)}{(\phi_n\,,\phi_n)} = \frac{1}{L}\int_{-L}^{L} f(x)\phi_n(x)dx\,,\quad n = 1, 2, \cdots.$$

Likewise,

$$b_n = \frac{(f\,,\psi_n)}{(\psi_n\,,\psi_n)} = \frac{1}{L}\int_{-L}^{L} f(x)\psi_n(x)dx\,, \quad n = 1, 2, \cdots .$$

Section 10.3

$1(a)$. The given function is assumed to be periodic with $2L = 2$. The Fourier *cosine* coefficients are given by

$$\begin{aligned} a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)dx \\ &= \int_{-1}^{0}(-1)dx + \int_{0}^{1}(1)dx \\ &= 0\,, \end{aligned}$$

and for $n > 0$,

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)cos\frac{n\pi x}{L}dx \\ &= -\int_{-1}^{0} cos\, n\pi x\, dx + \int_{0}^{1} cos\ n\pi x\, dx \\ &= 0\,. \end{aligned}$$

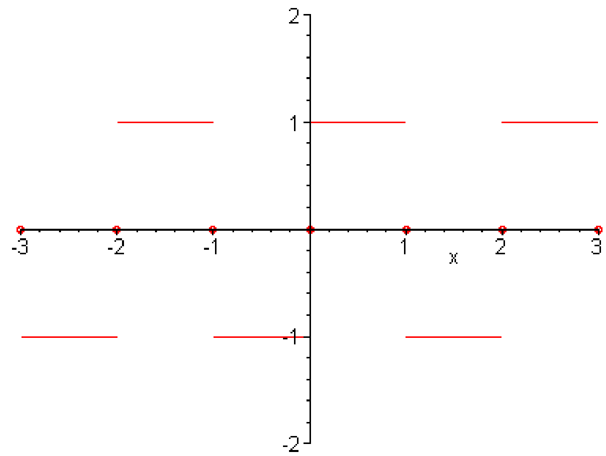
The Fourier *sine* coefficients are given by

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(x)sin\frac{n\pi x}{L}dx \\ &= -\int_{-1}^{0} sin\ n\pi x\, dx + \int_{0}^{1} sin\ n\pi x\, dx \\ &= 2\frac{1 - \,cos\, n\pi}{n\pi}\,. \end{aligned}$$

Therefore the Fourier series for the specified function is

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\,sin\,(2n-1)\pi x\,.$$

(b).



The function is piecewise continuous on each finite interval. The points of discontinuity are at *integer* values of x. At these points, the series converges to

$$|f(x-)+f(x+)|=0\,.$$

$3(a)$. The given function is assumed to be periodic with $T=2L$. The Fourier *cosine* coefficients are given by

$$\begin{aligned} a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)dx \\ &= \frac{1}{L}\int_{-L}^{0}(L+x)dx + \frac{1}{L}\int_{0}^{L}(L-x)dx \\ &= L\,, \end{aligned}$$

and for $n>0$,

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)cos\frac{n\pi x}{L}dx \\ &= \frac{1}{L}\int_{-L}^{0}(L+x)cos\frac{n\pi x}{L}\,dx + \frac{1}{L}\int_{0}^{L}(L-x)cos\frac{n\pi x}{L}\,dx \\ &= 2L\,\frac{1-\,cos\,n\pi}{n^2\pi^2}\,. \end{aligned}$$

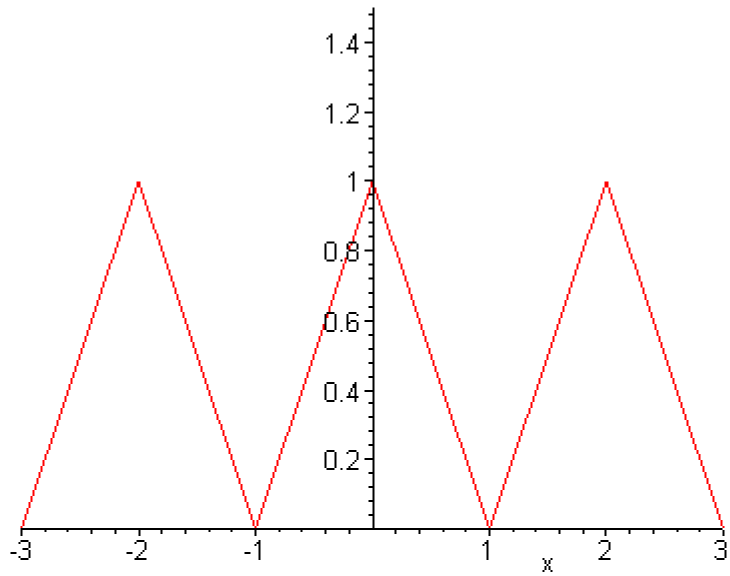
The Fourier *sine* coefficients are given by

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(x)sin\frac{n\pi x}{L}dx \\ &= \frac{1}{L}\int_{-L}^{0}(L+x)sin\frac{n\pi x}{L}\,dx + \frac{1}{L}\int_{0}^{L}(L-x)sin\frac{n\pi x}{L}\,dx \\ &= 0\,. \end{aligned}$$

Therefore the Fourier series of the specified function is

$$f(x) = \frac{L}{2} + \frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\,cos\frac{(2n-1)\pi x}{L}\,.$$

(b). For $L = 1$,



Note that $f(x)$ is *continuous*. Based on Theorem 10.3.1 , the series converges to the continuous function $f(x)$.

$5(a)$. The given function is assumed to be periodic with $2L = 2\pi$. The Fourier *cosine* coefficients are given by

$$\begin{aligned} a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)dx \\ &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}(1)dx \\ &= 1\,, \end{aligned}$$

and for $n > 0$,

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)cos\frac{n\pi x}{L}dx \\ &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}(1)cos\;nx\,dx \\ &= \frac{2}{n\pi}\,sin\left(\frac{n\pi}{2}\right). \end{aligned}$$

The Fourier *sine* coefficients are given by

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(x) sin \frac{n\pi x}{L} dx \\ &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} (1) sin\ nx\, dx \\ &= 0\,. \end{aligned}$$

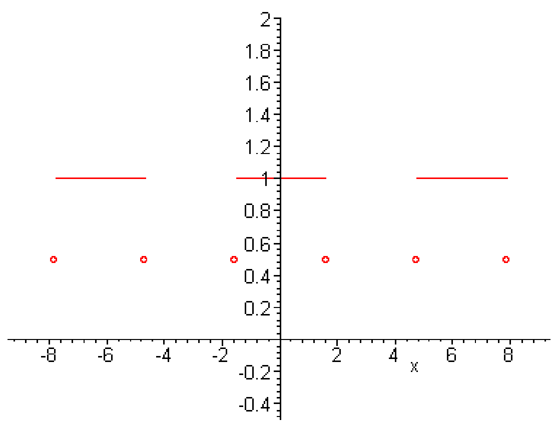
Observe that

$$sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0\,, & n = 2k \\ (-1)^{k+1}\,, & n = 2k-1 \end{cases} \quad , k = 1, 2, \cdots.$$

Therefore the Fourier series of the specified function is

$$f(x) = \frac{1}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} cos\,(2n-1)x\,.$$

(b).



The given function is piecewise continuous, with discontinuities at *odd* multiples of $\pi/2$. At $x_d = (2k-1)\pi/2$, $k = 0, 1, 2, \cdots$, the series converges to

$$|f(x_d-) + f(x_d+)| = 1/2\,.$$

$6(a)$. The given function is assumed to be periodic with $2L = 2$. The Fourier *cosine* coefficients are given by

$$\begin{aligned} a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)dx \\ &= \int_0^1 x^2 dx \\ &= 1/3\,, \end{aligned}$$

and for $n > 0$,

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x) cos \frac{n\pi x}{L} dx \\ &= \int_0^1 x^2 cos\ n\pi x\, dx \\ &= \frac{2\, cos\, n\pi}{n^2\pi^2} . \end{aligned}$$

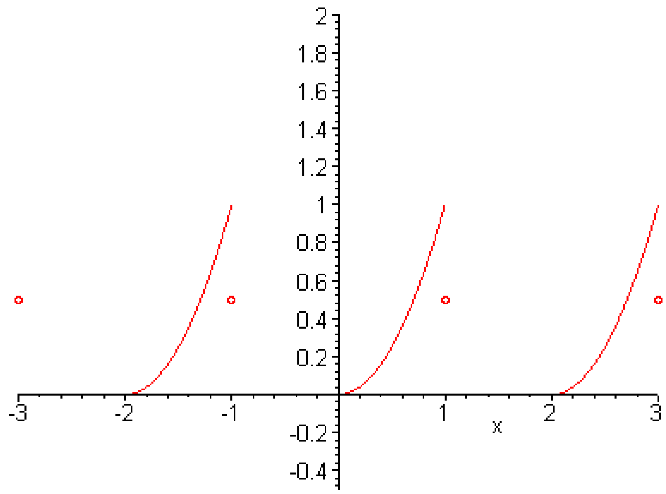
The Fourier *sine* coefficients are given by

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(x) sin \frac{n\pi x}{L} dx \\ &= \int_0^1 x^2 sin\ n\pi x\, dx \\ &= -\ \frac{2 - 2\, cos\, n\pi + n^2\pi^2\, cos\, n\pi}{n^3\pi^3} . \end{aligned}$$

Therefore the Fourier series for the specified function is

$$\begin{aligned} f(x) = \frac{1}{6} + \frac{2}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} cos\, n\pi x - \\ - \sum_{n=1}^{\infty}\left[\frac{2[1-(-1)^n]}{n^3\pi^3} + \frac{(-1)^n}{n\pi}\right] sin\, n\pi x\, . \end{aligned}$$

(b).



The given function is piecewise continuous, with discontinuities at the *odd* integers .

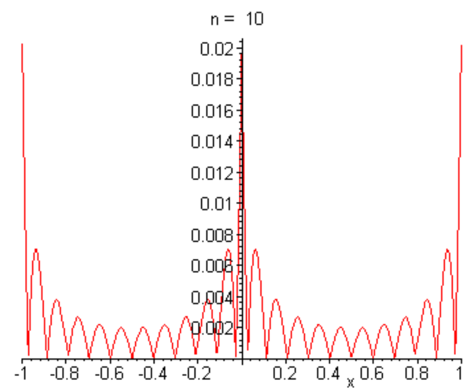
At $x_d = 2k - 1$, $k = 0, 1, 2, \cdots$, the series converges to

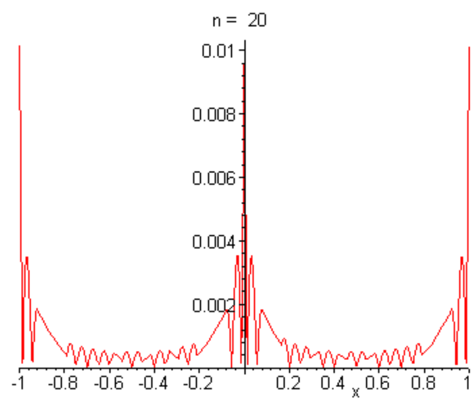
$$|f(x_d-) + f(x_d+)| = 1/2 .$$

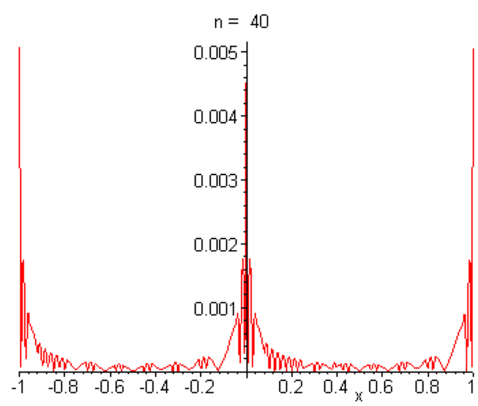
$8(a)$. As shown in Problem 16 of Section 10.2 ,

$$f(x) = \frac{1}{2} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\,cos(2n-1)\pi x .$$

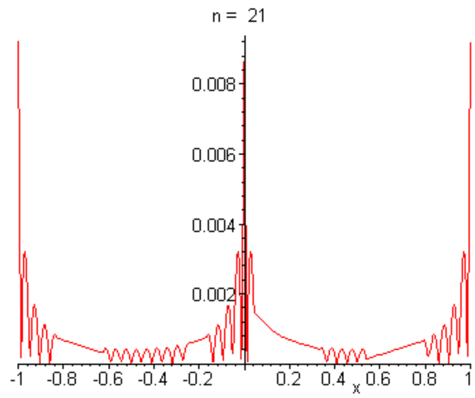
(b).







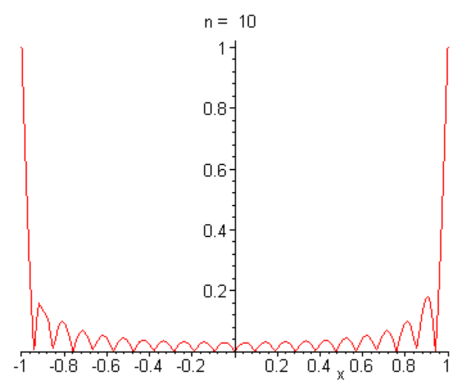
(c).

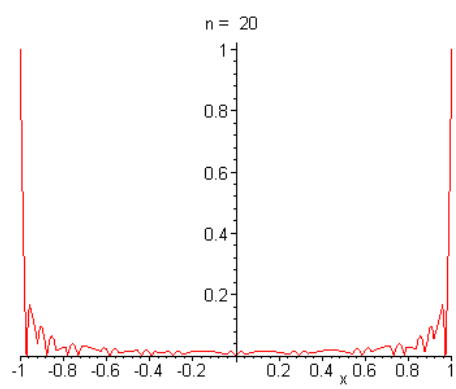


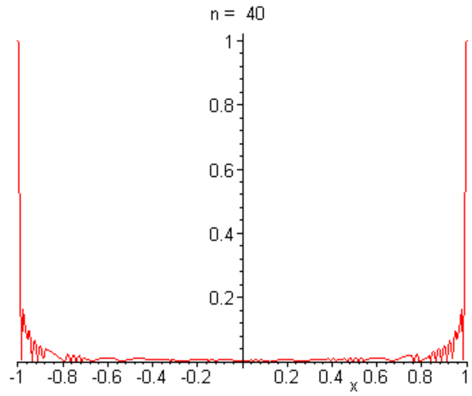
$9(a)$. As shown in Problem 20 of Section 10.2 ,

$$f(x) = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n} sin\, n\pi x\,.$$

(b).





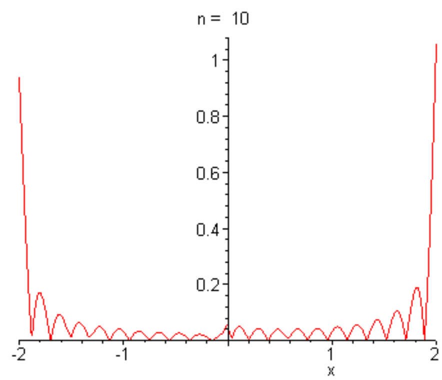


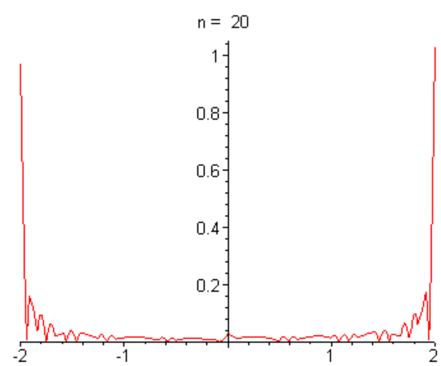
(c). The given function is discontinuous at $x = \pm 1$. At these points, the series will converge to a value of *zero*. The error can never be made arbitrarily small.

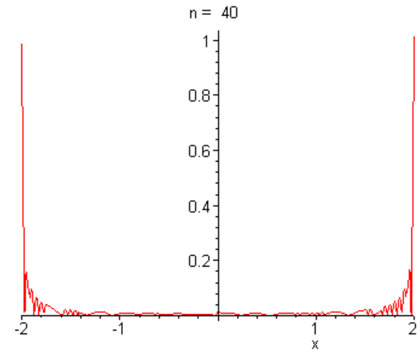
$10(a)$. As shown in Problem 22 of Section 10.2,

$$f(x) = \frac{1}{2} + \frac{12}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\,cos\frac{(2n-1)\pi x}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}sin\frac{n\pi x}{2}\,.$$

(b).





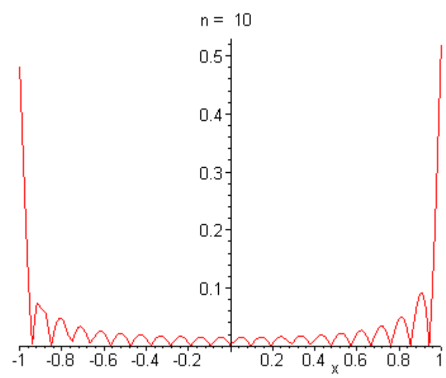


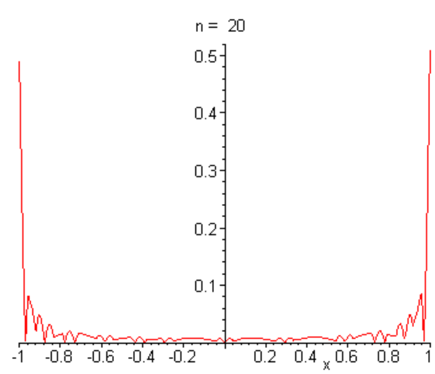
(c). The given function is discontinuous at $x = \pm 2$. At these points, the series will converge to a value of -1. The error can never be made arbitrarily small.

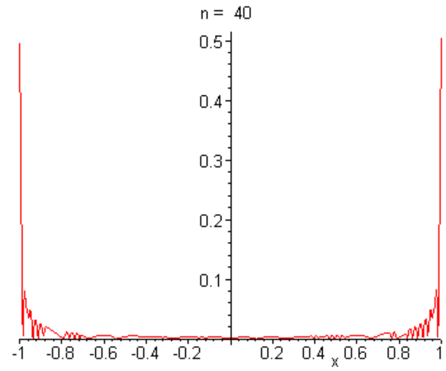
$11(a)$. As shown in Problem 6, above,

$$f(x) = \frac{1}{6} + \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi x - $$
$$- \sum_{n=1}^{\infty} \left[\frac{2[1-(-1)^n]}{n^3\pi^3} + \frac{(-1)^n}{n\pi} \right] \sin n\pi x \, .$$

(b).







(c). The given function is piecewise continuous, with discontinuities at the *odd* integers. At $x_d = 2k - 1$, $k = 0, 1, 2, \cdots$, the series converges to

$$|f(x_d -) + f(x_d +)| = 1/2\,.$$

At these points the error can never be made arbitrarily small.

13. The solution of the *homogenous* differential equation is

$$y_c(t) = c_1 \, cos\, \omega t + c_2 \, sin\, \omega t\,.$$

Given that $\omega^2 \neq n^2$, we can use the *method of undetermined coefficients* to find a particular solution

$$Y(t) = \frac{1}{\omega^2 - n^2} \, sin\, nt\,.$$

Hence the general solution of the ODE is

$$y(t) = c_1 \, cos\, \omega t + c_2 \, sin\, \omega t + \frac{1}{\omega^2 - n^2} \, sin\, nt\,.$$

Imposing the initial conditions, we obtain the equations

$$c_1 = 0$$
$$\omega \, c_2 + \frac{n}{\omega^2 - n^2} = 0\,.$$

It follows that $c_2 = \; - n/[\omega(\omega^2 - n^2)]$. The solution of the IVP is

$$y(t) = \frac{1}{\omega^2 - n^2} \, sin\, nt - \frac{n}{\omega(\omega^2 - n^2)} \, sin\, \omega t\,.$$

If $\omega^2 = n^2$, then the forcing function is also one of the fundamental solutions of the ODE.
The method of undetermined coefficients may still be used, with a more elaborate trial solution. Using the *method of variation of parameters*, we obtain

$$\begin{aligned} Y(t) &= -\cos nt \int \frac{\sin^2 nt}{n} dt + \sin nt \int \frac{\cos nt \sin nt}{n} dt \\ &= \frac{\sin nt - nt \cos nt}{2n^2} . \end{aligned}$$

In this case, the general solution is

$$y(t) = c_1 \cos nt + c_2 \sin nt - \frac{t}{2n} \cos nt .$$

Invoking the initial conditions, we obtain $c_1 = 0$ and $c_2 = 1/2n^2$. Therefore the solution of the IVP is

$$y(t) = \frac{1}{2n^2} \sin nt - \frac{t}{2n} \cos nt .$$

16. Note that the function $f(t)$ and the function given in Problem 8 have the same Fourier series. Therefore

$$f(t) = \frac{1}{2} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos(2n-1)\pi t .$$

The solution of the homogeneous problem is

$$y_c(t) = c_1 \cos \omega t + c_2 \sin \omega t .$$

Using the method of undetermined coefficients, we assume a particular solution of the form

$$Y(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos n\pi t .$$

Substitution into the ODE and equating like terms results in $A_0 = 1/2\omega^2$ and

$$A_n = \frac{a_n}{\omega^2 - n^2\pi^2} .$$

It follows that the general solution is

$$y(t) = c_1 \cos \omega t + c_2 \sin \omega t + \frac{1}{2\omega^2} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos(2n-1)\pi t}{(2n-1)^2 \left[\omega^2 - (2n-1)^2 \pi^2\right]} .$$

Setting $y(0) = 1$, we find that

$$c_1 = 1 - \frac{1}{2\omega^2} - \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos(2n-1)\pi t}{(2n-1)^2 \left[\omega^2 - (2n-1)^2 \pi^2\right]} .$$

Invoking the initial condition $y'(0) = 0$, we obtain $c_2 = 0$. Hence the solution of the initial value problem is

$$y(t) = cos\,\omega t - \frac{1}{2\omega^2}cos\,\omega t + \frac{1}{2\omega^2} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{cos(2n-1)\pi t - cos\,\omega t}{(2n-1)^2\left[\omega^2 - (2n-1)^2\pi^2\right]} .$$

17. Let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\,cos\frac{n\pi x}{L} + b_n\,sin\frac{n\pi x}{L}\right].$$

Squaring both sides of the equation, we *formally* have

$$\begin{aligned}|f(x)|^2 &= \frac{a_0^2}{4} + \sum_{n=1}^{\infty}\left[a_n^2\,cos^2\frac{n\pi x}{L} + b_n^2\,sin^2\frac{n\pi x}{L}\right] + a_0\sum_{n=1}^{\infty}\left[a_n\,cos\frac{n\pi x}{L} + b_n\,sin\frac{n\pi x}{L}\right] + \\ &+ \sum_{m\neq n}\left[c_{mn}\,cos\frac{m\pi x}{L}\,sin\frac{n\pi x}{L}\right].\end{aligned}$$

Integrating both sides of the last equation, and using the *orthogonality conditions*,

$$\begin{aligned}\int_{-L}^{L}|f(x)|^2dx &= \int_{-L}^{L}\frac{a_0^2}{4}dx + \sum_{n=1}^{\infty}\left[\int_{-L}^{L}a_n^2\,cos^2\frac{n\pi x}{L}dx + \int_{-L}^{L}b_n^2\,sin^2\frac{n\pi x}{L}dx\right] \\ &= \frac{a_0^2}{2}L + \sum_{n=1}^{\infty}\left[a_n^2\,L + b_n^2 L\right].\end{aligned}$$

Therefore,

$$\frac{1}{L}\int_{-L}^{L}|f(x)|^2dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$

19(a). As shown in the Example, the Fourier series of the function

$$f(x) = \begin{cases} 0, & -L < x < 0 \\ L, & 0 < x < L, \end{cases}$$

is given by

$$f(x) = \frac{L}{2} + \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}sin\frac{(2n-1)\pi x}{L}.$$

Setting $L = 1$,

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}sin(2n-1)\pi x.$$

It follows that

$$\sum_{n=1}^{\infty}\frac{1}{2n-1}sin(2n-1)\pi x = \frac{\pi}{2}\left[f(x)-\frac{1}{2}\right]. \qquad (ii)$$

(b). Given that

$$g(x) = \sum_{n=1}^{\infty}\frac{2n-1}{1+(2n-1)^2}\,sin(2n-1)\pi x\,, \qquad (i)$$

and subtracting Eq.(ii) from Eq.(i), we find that

$$g(x)-\frac{\pi}{2}\left[f(x)-\frac{1}{2}\right] = \sum_{n=1}^{\infty}\frac{2n-1}{1+(2n-1)^2}\,sin(2n-1)\pi x - \\ - \sum_{n=1}^{\infty}\frac{1}{2n-1}sin(2n-1)\pi x\,.$$

Based on the fact that

$$\frac{2n-1}{1+(2n-1)^2}-\frac{1}{2n-1} = -\frac{1}{(2n-1)\left[1+(2n-1)^2\right]}\,,$$

and the fact that we can combine the two series, it follows that

$$g(x) = \frac{\pi}{2}\left[f(x)-\frac{1}{2}\right] - \sum_{n=1}^{\infty}\frac{sin(2n-1)\pi x}{(2n-1)\left[1+(2n-1)^2\right]}\,.$$

Section 10.4

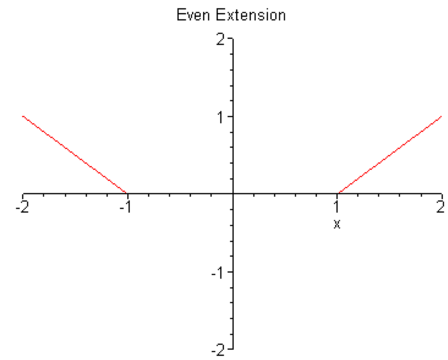
1. Since the function contains only odd powers of x, the function is *odd*.

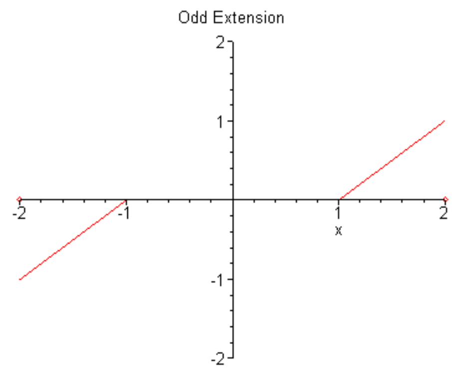
2. Since the function contains both odd and even powers of x, the function is *neither* even nor odd.

4. We have $sec\,x = 1/cos\,x$. Since the *quotient* of two even functions is even, the function is *even*.

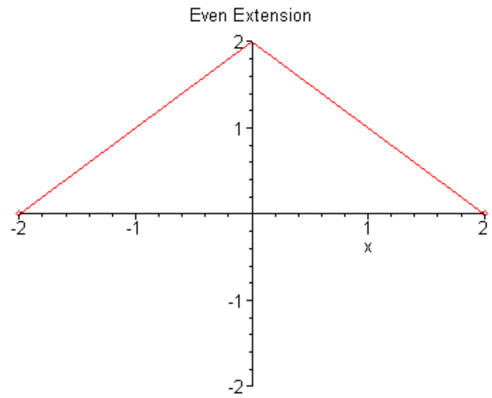
5. We can write $|x|^3 = |x| \cdot |x|^2 = |x| \cdot x^2$. Since both factors are even, it follows that the function is *even*.

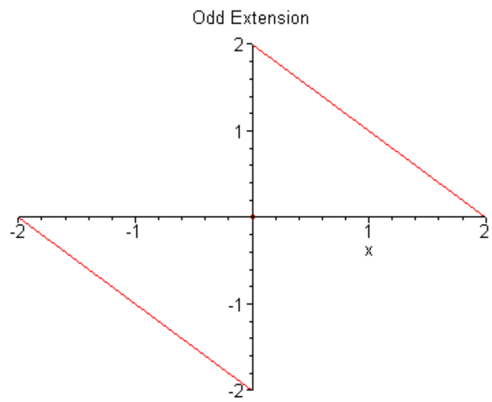
8. $L = 2$.



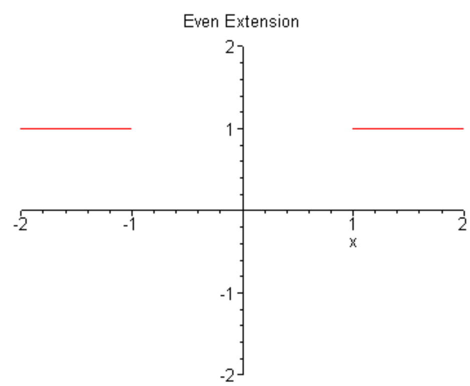


9. $L = 2$.


Even Extension
2
1
-2
-1
1
2
x
-1
-2


Odd Extension
2
1
-2
-1
1
2
x
-1
-2

11. $L = 2$.


Even Extension
2
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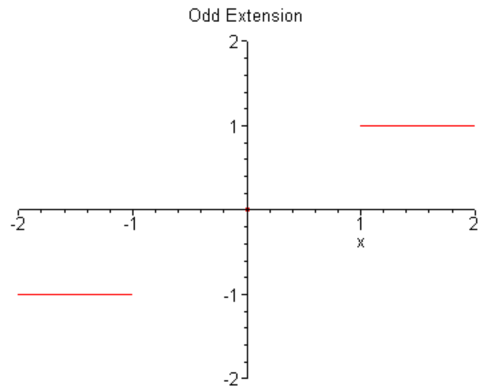
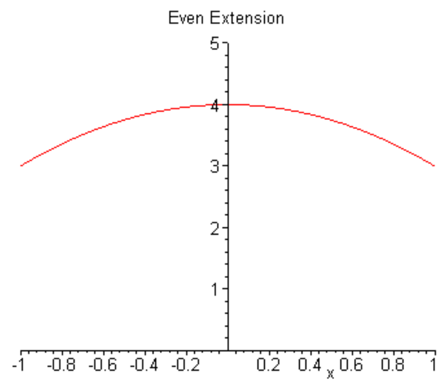
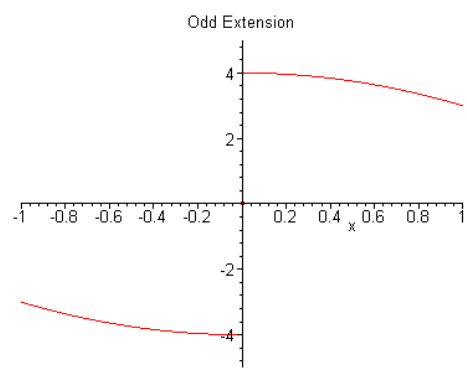
x
-1
-2



12. $L = 1$.





16. $L = 2$. For an *odd* extension of the function, the cosine coefficients are *zero*. The sine coefficients are given by

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L f(x) sin\frac{n\pi x}{L}dx \\ &= \int_0^1 x\, sin\frac{n\pi x}{2}dx + \int_1^2 sin\frac{n\pi x}{2}dx \\ &= 2\frac{2\, sin\frac{n\pi}{2} - n\pi\, cos\, n\pi}{n^2\pi^2}\,. \end{aligned}$$

Observe that

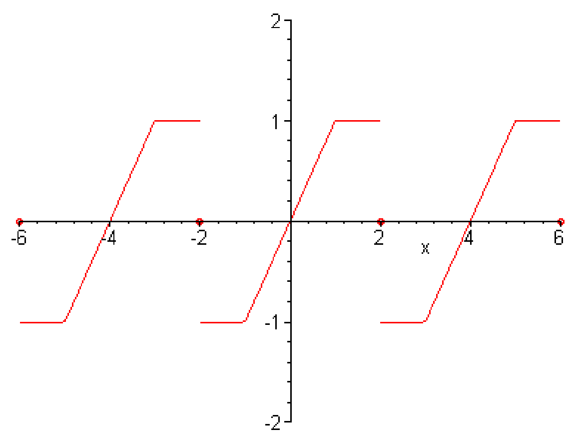
$$sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0\,, & n = 2k \\ (-1)^{k+1}\,, & n = 2k-1 \end{cases} \quad , k = 1, 2, \cdots.$$

Likewise,

$$cos\, n\pi = \begin{cases} 1\,, & n = 2k \\ -1\,, & n = 2k-1 \end{cases} \quad , k = 1, 2, \cdots.$$

Therefore the Fourier sine series of the specified function is

$$f(x) = -\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\, sin\, n\pi x + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{2(-1)^{n+1} + (2n-1)\pi}{(2n-1)^2}\, sin\frac{(2n-1)\pi x}{2}.$$



17. $L = \pi$. For an *even* extension of the function, the sine coefficients are *zero*. The cosine coefficients are given by

$$\begin{aligned} a_0 &= \frac{2}{L}\int_0^L f(x)dx \\ &= \frac{2}{\pi}\int_0^{\pi}(1)dx \\ &= 2\,, \end{aligned}$$

and for $n > 0$,

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)cos\frac{n\pi x}{L}dx \\ &= \frac{2}{\pi}\int_0^\pi (1)cos\ nx\,dx \\ &= 0\,. \end{aligned}$$

The even extension of the given function is a *constant* function. As expected, the Fourier cosine series is

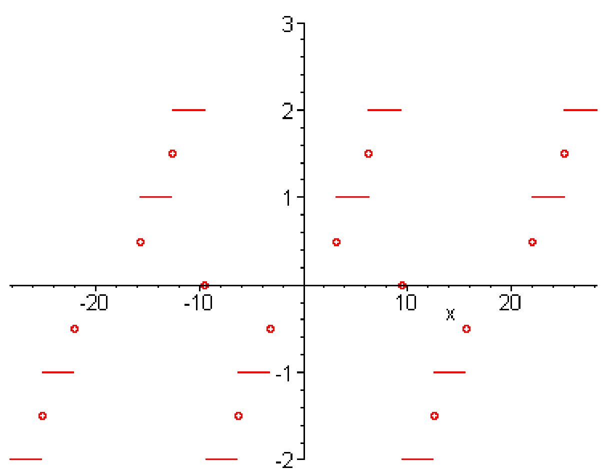
$$f(x) = \frac{a_0}{2} = 1\,.$$

19. $L = 3\pi$. For an *odd* extension of the function, the cosine coefficients are *zero*. The sine coefficients are given by

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L f(x)sin\frac{n\pi x}{L}dx \\ &= \frac{2}{3\pi}\int_\pi^{2\pi} sin\frac{nx}{3}dx + \frac{2}{3\pi}\int_{2\pi}^{3\pi} 2\,sin\frac{nx}{3}dx \\ &= -2\frac{2\,cos\,n\pi - cos\frac{n\pi}{3} - cos\frac{2n\pi}{3}}{n\pi}\,. \end{aligned}$$

Therefore the Fourier sine series of the specified function is

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[cos\frac{n\pi}{3} + cos\frac{2n\pi}{3} - 2\,cos\,n\pi\right]sin\frac{nx}{3}.$$



21. Extend the function over the interval $[-L, L]$ as

$$f(x) = \begin{cases} x + L\,, & -L \le x < 0 \\ L - x\,, & 0 \le x \le L\,. \end{cases}$$

Since the extended function is *even*, the sine coefficients are *zero*. The cosine coefficients
are given by

$$\begin{aligned} a_0 &= \frac{2}{L}\int_0^L f(x)dx \\ &= \frac{2}{L}\int_0^L (L-x)dx \\ &= L\,, \end{aligned}$$

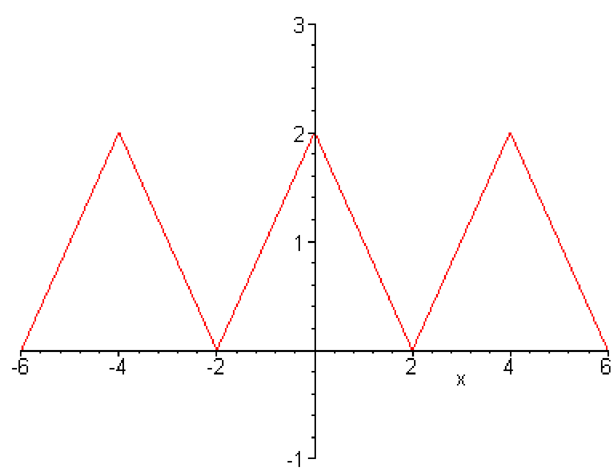
and for $n > 0$,

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)cos\frac{n\pi x}{L}dx \\ &= \frac{2}{L}\int_0^L (L-x)cos\frac{n\pi x}{L}\,dx \\ &= 2L\frac{1-cos\,n\pi}{n^2\pi^2}\,. \end{aligned}$$

Therefore the Fourier cosine series of the extended function is

$$f(x) = \frac{L}{2} + \frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\,cos\frac{(2n-1)\pi x}{L}.$$

In order to compare the result with Example 1 of Section 10.2, set $L = 2$. The cosine series converges to the function graphed below:



This function is a *shift* of the function in Example 1 of Section 10.2.

22. Extend the function over the interval $[-L, L]$ as

$$f(x) = \begin{cases} -x - L, & -L \leq x < 0 \\ L - x, & 0 < x \leq L, \end{cases}$$

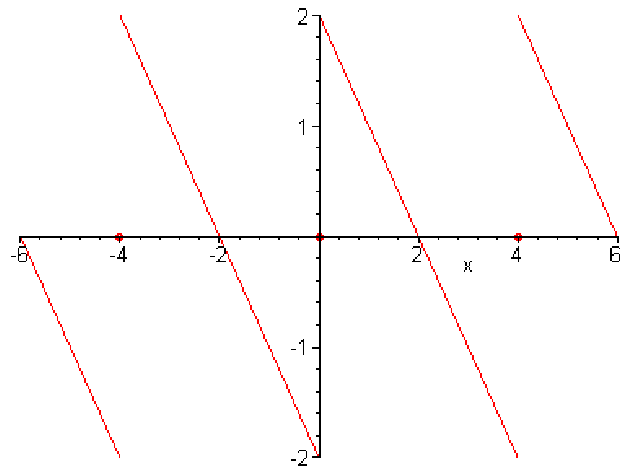
with $f(0) = 0$. Since the extended function is *odd*, the cosine coefficients are *zero*. The sine coefficients are given by

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L f(x) sin\frac{n\pi x}{L} dx \\ &= \frac{2}{L}\int_0^L (L - x) sin\frac{n\pi x}{L}\, dx \\ &= \frac{2L}{n\pi}. \end{aligned}$$

Therefore the Fourier cosine series of the extended function is

$$f(x) = \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\, sin\frac{n\pi x}{L}.$$

Setting $L = 2$, for example, the series converges to the function graphed below:



$23(a)$. $L = 2\pi$. For an *even* extension of the function, the sine coefficients are *zero*. The cosine coefficients are given by

$$\begin{aligned} a_0 &= \frac{2}{L}\int_0^L f(x)dx \\ &= \frac{1}{\pi}\int_0^\pi x\,dx \\ &= \pi/2, \end{aligned}$$

and for $n > 0$,

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)cos\frac{n\pi x}{L}dx \\ &= \frac{1}{\pi}\int_0^\pi x\,cos\frac{nx}{2}\,dx \\ &= 2\frac{2\,cos\left(\frac{n\pi}{2}\right) + n\pi sin\left(\frac{n\pi}{2}\right) - 2}{n^2\pi}. \end{aligned}$$

Therefore the Fourier cosine series of the given function is

$$f(x) = \frac{\pi}{4} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{\pi}{n}sin\frac{n\pi}{2} + \frac{2}{n^2}\left(cos\frac{n\pi}{2} - 1\right)\right]cos\frac{nx}{2}.$$

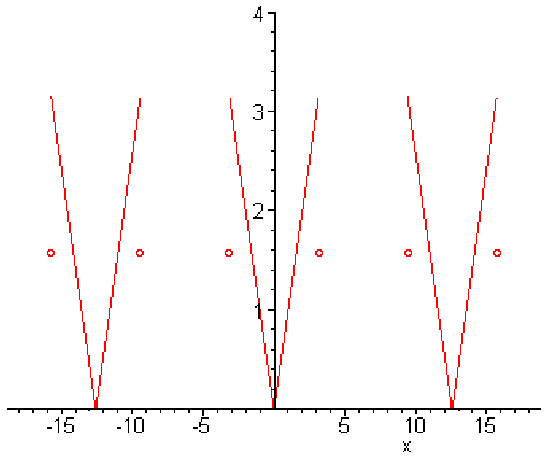
Observe that

$$sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0, & n = 2k \\ (-1)^{k+1}, & n = 2k-1 \end{cases}, \; k = 1, 2, \cdots.$$

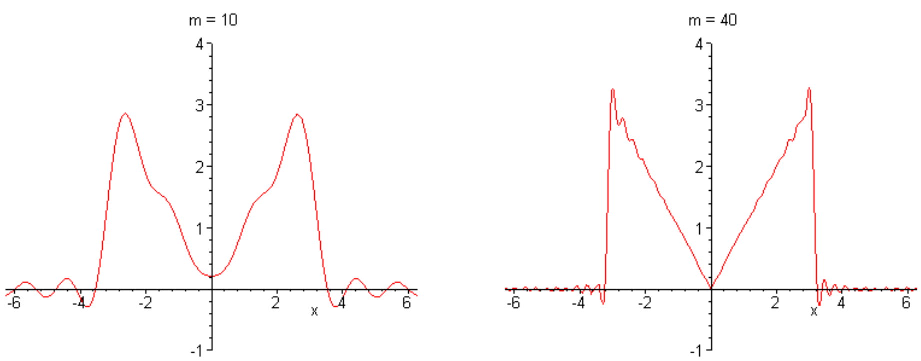
Likewise,

$$cos\left(\frac{n\pi}{2}\right) = \begin{cases} (-1)^k, & n = 2k \\ 0, & n = 2k-1 \end{cases} \quad , k = 1, 2, \cdots .$$

(b).



(c).



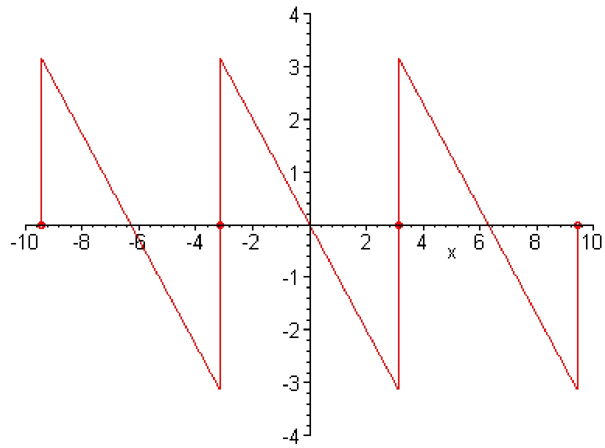
24(a). $L = \pi$. For an *odd* extension of the function, the cosine coefficients are *zero*. Note that $f(x) = -x$ on $0 \le x < \pi$. The sine coefficients are given by

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L f(x) sin \frac{n\pi x}{L} dx \\ &= -\frac{2}{\pi}\int_0^\pi x\, sin\, nx\, dx \\ &= \frac{2\, cos\, n\pi}{n} . \end{aligned}$$

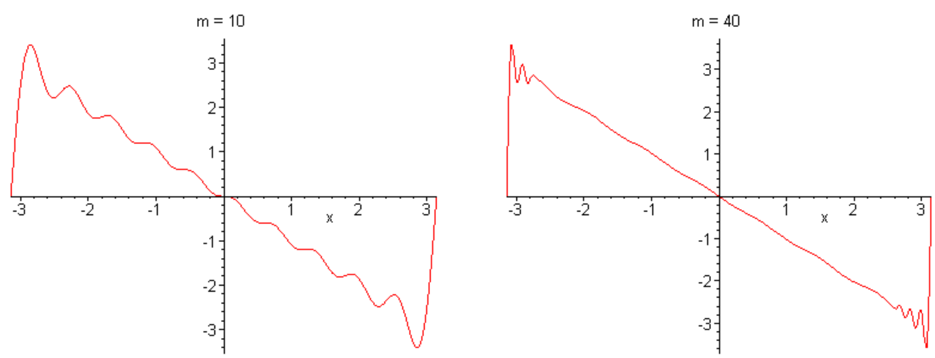
Therefore the Fourier sine series of the given function is

$$f(x) = 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n} sin\, nx.$$

(b).



(c).



$26(a)$. $L = 4$. For an *even* extension of the function, the sine coefficients are *zero*. The cosine coefficients are given by

$$\begin{aligned} a_0 &= \frac{2}{L}\int_0^L f(x)dx \\ &= \frac{1}{2}\int_0^4 \left(x^2 - 2x\right)dx \\ &= 8/3\,, \end{aligned}$$

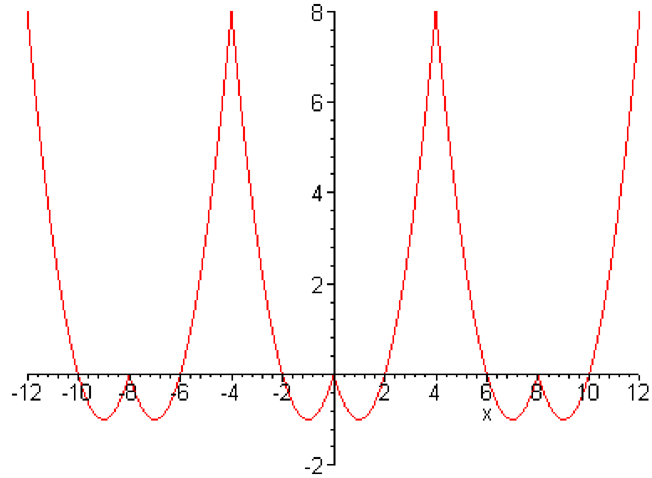
and for $n > 0$,

$$
\begin{aligned}
a_n &= \frac{2}{L}\int_0^L f(x) cos\frac{n\pi x}{L} dx \\
&= \frac{1}{2}\int_0^4 (x^2 - 2x) cos\frac{n\pi x}{4}\, dx \\
&= 16\,\frac{1+3\,cos\, n\pi}{n^2\pi^2}\,.
\end{aligned}
$$

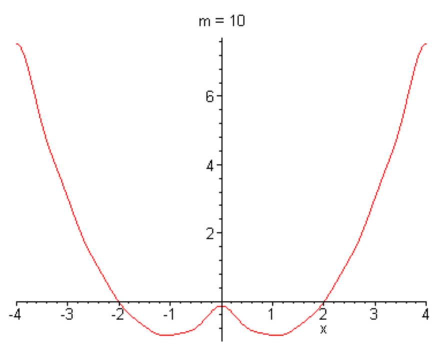
Therefore the Fourier cosine series of the given function is

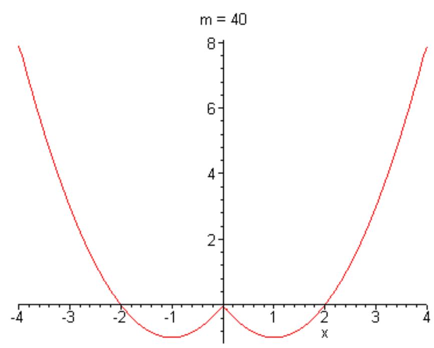
$$
f(x) = \frac{4}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1+3(-1)^n}{n^2} cos\frac{n\pi x}{4}.
$$

(b).

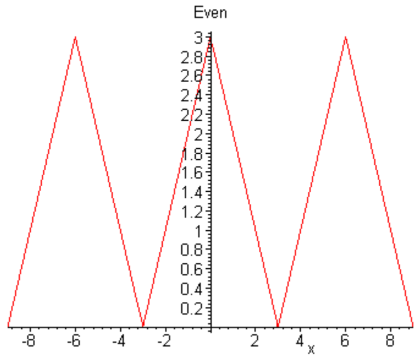


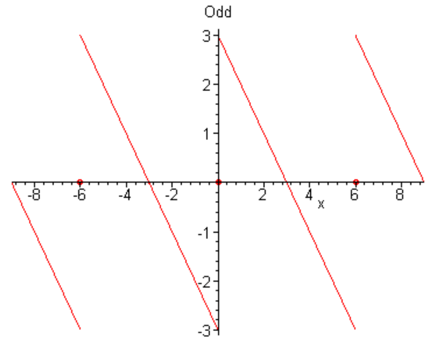
(c).





$27(a)$.





(b). $L = 3$. For an *even* extension of the function, the cosine coefficients are given by

$$\begin{aligned} a_0 &= \frac{2}{L}\int_0^L f(x)dx \\ &= \frac{2}{3}\int_0^3 (3-x)dx \\ &= 3\,, \end{aligned}$$

and for $n > 0$,

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)cos\frac{n\pi x}{L}dx \\ &= \frac{2}{3}\int_0^3 (3-x)cos\frac{n\pi x}{3}\,dx \\ &= 6\,\frac{1-cos\,n\pi}{n^2\pi^2}\,. \end{aligned}$$

Therefore the Fourier cosine series of the given function is

$$g(x) = \frac{3}{2} + \frac{6}{\pi^2}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2}cos\frac{n\pi x}{3}.$$

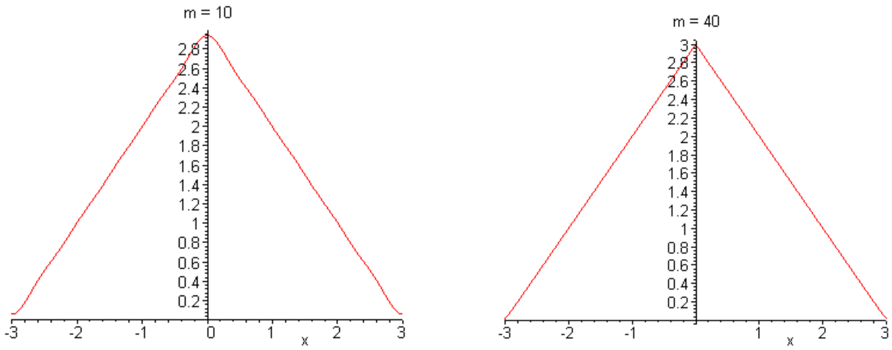
For an *odd* extension of the function, the sine coefficients are given by

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L f(x)sin\frac{n\pi x}{L}dx \\ &= \frac{2}{3}\int_0^3 (3-x)sin\frac{n\pi x}{3}\,dx \\ &= \frac{6}{n\pi}\,. \end{aligned}$$

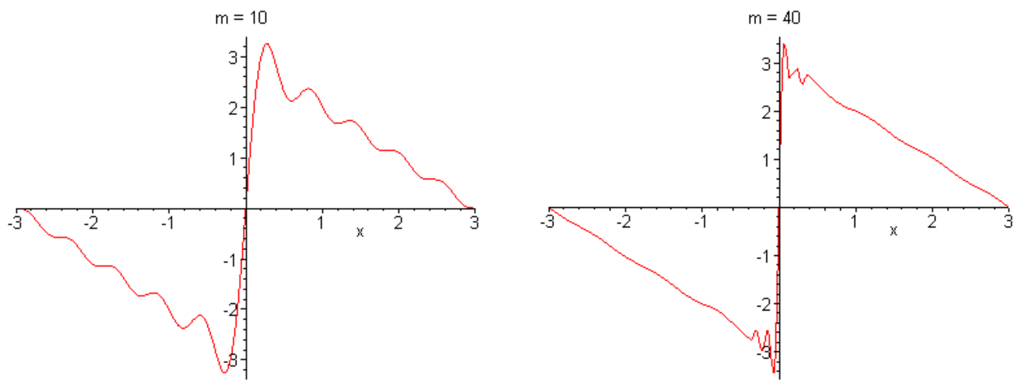
Therefore the Fourier sine series of the given function is

$$h(x) = \frac{6}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} sin \frac{n\pi x}{3}.$$

(c). For the *even* extension:

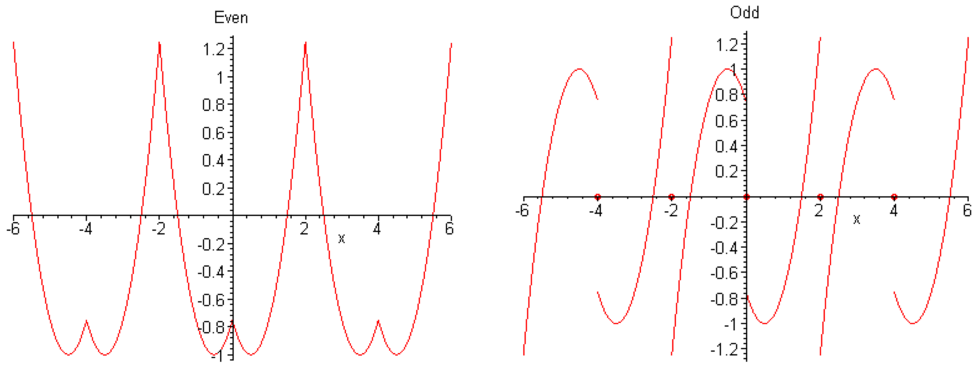


For the *odd* extension:



(d). Since the *even* extension is *continuous*, the series converges uniformly. On the other
hand, the *odd* extension is *discontinuous*. Gibbs' phenomenon results in a finite error for all values of n .

29(a).



(b). $L = 2$. For an *even* extension of the function, the cosine coefficients are given by

$$\begin{aligned} a_0 &= \frac{2}{L}\int_0^L f(x)dx \\ &= \int_0^2 \left[\frac{4x^2 - 4x - 3}{4}\right] dx \\ &= -5/6\,, \end{aligned}$$

and for $n > 0$,

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x) cos\frac{n\pi x}{L} dx \\ &= \int_0^2 \left[\frac{4x^2 - 4x - 3}{4}\right] cos\frac{n\pi x}{2}\,dx \\ &= 4\,\frac{1 + 3\,cos\,n\pi}{n^2\pi^2}\,. \end{aligned}$$

Therefore the Fourier cosine series of the given function is

$$g(x) = -\frac{5}{12} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1 + 3(-1)^n}{n^2} cos\frac{n\pi x}{2}.$$

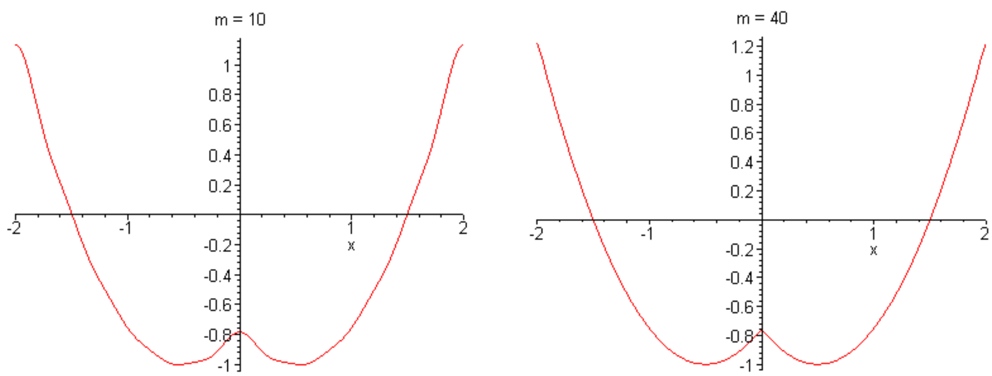
For an *odd* extension of the function, the sine coefficients are given by

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L f(x) sin\frac{n\pi x}{L} dx \\ &= \int_0^2 \left[\frac{4x^2 - 4x - 3}{4}\right] sin\frac{n\pi x}{2}\,dx \\ &= -\frac{32 + 3n^2\pi^2 + 5n^2\pi^2 cos\,n\pi - 32\,cos\,n\pi}{2\,n^3\pi^3}\,. \end{aligned}$$

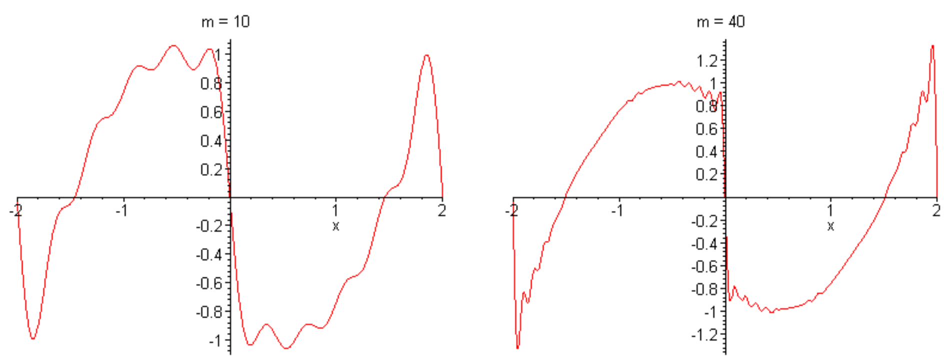
Therefore the Fourier sine series of the given function is

$$h(x) = -\frac{1}{2\pi^3}\sum_{n=1}^{\infty}\frac{32(1-\cos n\pi)+n^2\pi^2(3+5\cos n\pi)}{n^3}\sin\frac{n\pi x}{2}.$$

(c). For the *even* extension:

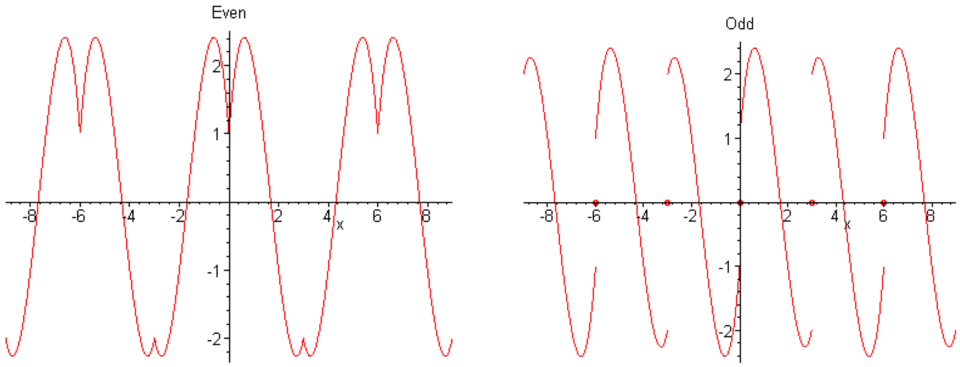


For the odd extension:



(d). Since the *even* extension is *continuous*, the series converges uniformly. On the other
hand, the *odd* extension is *discontinuous*. Gibbs' phenomenon results in a finite error for all values of n .

30(a).



(b). $L = 3$. For an *even* extension of the function, the cosine coefficients are given by

$$\begin{aligned} a_0 &= \frac{2}{L}\int_0^L f(x)dx \\ &= \frac{2}{3}\int_0^3 \left(x^3 - 5x^2 + 5x + 1\right)dx \\ &= 1/2\,, \end{aligned}$$

and for $n > 0$,

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)cos\frac{n\pi x}{L}dx \\ &= \frac{2}{3}\int_0^3 \left(x^3 - 5x^2 + 5x + 1\right)cos\frac{n\pi x}{3}\,dx \\ &= 2\,\frac{162 - 15\,n^2\pi^2 + 6\,n^2\pi^2 cos\,n\pi - 162\,cos\,n\pi}{n^4\pi^4}\,. \end{aligned}$$

Therefore the Fourier cosine series of the given function is

$$g(x) = \frac{1}{4} + \frac{2}{\pi^4}\sum_{n=1}^{\infty}\frac{162(1 - \ cos\,n\pi) - 3\,n^2\pi^2(5 - 2\,cos\,n\pi)}{n^4}cos\frac{n\pi x}{3}.$$

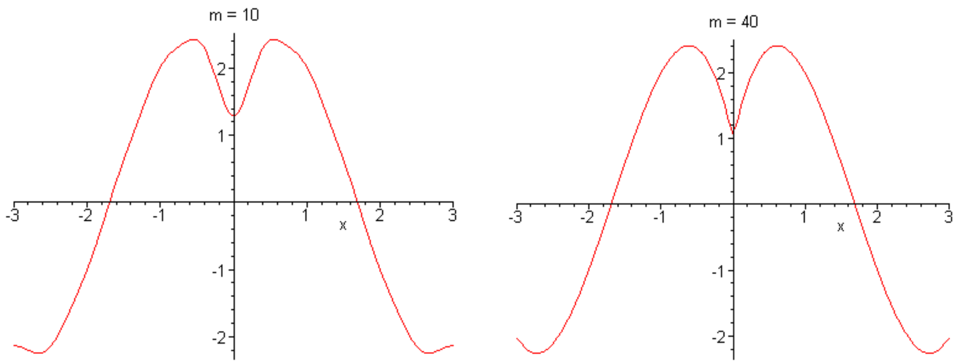
For an *odd* extension of the function, the sine coefficients are given by

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L f(x)sin\frac{n\pi x}{L}dx \\ &= \frac{2}{3}\int_0^3 \left(x^3 - 5x^2 + 5x + 1\right)sin\frac{n\pi x}{3}\,dx \\ &= 2\,\frac{90 + \ n^2\pi^2 + 2\,n^2\pi^2 cos\,n\pi + 72\,cos\,n\pi}{n^3\pi^3}\,. \end{aligned}$$

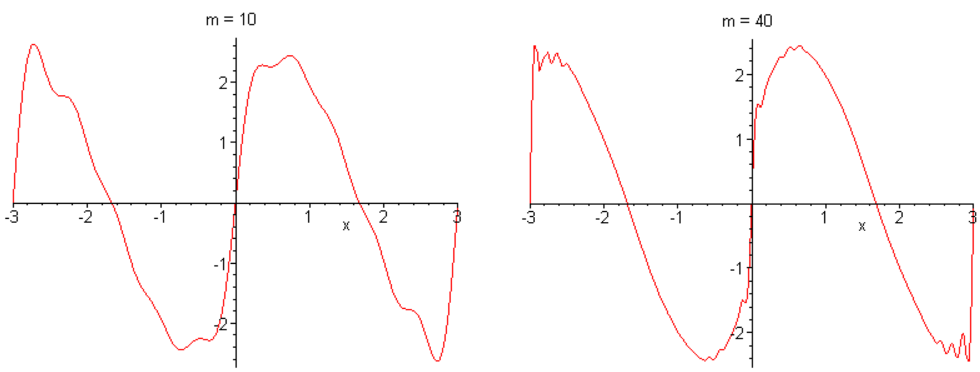
Therefore the Fourier sine series of the given function is

$$h(x) = \frac{2}{\pi^3} \sum_{n=1}^{\infty} \frac{18(5 + 4\cos n\pi) + n^2\pi^2(1 + 2\cos n\pi)}{n^3} \sin\frac{n\pi x}{3}.$$

(c). For the *even* extension:



For the *odd* extension:



(d). Since the *even* extension is *continuous*, the series converges uniformly. On the other
hand, the *odd* extension is *discontinuous*. Gibbs' phenomenon results in a finite error for all values of n; particularly at $x = \pm 3$.

33. Let $f(x)$ be a differentiable *even* function. For any x in its domain,

$$f(-x+h) - f(-x) = f(x-h) - f(x).$$

It follows that

$$\begin{aligned} f'(-x) &= \lim_{h\to 0}\frac{f(-x+h)-f(-x)}{h} \\ &= \lim_{h\to 0}\frac{f(x-h)-f(x)}{h} \\ &= -\lim_{h\to 0}\frac{f(x-h)-f(x)}{(-h)}\,. \end{aligned}$$

Setting $h=-\delta$, we have

$$\begin{aligned} f'(-x) &= -\lim_{h\to 0}\frac{f(x+\delta)-f(x)}{\delta} \\ &= -\lim_{-\delta\to 0}\frac{f(x+\delta)-f(x)}{\delta} \\ &= -f'(x)\,. \end{aligned}$$

Therefore $f'(-x)=-f'(x)$.

If $f(x)$ is a differentiable *odd* function, for any x in its domain,

$$f(-x+h)-f(-x)=-f(x-h)+f(x)\,.$$

It follows that

$$\begin{aligned} f'(-x) &= \lim_{h\to 0}\frac{f(-x+h)-f(-x)}{h} \\ &= \lim_{h\to 0}\frac{-f(x-h)+f(x)}{h} \\ &= \lim_{h\to 0}\frac{f(x-h)-f(x)}{(-h)}\,. \end{aligned}$$

Setting $h=-\delta$, we have

$$\begin{aligned} f'(-x) &= \lim_{h\to 0}\frac{f(x+\delta)-f(x)}{\delta} \\ &= \lim_{-\delta\to 0}\frac{f(x+\delta)-f(x)}{\delta} \\ &= f'(x)\,. \end{aligned}$$

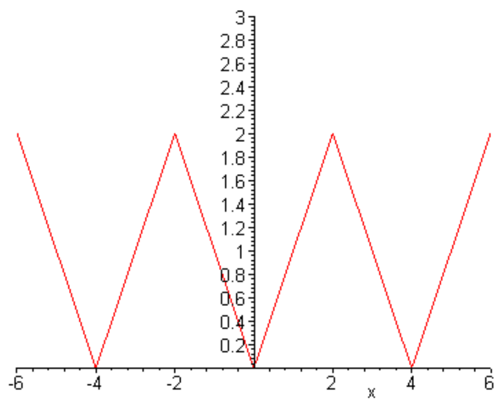
Therefore $f'(-x)=f'(x)$.

36. From Example 1 of Section 10.2, the function

$$f(x)=\begin{cases} -x\,, & -2\le x<0 \\ x, & 0\le x<2\,, \end{cases}$$

$(L=2)$ has a convergent Fourier series

$$f(x) = 1 - \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} cos \frac{(2n-1)\pi x}{2}.$$



Since $f(x)$ is continuous, the series converges everywhere. In particular, at $x = 0$, we have

$$0 = f(0) = 1 - \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}.$$

It follows immediately that

$$\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots.$$

40. Since one objective is to obtain a Fourier series containing only *cosine* terms, any extension of $f(x)$ should be an *even* function. Another objective is to derive a series containing only the terms

$$cos \frac{(2n-1)\pi x}{2L}, \; n = 1, 2, \cdots.$$

First note that the functions

$$cos \frac{n\pi x}{L}, \; n = 1, 2, \cdots$$

are *symmetric* about $x = L$. Indeed,

$$
\begin{aligned}
cos\frac{n\pi(2L-x)}{L} &= cos\Big(2n\pi - \frac{n\pi x}{L}\Big) \\
&= cos\Big(-\frac{n\pi x}{L}\Big) \\
&= cos\frac{n\pi x}{L}\,.
\end{aligned}
$$

It follows that if $f(x)$ is extended into $(L\,,2L)$ as an *antisymmetric* function about $x = L$,
that is, $f(2L - x) = -f(x)$ for $0 \leq x \leq 2L$, then

$$
\int_0^{2L} f(x)cos\frac{n\pi x}{L}\,dx = 0\,.
$$

This follows from the fact that the integrand is *antisymmetric* function about $x = L$. Now
extend the function $f(x)$ to obtain

$$
\tilde{f}\,(x) = \begin{cases} f(x), & 0 \leq x < L \\ -f(2L-x), & L < x < 2L\,. \end{cases}
$$

Finally, extend the resulting function into $(-2L\,,0)$ as an *even* function, and then as a periodic function of period $4L$.

By construction, the Fourier series will contain only *cosine* terms. We first note that

$$
\begin{aligned}
a_0 &= \frac{2}{2L}\int_0^{2L}\tilde{f}\,(x)\,dx \\
&= \frac{1}{L}\int_0^{L} f(x)dx - \frac{1}{L}\int_L^{2L} f(2L-x)dx \\
&= \frac{1}{L}\int_0^{L} f(x)dx - \frac{1}{L}\int_0^{L} f(u)du \\
&= 0\,.
\end{aligned}
$$

For $n > 0$,

$$
\begin{aligned}
a_n &= \frac{2}{2L}\int_0^{2L}\tilde{f}\,(x)cos\frac{n\pi x}{2L}\,dx \\
&= \frac{1}{L}\int_0^{L} f(x)cos\frac{n\pi x}{2L}dx - \frac{1}{L}\int_L^{2L} f(2L-x)cos\frac{n\pi x}{2L}dx\,.
\end{aligned}
$$

For the second integral, let $u = 2L - x$. Then

$$
cos\frac{n\pi x}{2L} = cos\frac{n\pi(2L+u)}{2L} = (-1)^n cos\frac{n\pi u}{2L}
$$

and therefore

$$\int_L^{2L} f(2L-x)\cos\frac{n\pi x}{2L}dx = (-1)^n \int_0^L f(u)\cos\frac{n\pi u}{2L}du\,.$$

Hence

$$a_n = \frac{1-(-1)^n}{L}\int_0^L f(x)\cos\frac{n\pi x}{2L}dx\,.$$

It immediately follows that $a_n = 0$ for $n = 2k$, $k = 0, 1, 2, \cdots$, and

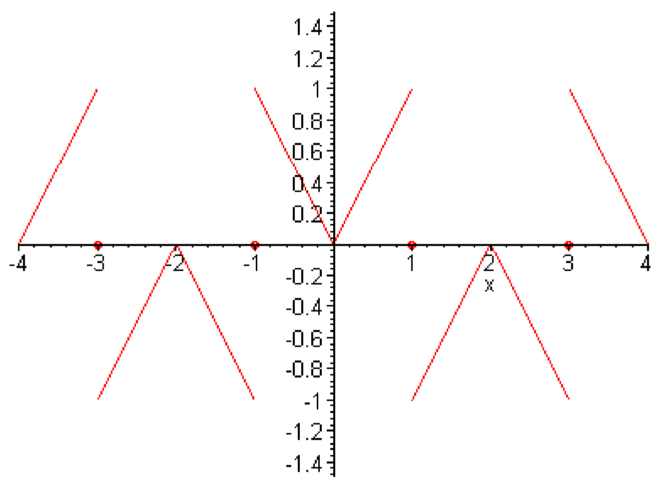
$$a_{2k-1} = \frac{2}{L}\int_0^L f(x)\cos\frac{(2k-1)\pi x}{2L}dx\,,\ \text{for}\ k = 1, 2, \cdots.$$

The associated Fourier series representation

$$f(x) = \sum_{n=0}^{\infty} a_{2n-1}\cos\frac{(2n-1)\pi x}{2L}$$

converges almost everywhere on $(-2L, 2L)$ and hence on $(0, L)$.

For example, if $f(x) = x$ for $0 \le x \le L = 1$, the graph of the extended function is:



Section 10.5

1. We consider solutions of the form $u(x\,,t) = X(x)T(t)$. Substitution into the partial differential equation results in

$$xX''T + XT' = 0\,.$$

Divide both sides of the differential equation by the product XT to obtain

$$x\frac{X''}{X} + \frac{T'}{T} = 0\,,$$

so that

$$x\frac{X''}{X} = -\frac{T'}{T}\,.$$

Since both sides of the resulting equation are functions of different variables, each must be equal to a constant, say λ. We obtain the ordinary differential equations

$$xX'' - \lambda X = 0 \text{ and } T' + \lambda T = 0\,.$$

2. In order to apply the method of separation of variables, we consider solutions of the form $u(x\,,t) = X(x)T(t)$. Substituting the assumed form of the solution into the partial differential equation, we obtain

$$t\,X''T + x\,XT' = 0\,.$$

Divide both sides of the differential equation by the product $xtXT$ to obtain

$$\frac{X''}{xX} + \frac{T'}{t\,T} = 0\,,$$

so that

$$\frac{X''}{xX} = -\frac{T'}{t\,T}\,.$$

Since both sides of the resulting equation are functions of *different* variables, it follows that

$$\frac{X''}{xX} = -\frac{T'}{t\,T} = \lambda\,.$$

Therefore $X(x)$ and $T(t)$ are solutions of the ordinary differential equations

$$X'' - \lambda x\,X = 0 \text{ and } T' + \lambda t\,T = 0\,.$$

4. Assume that the solution of the PDE has the form $u(x\,,t) = X(x)T(t)$. Substitution into the partial differential equation results in

$$[p(x)X']'T - r(x)X\,T'' = 0\,.$$

Divide both sides of the differential equation by the product $r(x)XT$ to obtain

$$\frac{[p(x)X']'}{r(x)X} - \frac{T''}{T} = 0\,,$$

that is,

$$\frac{[p(x)X']'}{r(x)X} = \frac{T''}{T}\,.$$

Since both sides of the resulting equation are functions of different variables, each must be equal to a constant, say $-\lambda$. We obtain the ordinary differential equations

$$[p(x)X']' + \lambda r(x)X = 0 \text{ and } T'' + \lambda T = 0\,.$$

6. We consider solutions of the form $u(x\,,y) = X(x)Y(y)$. Substitution into the partial differential equation results in

$$X''Y + XY'' + xXY = 0\,.$$

Divide both sides of the differential equation by the product XY to obtain

$$\frac{X''}{X} + \frac{Y''}{Y} + x = 0\,,$$

that is,

$$\frac{X''}{X} + x = -\frac{Y''}{Y}\,.$$

Since both sides of the resulting equation are functions of *different* variables, it follows that

$$\frac{X''}{X} + x = -\frac{Y''}{Y} = -\lambda\,.$$

We obtain the ordinary differential equations

$$X'' + (x+\lambda)X = 0 \text{ and } Y'' - \lambda Y = 0\,.$$

7. The heat conduction equation, $100\,u_{xx} = u_t$, and the given boundary conditions are homogeneous. We consider solutions of the form $u(x\,,t) = X(x)T(t)$. Substitution into
the partial differential equation results in

$$100\,X''T = XT'.$$

Divide both sides of the differential equation by the product XT to obtain

$$\frac{X''}{X} = \frac{T'}{100\,T}\,.$$

Since both sides of the resulting equation are functions of *different* variables, it follows that

$$\frac{X''}{X} = \frac{T'}{100\,T} \; = -\,\lambda\,.$$

Therefore $X(x)$ and $T(t)$ are solutions of the ordinary differential equations

$$X'' + \lambda\,X = 0 \text{ and } T' + 100\lambda\,T = 0\,.$$

The general solution of the *spatial* equation is $X = c_1 cos\,\lambda^{1/2}x + c_2\,sin\,\lambda^{1/2}x$. In order to satisfy the homogeneous boundary conditions, we require that $c_1 = 0$, and

$$\lambda^{1/2} = n\pi\,.$$

Hence the eigenfunctions are $X_n = sin\,n\pi x$, with associated eigenvalues $\lambda_n = n^2\pi^2$.

We thus obtain the family of equations $T' + 100\lambda_n\,T = 0$. Solution are given by

$$T_n = e^{-100\lambda_n t}.$$

Hence the fundamental solutions of the PDE are

$$u_n(x\,,t) = e^{-100n^2\pi^2 t} sin\,n\pi x\,,$$

which yield the general solution

$$u(x\,,t) = \sum_{n=1}^{\infty} c_n\,e^{-100n^2\pi^2 t} sin\,n\pi x\,.$$

Finally, the initial condition $u(x\,,0) = sin\,2\pi x - sin\,5\pi x$ must be satisfied. Therefore is it necessary that

$$\sum_{n=1}^{\infty} c_n\,sin\,n\pi x = sin\,2\pi x - sin\,5\pi x\,.$$

It follows from the *othogonality* conditions that $c_2 = -\,c_5 = 1$, with all other $c_n = 0$. Therefore the solution of the given heat conduction problem is

$$u(x\,,t) = e^{-400\pi^2 t} sin\,2\pi x - e^{-2500\pi^2 t} sin\,5\pi x\,.$$

9. The heat conduction problem is formulated as

$$\begin{aligned} u_{xx} &= u_t\,, & 0 < x < 40\,,\ t > 0\,; \\ u(0\,,t) &= 0\,, & u(40\,,t) = 0\,,\ t > 0\,; \\ u(x\,,0) &= 50\,, & 0 < x < 40\,. \end{aligned}$$

Assume a solution of the form $u(x\,,t) = X(x)T(t)$. Following the procedure in this section, we obtain the eigenfunctions $X_n = sin\, n\pi x/40$, with associated eigenvalues $\lambda_n = n^2\pi^2/1600$. The solutions of the *temporal* equations are

$$T_n = e^{-\lambda_n t}.$$

Hence the general solution of the given problem is

$$u(x\,,t) = \sum_{n=1}^{\infty} c_n\, e^{-n^2\pi^2 t/1600} sin\frac{n\pi x}{40}\,.$$

The coefficients c_n are the *Fourier sine* coefficients of $u(x\,,0) = 50$. That is,

$$\begin{aligned} c_n &= \frac{2}{L}\int_0^L f(x) sin\frac{n\pi x}{L} dx \\ &= \frac{5}{2}\int_0^{40} sin\frac{n\pi x}{40} dx \\ &= 100\,\frac{1 - \ cos\, n\pi}{n\pi}\,. \end{aligned}$$

The sine series of the initial condition is

$$50 = \frac{100}{\pi}\sum_{n=1}^{\infty}\frac{1 - cos\, n\pi}{n} sin\frac{n\pi x}{40}\,.$$

Therefore the solution of the given heat conduction problem is

$$u(x\,,t) = \frac{100}{\pi}\sum_{n=1}^{\infty}\frac{1 - cos\, n\pi}{n}\ e^{-n^2\pi^2 t/1600} sin\frac{n\pi x}{40}\,.$$

11. Refer to Prob. 9 for the formulation of the problem. In this case, the initial condition is given by

$$u(x\,,0) = \begin{cases} 0\,, & 0 \le x < 10\,, \\ 50\,, & 10 \le x \le 30\,, \\ 0\,, & 30 < x \le 40\,. \end{cases}$$

All other data being the same, the solution of the given problem is

$$u(x\,,t) = \sum_{n=1}^{\infty} c_n\, e^{-n^2\pi^2 t/1600} sin\frac{n\pi x}{40}\,.$$

The coefficients c_n are the *Fourier sine* coefficients of $u(x\,,0)$. That is,

$$\begin{aligned} c_n &= \frac{2}{L}\int_0^L f(x) sin\frac{n\pi x}{L} dx \\ &= \frac{5}{2}\int_{10}^{30} sin\frac{n\pi x}{40} dx \\ &= 100\,\frac{cos\frac{n\pi}{4} - cos\frac{3n\pi}{4}}{n\pi}\,. \end{aligned}$$

Therefore the solution of the given heat conduction problem is

$$u(x\,,t) = \frac{100}{\pi}\sum_{n=1}^{\infty}\frac{cos\frac{n\pi}{4} - cos\frac{3n\pi}{4}}{n}\; e^{-n^2\pi^2 t/1600} sin\frac{n\pi x}{40}\,.$$

12. Refer to Prob. 9 for the formulation of the problem. In this case, the initial condition is given by

$$u(x\,,0) = x\,, \quad 0 < x < 40\,.$$

All other data being the same, the solution of the given problem is

$$u(x\,,t) = \sum_{n=1}^{\infty} c_n\, e^{-n^2\pi^2 t/1600} sin\frac{n\pi x}{40}\,.$$

The coefficients c_n are the *Fourier sine* coefficients of $u(x\,,0) = x$. That is,

$$\begin{aligned} c_n &= \frac{2}{L}\int_0^L f(x) sin\frac{n\pi x}{L} dx \\ &= \frac{1}{20}\int_0^{40} x\, sin\frac{n\pi x}{40} dx \\ &= -\,80\,\frac{cos\, n\pi}{n\pi}\,. \end{aligned}$$

Therefore the solution of the given heat conduction problem is

$$u(x\,,t) = \frac{80}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\; e^{-n^2\pi^2 t/1600} sin\frac{n\pi x}{40}\,.$$

13. Substituting $x = 20$, into the solution, we have

$$u(20\,,t) = \frac{100}{\pi}\sum_{n=1}^{\infty}\frac{1 - cos\,n\pi}{n}\;e^{-n^2\pi^2 t/1600} sin\frac{n\pi}{2}\;.$$

We can also write

$$u(20\,,t) = \frac{200}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{2k-1}\;e^{-(2k-1)^2\pi^2 t/1600}\,.$$

Therefore,

$$u(20\,,5) = \frac{200}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{2k-1}\;e^{-(2k-1)^2\pi^2/320}.$$

Let

$$A_k = \frac{(-1)^{n+1}200}{\pi(2k-1)}e^{-(2k-1)^2\pi^2/320}.$$

It follows that $|A_k| < 0.005$ for $k \geq 9$. So for $n = 2k - 1 \geq 17$, the summation is unaffected by additional terms.

For $t = 20$,

$$u(20\,,20) = \frac{200}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{2k-1}\;e^{-(2k-1)^2\pi^2/80}.$$

Let

$$A_k = \frac{(-1)^{n+1}200}{\pi(2k-1)}e^{-(2k-1)^2\pi^2/80}.$$

It follows that $|A_k| < 0.003$ for $k \geq 5$. So for $n = 2k - 1 \geq 9$, the summation is unaffected by additional terms.

For $t = 80$,

$$u(20\,,80) = \frac{200}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{2k-1}\;e^{-(2k-1)^2\pi^2/20}.$$

Let

$$A_k = \frac{(-1)^{n+1}200}{\pi(2k-1)}e^{-(2k-1)^2\pi^2/20}.$$

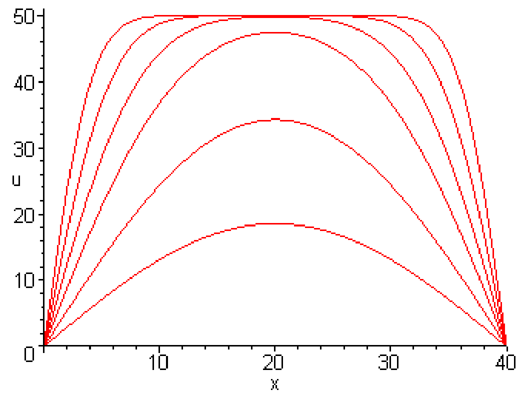
It follows that $|A_k| < 0.00005$ for $k \geq 3$. So for $n = 2k - 1 \geq 5$, the summation is unaffected by additional terms.

The series solution converges *faster* as t increases.

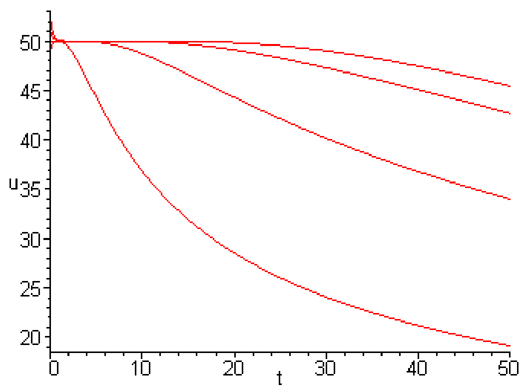
14(a). The solution of the given heat conduction problem is

$$u(x\,,t) = \frac{100}{\pi}\sum_{n=1}^{\infty}\frac{1-cos\,n\pi}{n}\;e^{-n^2\pi^2t/1600}sin\frac{n\pi x}{40}\;.$$

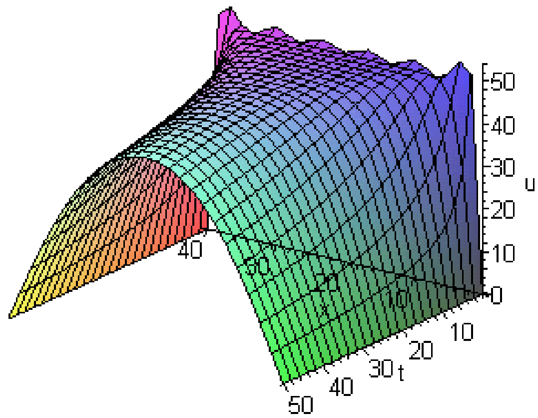
Setting $t = 5, 10, 20, 40, 100, 200$:



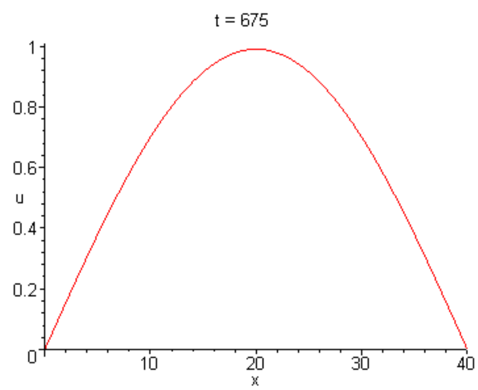
(b). Setting $x = 5, 10, 15, 20$:



(c). Surface plot of $u(x\,,t)$:



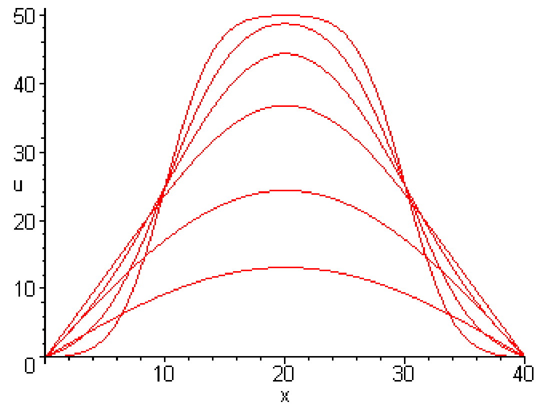
(d). $0 \le u(x\,,t) \le 1$ for $t \ge 675\,sec$.



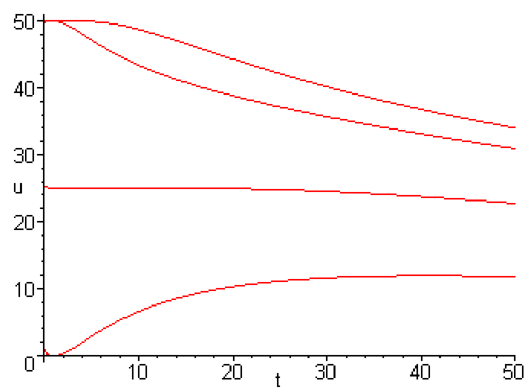
$16(a)$. The solution of the given heat conduction problem is

$$u(x\,,t) = \frac{100}{\pi}\sum_{n=1}^{\infty}\frac{cos\frac{n\pi}{4} - cos\frac{3n\pi}{4}}{n}\,e^{-n^2\pi^2 t/1600}sin\frac{n\pi x}{40}\,.$$

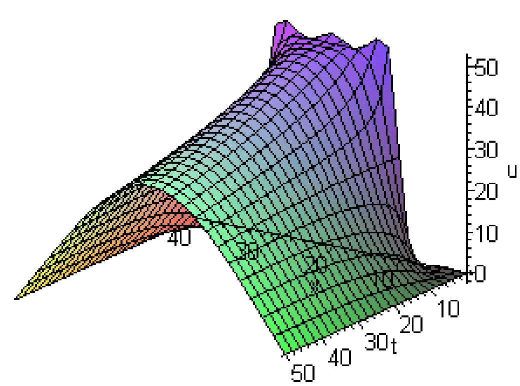
Setting $t = 5, 10, 20, 40, 100, 200$:



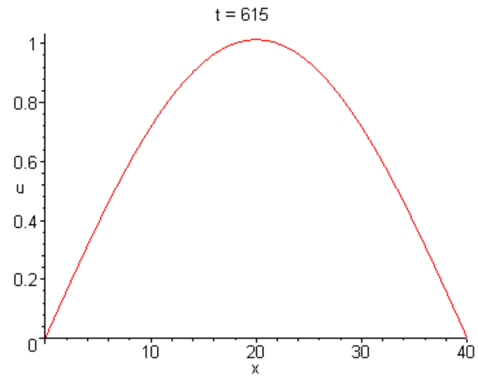
(b). Setting $x = 5, 10, 15, 20$:



(c). Surface plot of $u(x\,, t)$:



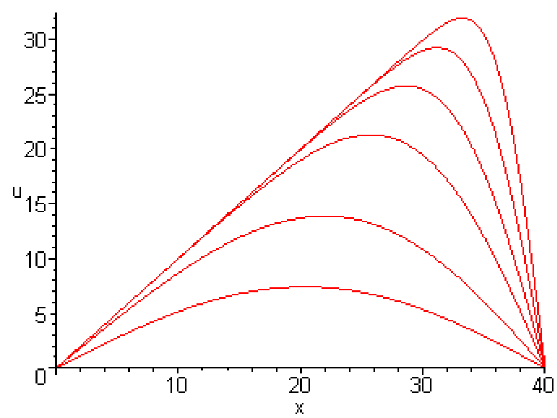
(d). $0 \leq u(x\,,t) \leq 1$ for $t \geq 615\,sec$.



$17(a)$. The solution of the given heat conduction problem is

$$u(x\,,t) = \frac{80}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\,e^{-n^2\pi^2 t/1600}sin\frac{n\pi x}{40}\,.$$

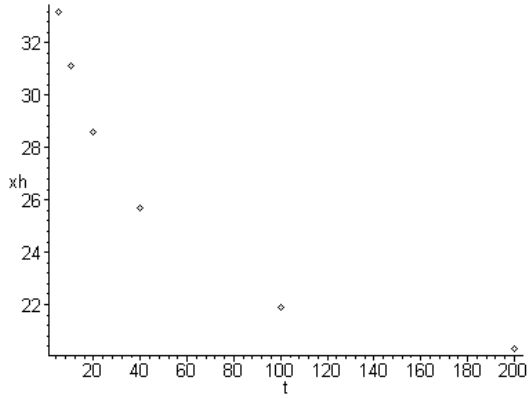
Setting $t = 5, 10, 20, 40, 100, 200$:



(b). Analyzing the individual plots, we find that the 'hot spot' varies with time:

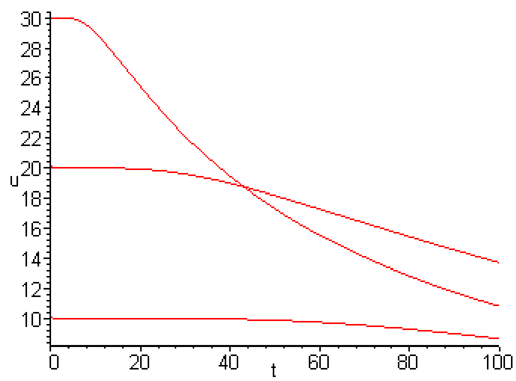
t	5	10	20	40	100	200
x_h	33	31	29	26	22	21

Location of the 'hot spot', x_h , versus *time* :

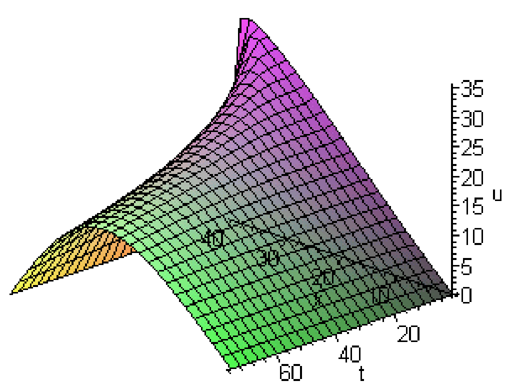


Evidently, the location of the greatest temperature migrates to the center of the rod.

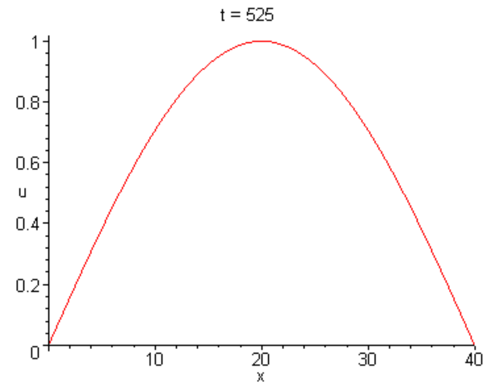
(c). Setting $x = 5, 10, 15, 20$:



(d). Surface plot of $u(x\,,t)$:



(e). $0 \le u(x\,,t) \le 1$ for $t \ge 525\,sec$.



19. The solution of the given heat conduction problem is

$$u(x\,,t) = \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{1-cos\,n\pi}{n}\;e^{-n^2\pi^2\alpha^2 t/400}sin\frac{n\pi x}{20}\,.$$

Setting $x = 10\,cm$,

$$u(10\,,t) = \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{1-cos\,n\pi}{n}\;e^{-n^2\pi^2\alpha^2 t/400}sin\frac{n\pi}{2}\,.$$

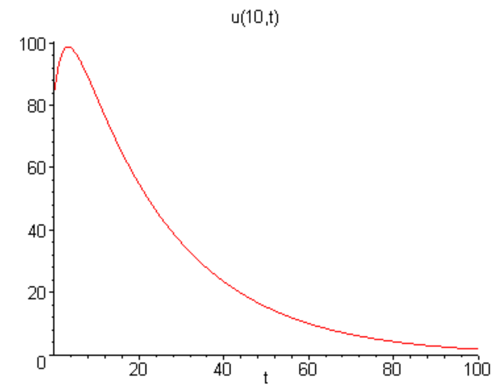
A *two-term* approximation is given by

$$u(10\,,t) \approx \frac{400}{3\pi}\left[3\,e^{-\pi^2\alpha^2 t/400} - e^{-9\pi^2\alpha^2 t/400}\right].$$

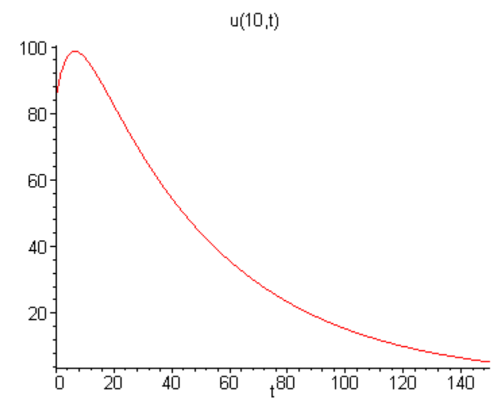
From Table 10.5.1 :

	α^2
silver	1.71
aluminum	0.86
cast iron	0.12

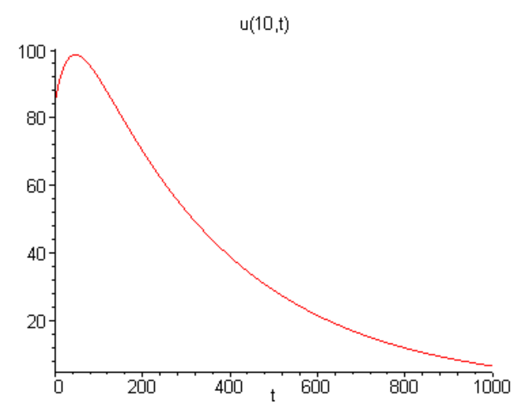
(a). $\alpha^2 = 1.71$:



(b). $\alpha^2 = 0.86$:



(c). $\alpha^2 = 0.12$:



$21(a)$. Given the partial differential equation

$$a\,u_{xx} - b\,u_t + c\,u = 0\,,$$

in which a , b , and c are constants, set $u(x\,,t) = e^{\delta t}w(x\,,t)$. Substitution into the PDE results in

$$a\,e^{\delta t}w_{xx} - b\big(\delta\,e^{\delta t}w + e^{\delta t}w_t\big) + c\,e^{\delta t}w = 0\,.$$

Dividing both sides of the equation by $e^{\delta t}$, we obtain

$$a\,w_{xx} - b\,w_t + (c - b\delta)\,w = 0\,.$$

As long as $b \neq 0$, choosing $\delta = c/b$ yields

$$\frac{a}{b}\,w_{xx} - w_t = 0\,,$$

which is the *heat conduction equation* with dependent variable w .

23. The heat conduction equation in *polar coordinates* is given by

$$\alpha^2\left[u_{rr} + \frac{1}{r}\,u_r + \frac{1}{r^2}\,u_{\theta\theta}\right] = u_t\,.$$

We consider solutions of the form $u(r\,,\theta\,,t) = R(r)\Theta(\theta)T(t)$. Substitution into the PDE
results in

$$\alpha^2\left[R''\Theta T + \frac{1}{r}\,R'\Theta T + \frac{1}{r^2}\,R\Theta'' T\right] = R\Theta T'\,.$$

Dividing both sides of the equation by the factor $R\Theta T$, we obtain

$$\frac{R''}{R} + \frac{1}{r}\,\frac{R'}{R} + \frac{1}{r^2}\,\frac{\Theta''}{\Theta} = \frac{T'}{\alpha^2 T}\,.$$

Since both sides of the resulting differential equation depend on *different* variables, each side must be equal to a constant, say $-\lambda$. That is,

$$\frac{R''}{R} + \frac{1}{r}\,\frac{R'}{R} + \frac{1}{r^2}\,\frac{\Theta''}{\Theta} = \frac{T'}{\alpha^2 T} = -\lambda^2\,.$$

It follows that $T' + \alpha^2\lambda^2 T = 0$, and

$$\frac{R''}{R} + \frac{1}{r}\,\frac{R'}{R} + \frac{1}{r^2}\,\frac{\Theta''}{\Theta} = -\lambda^2\,.$$

Multiplying both sides of this differential equation by r^2, we find that

$$r^2\frac{R''}{R} + r\,\frac{R'}{R} + \frac{\Theta''}{\Theta} = -\lambda^2 r^2\,,$$

which can be written as

$$r^2\frac{R''}{R} + r\,\frac{R'}{R} + \lambda^2 r^2 = \,-\,\frac{\Theta''}{\Theta}\,.$$

Once again, since both sides of the resulting differential equation depend on *different* variables, each side must be equal to a constant. Hence

$$r^2\frac{R''}{R} + r\,\frac{R'}{R} + \lambda^2 r^2 = \mu^2 \text{ and } \,-\,\frac{\Theta''}{\Theta} = \mu^2.$$

The resulting ordinary equations are

$$\begin{aligned} r^2 R'' + rR' + \left(\lambda^2 r^2 - \mu^2\right)R &= 0 \\ \Theta'' + \mu^2\Theta &= 0 \\ T' + \alpha^2\lambda^2 T &= 0\,. \end{aligned}$$

Section 10.6

1. The steady-state solution, $v(x)$, satisfies the boundary value problem

$$v''(x) = 0\,,\ 0 < x < 50\,,\ v(0) = 10\ ,v(50) = 40\,.$$

The general solution of the ODE is $v(x) = Ax + B$. Imposing the boundary conditions, we have

$$v(x) = \frac{40 - 10}{50}x + 10 = \frac{3x}{5} + 10\,.$$

2. The steady-state solution, $v(x)$, satisfies the boundary value problem

$$v''(x) = 0\,,\ 0 < x < 40\,,\ v(0) = 30\ ,v(40) = -20\,.$$

The solution of the ODE is *linear*. Imposing the boundary conditions, we have

$$v(x) = \frac{-20 - 30}{40}x + 30 = -\frac{5x}{4} + 30\,.$$

4. The steady-state solution is also a solution of the boundary value problem given by $v''(x) = 0\,,\ 0 < x < L$, and the conditions $v'(0) = 0\,, v(L) = T$. The solution of the ODE is $v(x) = Ax + B$. The boundary condition $v'(0) = 0$ requires that $A = 0$. The other condition requires that $B = T$. Hence $v(x) = T$.

5. As in Prob. 4, the steady-state solution has the form $v(x) = Ax + B$. The boundary condition $v(0) = 0$ requires that $B = 0$. The boundary condition $v'(L) = 0$ requires that $A = 0$. Hence $v(x) = 0$.

6. The steady-state solution has the form $v(x) = Ax + B$. The first boundary condition, $v(0) = T$, requires that $B = T$. The other boundary condition, $v'(L) = 0$, requires that $A = 0$. Hence $v(x) = T$.

8. The steady-state solution, $v(x)$, satisfies the differential equation $v''(x) = 0$, along with the boundary conditions

$$v(0) = T\ ,\ v'(L) + v(L) = 0\,.$$

The general solution of the ODE is $v(x) = Ax + B$. The boundary condition $v'(0) = 0$ requires that $B = T$. It follows that $v(x) = Ax + T$, and

$$v'(L) + v(L) = A + AL + T\,.$$

The second boundary condition requires that $A = -T/(1 + L)$. Therefore

$$v(x) = -\frac{Tx}{1 + L} + T\,.$$

$10(a)$. Based on the *symmetry* of the problem, consider only *left* half of the bar. The steady-state solution satisfies the ODE $v''(x) = 0$, along with the boundary conditions $v(0) = 0$ and $v(50) = 100$. The solution of this boundary value problem is $v(x) = 2x$. It follows that the steady-state temperature is the *entire* rod is given by

$$f(x) = \begin{cases} 2x\,, & 0 \le x \le 50 \\ 200 - 2x\,, & 50 \le x \le 100\,. \end{cases}$$

(b). The heat conduction problem is formulated as

$$\begin{aligned} \alpha^2 u_{xx} &= u_t\,, \qquad 0 < x < 100\,,\ t > 0\,; \\ u(0\,,t) &= 20\,, \qquad u(100\,,t) = 0\,,\ t > 0\,; \\ u(x\,,0) &= f(x)\,, \qquad 0 < x < 100\,. \end{aligned}$$

First express the solution as $u(x\,,t) = g(x) + w(x\,,t)$, where $g(x) = -x/5 + 20$ and w satisfies the heat conduction problem

$$\begin{aligned} \alpha^2 w_{xx} &= w_t\,, \qquad 0 < x < 100\,,\ t > 0\,; \\ w(0\,,t) &= 0\,, \qquad w(100\,,t) = 0\,,\ t > 0\,; \\ w(x\,,0) &= f(x) - g(x)\,, \qquad 0 < x < 100\,. \end{aligned}$$

Based on the results in Section 10.5,

$$w(x\,,t) = \sum_{n=1}^{\infty} c_n\, e^{-n^2\pi^2\alpha^2 t/10000} sin\frac{n\pi x}{100}\,,$$

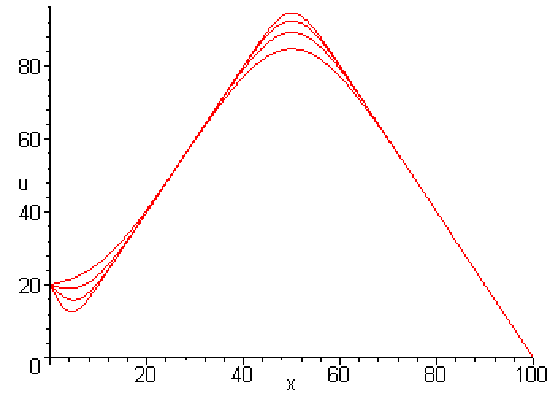
in which the coefficients c_n are the Fourier sine coefficients of $f(x) - g(x)$. That is,

$$\begin{aligned} c_n &= \frac{2}{L}\int_0^L [f(x) - g(x)] sin\frac{n\pi x}{L} dx \\ &= \frac{1}{50}\int_0^{100} [f(x) - g(x)] sin\frac{n\pi x}{100} dx \\ &= 40\,\frac{20\, sin\frac{n\pi}{2} - n\pi}{n^2\pi^2}\,. \end{aligned}$$

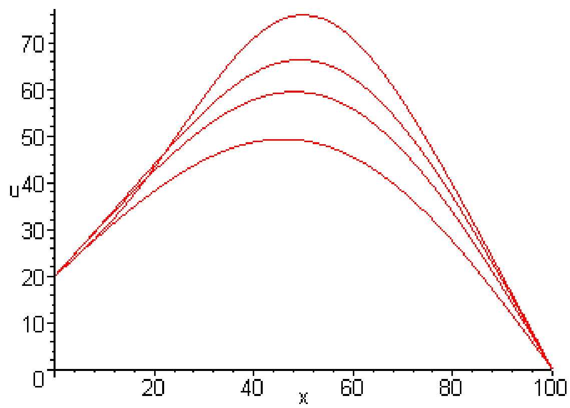
Finally, the *thermal diffusivity* of copper is $1.14\, cm^2/sec$. Therefore the temperature distribution in the rod is

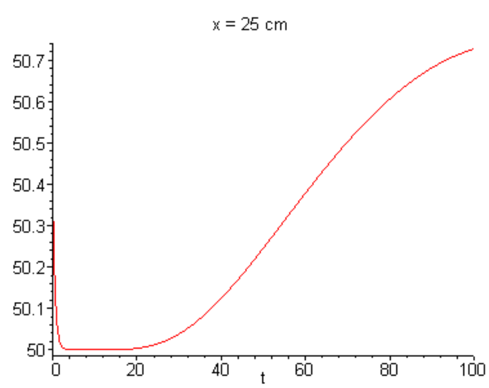
$$u(x\,,t) = 20 - \frac{x}{5} + \frac{40}{\pi^2}\sum_{n=1}^{\infty}\frac{20\, sin\frac{n\pi}{2} - n\pi}{n^2}\, e^{-1.14\, n^2\pi^2 t/10000} sin\frac{n\pi x}{100}\,.$$

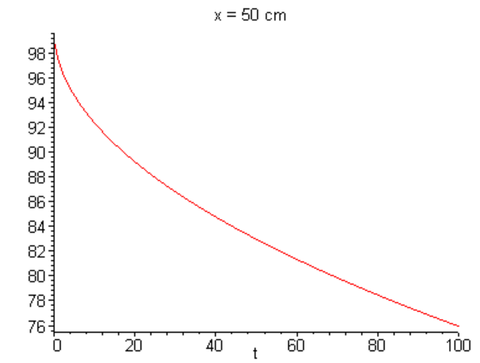
(c). $t = 5\,, 10, 20, 40\, sec$:

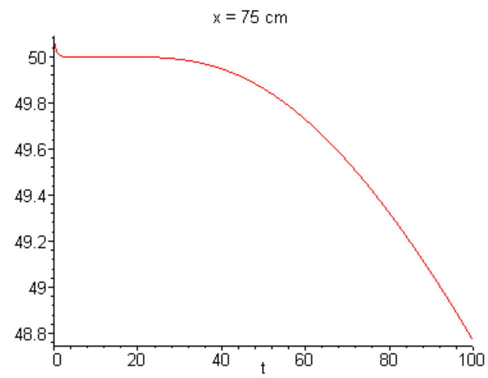


$t = 100, 200, 300, 500\, sec$:

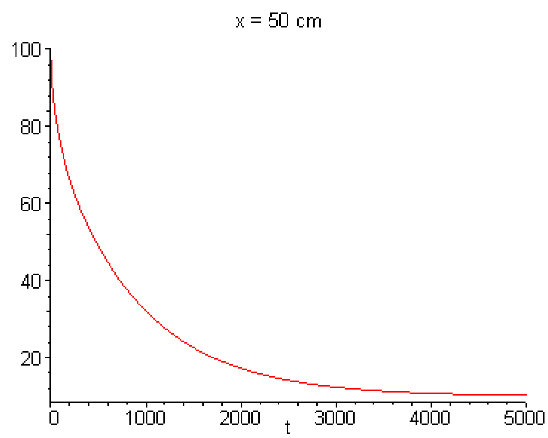








(d). The steady-state temperature of the center of the rod will be $g(50) = 10°C$.



Using a one-term approximation,

$$u(x\,,t) \approx 10 + \frac{800 - 40\,\pi}{\pi^2}\;e^{-1.14\,\pi^2 t/10000}.$$

Numerical investigation shows that $10 < u(50\,,t) < 11$ for $t \geq 3755\,sec$.

$11(a)$. The heat conduction problem is formulated as

$$u_{xx} = u_t\,, \qquad 0 < x < 30\,,\ t > 0\,;$$
$$u(0\,,t) = 30\,, \qquad u(30\,,t) = 0\,,\ t > 0\,;$$
$$u(x\,,0) = f(x)\,, \qquad 0 < x < 30\,,$$

in which the initial condition is given by $f(x) = x(60 - x)/30$. Express the solution as $u(x\,,t) = v(x) + w(x\,,t)$, where $v(x) = 30 - x$ and w satisfies the heat conduction problem

$$w_{xx} = w_t\,, \qquad 0 < x < 30\,,\ t > 0\,;$$
$$w(0\,,t) = 0\,, \qquad w(30\,,t) = 0\,,\ t > 0\,;$$
$$w(x\,,0) = f(x) - v(x)\,, \qquad 0 < x < 30\,.$$

As shown in Section 10.5,

$$w(x\,,t) = \sum_{n=1}^{\infty} c_n\, e^{-n^2\pi^2 t/900} sin\frac{n\pi x}{30}\,,$$

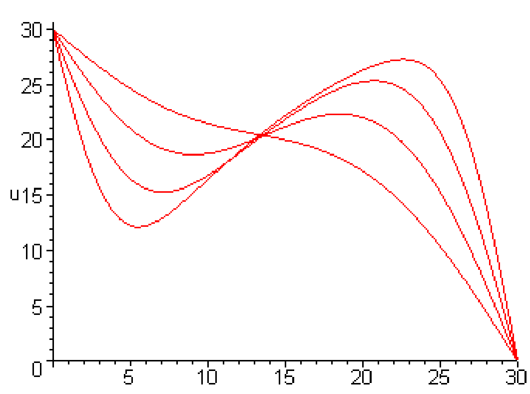
in which the coefficients c_n are the Fourier sine coefficients of $f(x) - v(x)$. That is,

$$\begin{aligned} c_n &= \frac{2}{L}\int_0^L [f(x) - g(x)]sin\frac{n\pi x}{L}dx \\ &= \frac{1}{15}\int_0^{30} [f(x) - g(x)]sin\frac{n\pi x}{30}dx \\ &= 60\,\frac{2(1 - cos\,n\pi) - n^2\pi^2(1 + cos\,n\pi)}{n^3\pi^3}\,. \end{aligned}$$

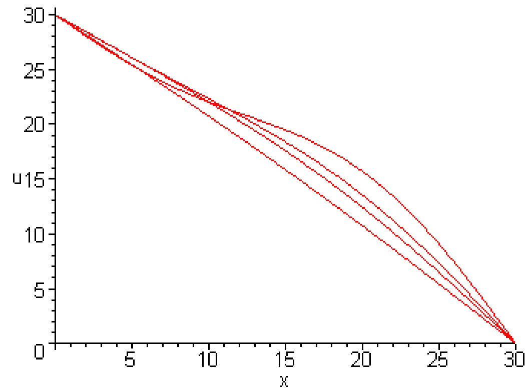
Therefore the temperature distribution in the rod is

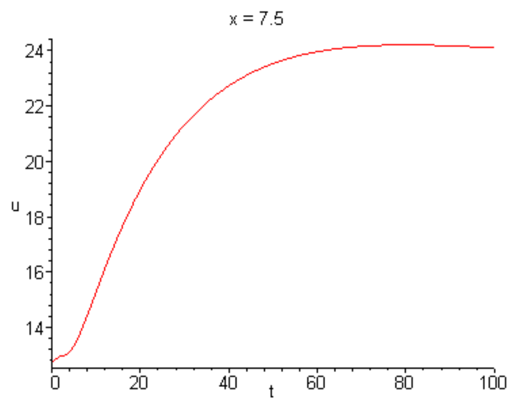
$$u(x\,,t) = 30 - x + \frac{60}{\pi^3}\sum_{n=1}^{\infty}\frac{2(1 - cos\,n\pi) - n^2\pi^2(1 + cos\,n\pi)}{n^3}\, e^{-n^2\pi^2 t/900} sin\frac{n\pi x}{30}\,.$$

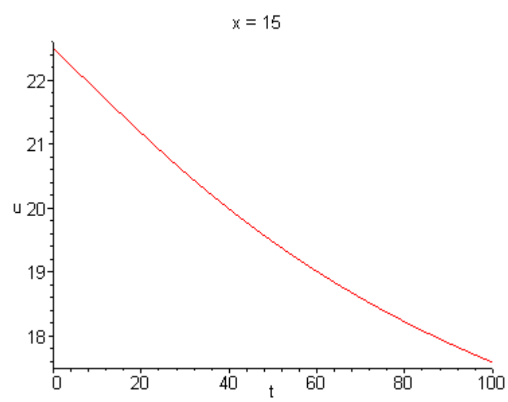
(b). $t = 5\,, 10, 20, 40\, sec$:

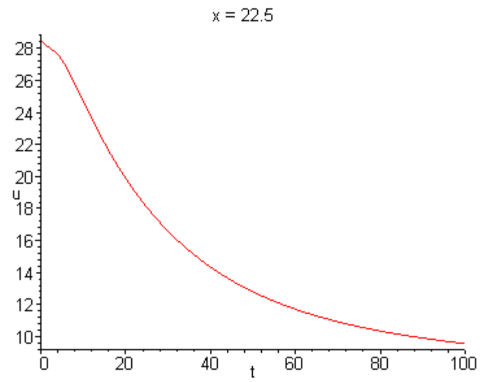


$t = 50, 75, 100, 200\,sec$:

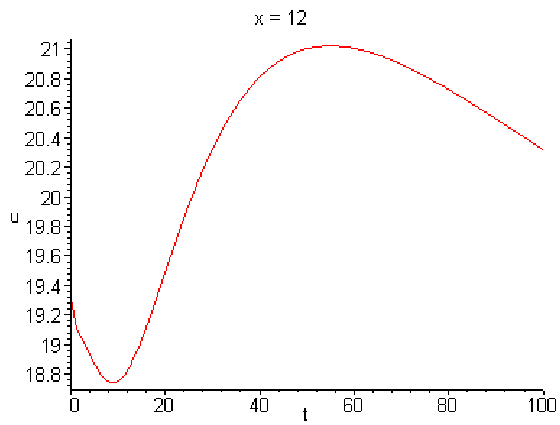








(c).



Based on the *heat conduction equation*, the rate of change of the temperature at any given point is proportional to the *concavity* of the graph of u versus x, that is, u_{xx}. Evidently, near $t = 60$, the concavity of $u(x, t)$ changes.

$13(a)$. The heat conduction problem is formulated as

$$\begin{aligned} u_{xx} = 4\,u_t\,, &\qquad 0 < x < 40\,,\ t > 0\,; \\ u_x(0\,,t) = 0\,, &\qquad u_x(40\,,t) = 0\,,\ t > 0\,; \\ u(x\,,0) = f(x)\,, &\qquad 0 < x < 40\,, \end{aligned}$$

in which the initial condition is given by $f(x) = x(60 - x)/30$.

As shown in the discussion on rods with *insulated ends*, the solution is given by

$$u(x\,,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n\, e^{-n^2\pi^2\alpha^2 t/1600} cos\frac{n\pi x}{40}\,,$$

where c_n are the Fourier cosine coefficients. In this problem,

$$
\begin{aligned}
c_0 &= \frac{2}{L}\int_0^L f(x)dx \\
&= \frac{1}{20}\int_0^{40} \frac{x(60-x)}{30}dx \\
&= 400/9\,,
\end{aligned}
$$

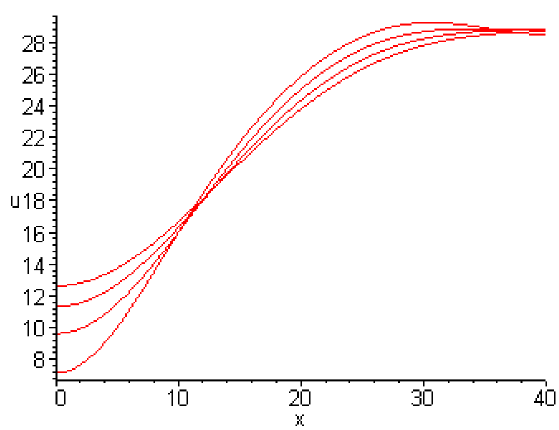
and for $n \geq 1$,

$$
\begin{aligned}
c_n &= \frac{2}{L}\int_0^L f(x)cos\frac{n\pi x}{L}dx \\
&= \frac{1}{20}\int_0^{40} \frac{x(60-x)}{30}\,cos\frac{n\pi x}{40}dx \\
&= -\,\frac{160(3+\,cos\,n\pi)}{3n^2\pi^2}\,.
\end{aligned}
$$

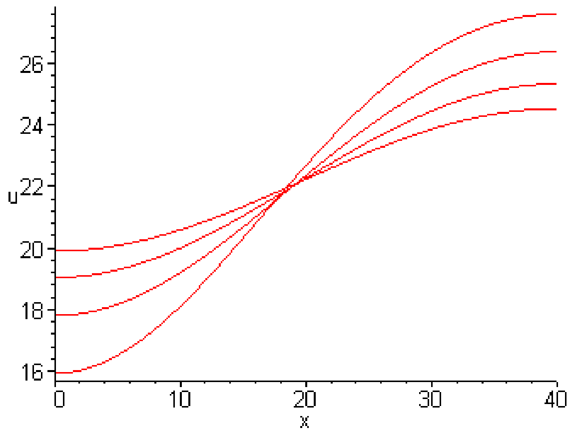
Therefore the temperature distribution in the rod is

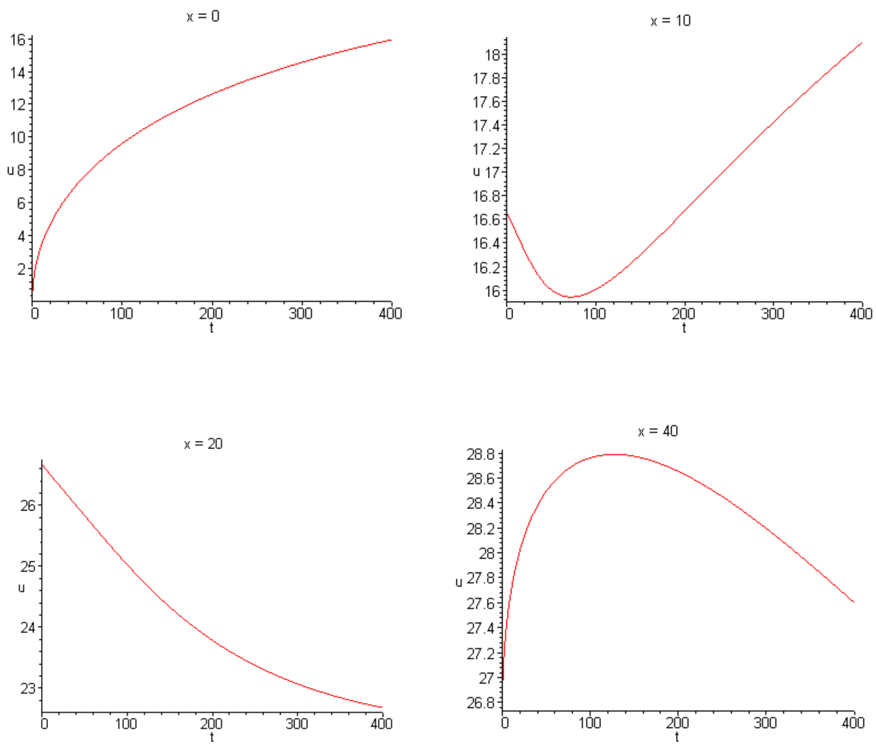
$$
u(x\,,t) = \frac{200}{9} - \frac{160}{3\pi^2}\sum_{n=1}^{\infty}\frac{(3+\,cos\,n\pi)}{n^2}\,e^{-n^2\pi^2 t/6400}cos\frac{n\pi x}{40}\,.
$$

(b). $t = 50, 100, 150, 200\,sec$:



$t = 40, 600, 800, 1000\, sec$:





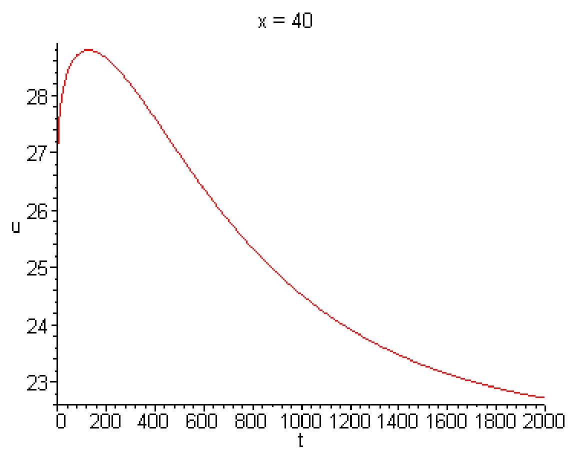
(c). Since

$$\lim_{t\to\infty} e^{-n^2\pi^2 t/6400} cos \frac{n\pi x}{40} = 0$$

for each x, it follows that the steady-state temperature is $u_\infty = 200/9$.

(d). We first note that

$$u(40\,,t)=\frac{200}{9}-\frac{160}{3\pi^2}\sum_{n=1}^{\infty}\frac{(\,-\,1)^n(3+\,\cos n\pi)}{n^2}\,e^{-n^2\pi^2t/6400}.$$



For large values of t, an approximation is given by

$$u(40\,,t)\approx\frac{200}{9}+\frac{320}{3\pi^2}\;e^{-\pi^2t/6400}.$$

Numerical investigation shows that $22.22<u(40\,,t)<23.22$ for $t\geq 1550\,sec$.

$16(a)$. The heat conduction problem is formulated as

$$\begin{aligned} u_{xx}&=u_t\,, & 0<x<30\,,\ t>0\,;\\ u(0\,,t)&=0\,, & u_x(30\,,t)=0\,,\ t>0\,;\\ u(x\,,0)&=f(x)\,, & 0<x<30\,,\end{aligned}$$

in which the initial condition is given by $f(x)=30-x$. Based on the results of Prob. 15,
the solution is given by

$$u(x\,,t)=\sum_{n=1}^{\infty}c_n\,e^{-(2n-1)^2\pi^2t/3600}sin\frac{n\pi x}{60}\,,$$

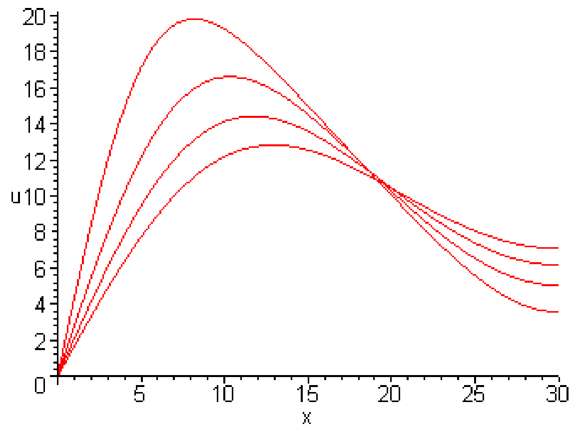
in which

$$\begin{aligned} c_n&=\frac{2}{L}\int_0^L f(x)sin\frac{(2n-1)\pi x}{60}dx\\ &=\frac{1}{15}\int_0^{30}(30-x)sin\frac{(2n-1)\pi x}{60}dx\\ &=120\frac{2\cos n\pi+(2n-1)\pi}{(2n-1)^2\pi^2}\,.\end{aligned}$$

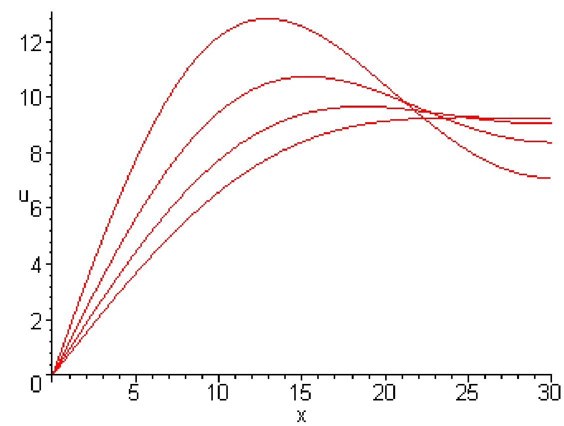
Therefore the solution of the heat conduction problem is

$$u(x\,,t) = 120\sum_{n=1}^{\infty}\frac{2\,cos\,n\pi + (2n-1)\pi}{(2n-1)^2\pi^2}\;e^{-(2n-1)^2\pi^2 t/3600}\,sin\frac{n\pi x}{60}\,.$$

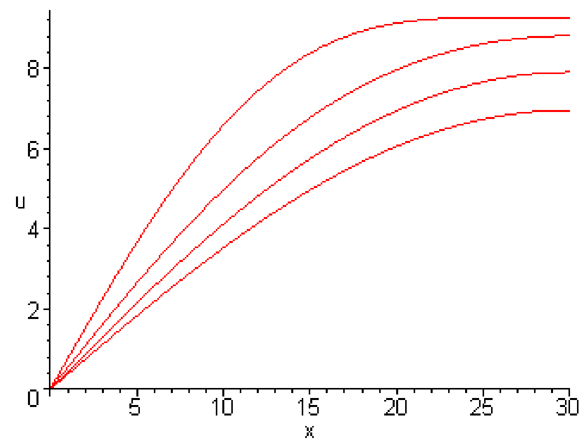
(b). $t = 10, 20, 30, 40\,sec$:

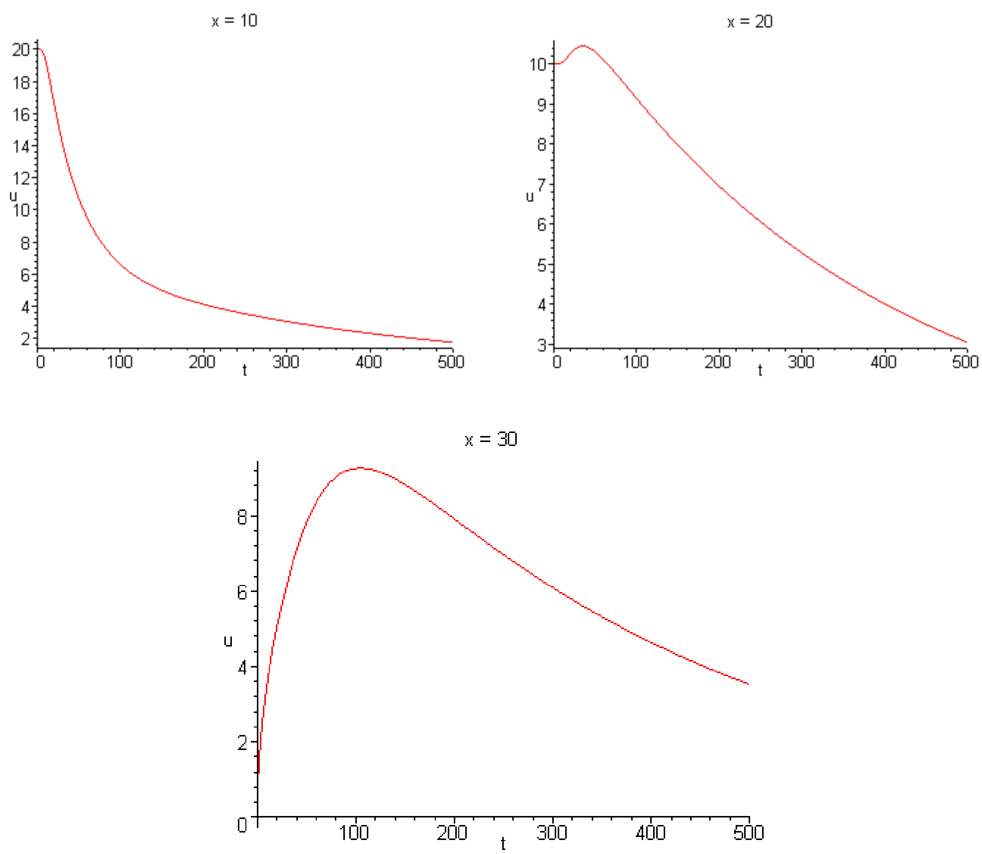


$t = 40, 60, 80, 100\,sec$:

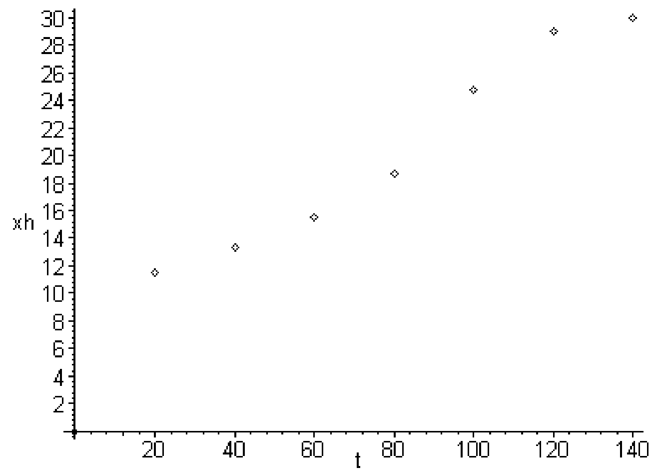


$t = 100, 150, 200, 250\, sec$:



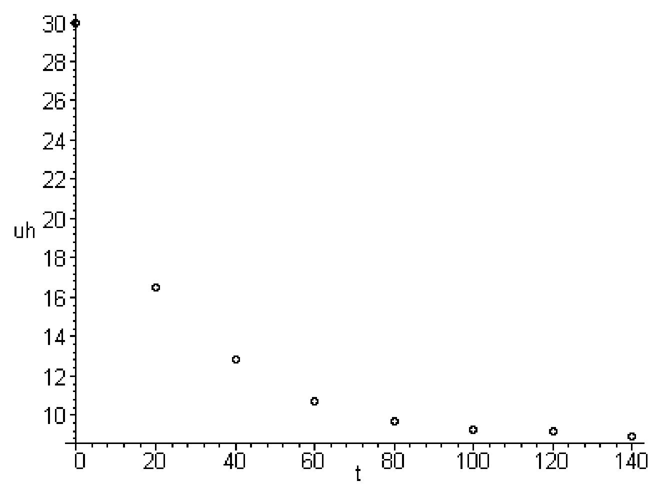


(c).



The location of x_h moves from $x = 0$ to $x = 30$.

(d).



$17(a)$. The heat conduction problem is formulated as

$$\begin{aligned} u_{xx} &= u_t\,, & 0 < x < 30\,,\ t > 0\,; \\ u(0\,,t) &= 40\,, & u_x(30\,,t) = 0\,,\ t > 0\,; \\ u(x\,,0) &= 30 - x\,, & 0 < x < 30\,, \end{aligned}$$

The steady-state temperature satisfies the boundary value problem

$$v'' = 0\,, v(0) = 40 \text{ and } v'(30) = 0\,.$$

It easy to see we must have $v(x) = 40$. Express the solution as

$$u(x\,,t) = 40 + w(x\,,t)\,,$$

in which w satisfies the heat conduction problem

$$\begin{aligned} w_{xx} &= w_t \,, && 0 < x < 30\,,\ t > 0\,; \\ w(0\,,t) &= 0\,, && w_x(30\,,t) = 0\,,\ t > 0\,; \\ w(x\,,0) &= -10 - x\,, && 0 < x < 30\,. \end{aligned}$$

As shown in Prob. 15, the solution is given by

$$w(x\,,t) = \sum_{n=1}^{\infty} c_n\, e^{-(2n-1)^2\pi^2 t/3600} sin\frac{n\pi x}{60}\,,$$

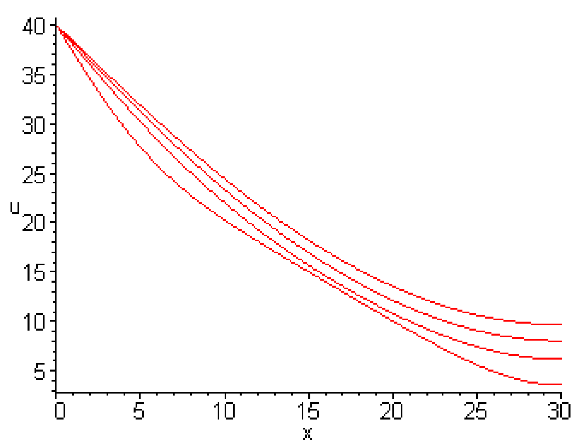
in which

$$\begin{aligned} c_n &= \frac{2}{L}\int_0^L f(x) sin\frac{(2n-1)\pi x}{60}dx \\ &= \frac{1}{15}\int_0^{30}(-10-x) sin\frac{(2n-1)\pi x}{60}dx \\ &= 40\frac{6\,cos\,n\pi - (2n-1)\pi}{(2n-1)^2\pi^2}\,. \end{aligned}$$

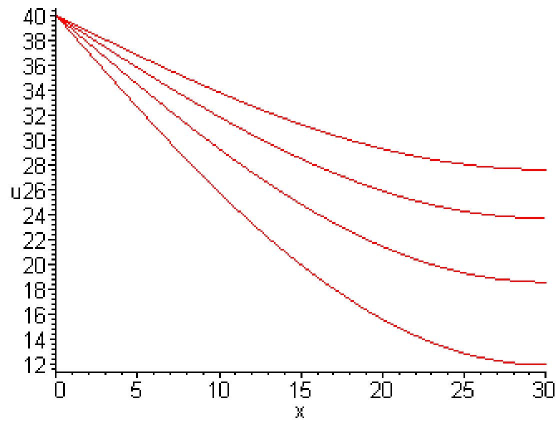
Therefore the solution of the *original* heat conduction problem is

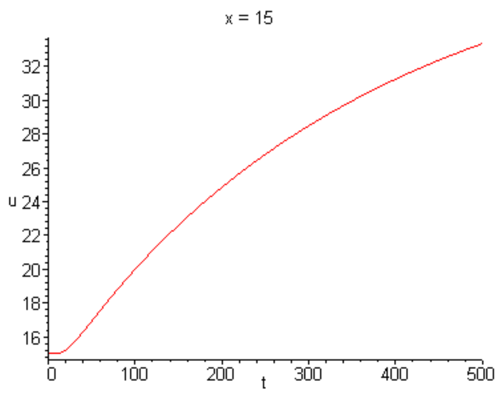
$$u(x\,,t) = 40 + 40\sum_{n=1}^{\infty}\frac{6\,cos\,n\pi - (2n-1)\pi}{(2n-1)^2\pi^2}\, e^{-(2n-1)^2\pi^2 t/3600} sin\frac{n\pi x}{60}\,.$$

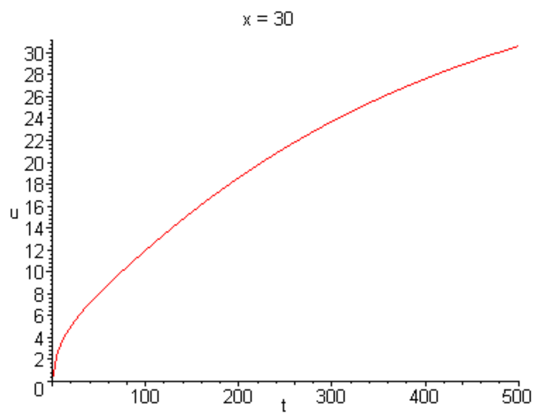
$(b).\ \ t = 10, 30, 50, 70\,sec$:



$t = 100, 200, 300, 400\, sec$:







(c). Observe the concavity of the curves. Note also that the temperature at the *insulated* end tends to the value of the fixed temperature at the boundary $x = 0$.

18. Setting $\lambda = \mu^2$, the general solution of the ODE $X'' + \mu^2 X = 0$ is

$$X(x) = k_1 e^{i\mu x} + k_2 e^{-i\mu x}.$$

The boundary conditions $y'(0) = y'(L) = 0$ lead to the system of equations

$$\begin{aligned} \mu k_1 - \mu k_2 &= 0 \\ \mu k_1 e^{i\mu L} - \mu k_2 e^{-i\mu L} &= 0\,. \end{aligned} \qquad (*)$$

If $\mu = 0$, then the solution of the ODE is $X = Ax + B$. The boundary conditions require that $X = B$.

If $\mu \neq 0$, then the system algebraic equations has a *nontrivial* solution if and only if the coefficient matrix is *singular*. Set the determinant equal to zero to obtain

$$e^{-i\mu L} - e^{i\mu L} = 0\,.$$

Let $\mu = \nu + i\sigma$. Then $i\mu L = i\nu L - \sigma L$, and the previous equation can be written as

$$e^{\sigma L}e^{-i\nu L} - e^{-\sigma L}e^{i\nu L} = 0\,.$$

Using Euler's relation, $e^{i\nu L} = cos\,\nu L + i\,sin\,\nu L$, we obtain

$$e^{\sigma L}(cos\,\nu - i\,sin\,\nu) - e^{-\sigma L}(cos\,\nu + i\,sin\,\nu) = 0\,.$$

Equating the real and imaginary parts of the equation,

$$\left(e^{\sigma L} - e^{-\sigma L}\right)cos\,\nu L = 0$$
$$\left(e^{\sigma L} + e^{-\sigma L}\right)sin\,\nu L = 0\,.$$

Based on the second equation, $\nu L = n\pi$, $n \in \mathbb{I}$. Since $cos\,nL \neq 0$, it follows that $e^{\sigma L} = e^{-\sigma L}$, or $e^{2\sigma L} = 1$. Hence $\sigma = 0$, and $\mu = n\pi/L$, $n \in \mathbb{I}$.

Note that if $\sigma \neq 0$, then the last two equations have no solution. It follows that the system
of equations $(*)$ has *no nontrivial solutions*.

20(a). Consider solutions of the form $u(x\,,t) = X(x)T(t)$. Substitution into the partial differential equation results in

$$\alpha^2 X''T = T'.$$

Divide both sides of the differential equation by the product XT to obtain

$$\frac{X''}{X} = \frac{T'}{\alpha^2\,T}\,.$$

Since both sides of the resulting equation are functions of different variables, each must be equal to a constant, say $-\lambda$. We obtain the ordinary differential equations

$$X'' + \lambda X = 0 \text{ and } T' + \lambda\alpha^2\,T = 0\,.$$

Invoking the first boundary condition,

$$u(0\,,t) = X(0)T(t) = 0\,.$$

At the other boundary,

$$u_x(L\,,t) + \gamma\,u(L\,,t) = [X'(L) + \gamma\,X(L)]T(t) = 0\,.$$

Since these conditions are valid for all $t > 0$, it follows that

$$X(0) = 0 \text{ and } X'(L) + \gamma\,X(L) = 0\,.$$

(b). We consider the boundary value problem

$$X'' + \lambda X = 0\,,\; 0 < x < L\,; \qquad (*)$$
$$X(0) = 0\;,\; X'(L) + \gamma\, X(L) = 0\,.$$

Assume that λ is real, with $\lambda = -\mu^2$. The general solution of the ODE is

$$X(x) = c_1 cosh(\mu x) + c_2 sinh(\mu x)\,.$$

The first boundary condition requires that $c_1 = 0$. Imposing the second boundary condition,

$$c_2\,\mu cosh(\mu L) + \gamma\, c_2 sinh(\mu L) = 0\,.$$

If $c_2 \neq 0$, then $\mu cosh(\mu L) + \gamma\, sinh(\mu L) = 0$, which can also be written as

$$(\mu + \gamma)e^{\mu L} - (\mu + \gamma)e^{-\mu L} = 0\,.$$

If $\gamma = -\mu$, then it follows that $cosh(\mu L) = sinh(\mu L)$, and hence $\mu = 0$. If $\gamma \neq -\mu$, then $e^{\mu L} = e^{-\mu L}$ again implies that $\mu = 0$. For the case $\mu = 0$, the general solution is $X(x) = Ax + B$. Imposing the boundary conditions, we have $B = 0$ and

$$A + \gamma\, AL = 0\,.$$

If $\gamma = -1/L$, then $X(x) = Ax$ is a solution of $(*)$. Otherwise $A = 0$.

(c). Let $\lambda = \mu^2$, with $\mu > 0$. The general solution of $(*)$ is

$$X(x) = c_1 cos(\mu x) + c_2 sin(\mu x)\,.$$

The first boundary condition requires that $c_1 = 0$. From the second boundary condition,

$$c_2\,\mu cos(\mu L) + \gamma\, c_2 sin(\mu L) = 0\,.$$

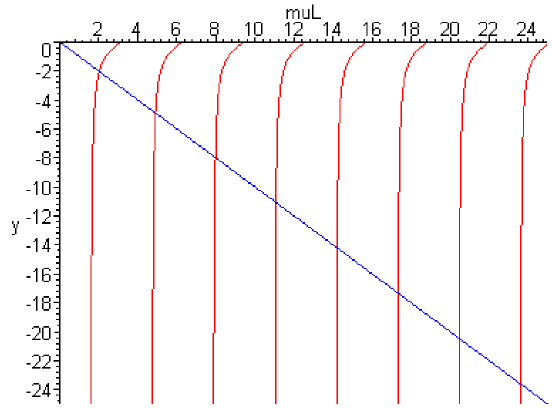
For a nontrivial solution, we must have

$$\mu cos(\mu L) + \gamma\, sin(\mu L) = 0\,.$$

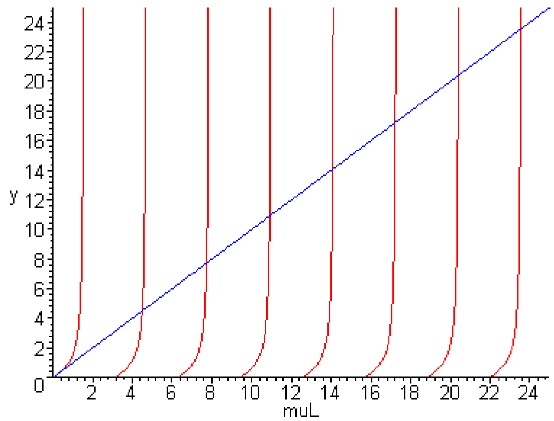
(d). The last equation can also be written as

$$tan\,\mu L = -\frac{\mu}{\gamma}\,. \qquad (**)$$

The eigenvalues λ obtained from the solutions of $(**)$, which are *infinite* in number. In the graph below, we assume $\gamma L = 1$.



For $\gamma L = -1$:



Denote the nonzero solutions of $(**)$ by $\mu_1 , \mu_2 , \mu_3 , \cdots$.

(e). We can in principle calculate the eigenvalues $\lambda_n = \mu_n^2$. Hence the associated eigenfunctions are $X_n = sin\, \mu_n x$. Furthermore, the solutions of the temporal equations are $T_n = exp(-\alpha^2 \mu_n^2\, t)$. The fundamental solutions of the heat conduction problem are given as

$$u_n(x\,, t) = e^{-\alpha^2 \mu_n^2\, t} sin\, \mu_n x\,,$$

which lead to the general solution

$$u(x\,, t) = \sum_{n=1}^{\infty} c_n\, e^{-\alpha^2 \mu_n^2\, t} sin\, \mu_n x\,.$$

Section 10.7

$2(a)$. The initial velocity is *zero*. Therefore the solution, as given by Eq. (20), is

$$u(x\,,t) = \sum_{n=1}^{\infty} c_n \, sin\frac{n\pi x}{L}\, cos\frac{n\pi a\, t}{L}\,,$$

in which the coefficients are the Fourier *sine* coefficients of $f(x)$. That is,

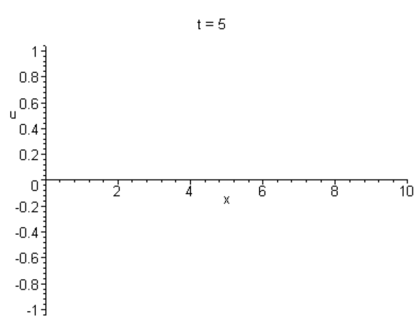
$$\begin{aligned} c_n &= \frac{2}{L}\int_0^L f(x) sin\frac{n\pi x}{L} dx \\ &= \frac{2}{L}\left[\int_0^{L/4} \frac{4x}{L} sin\frac{n\pi x}{L} dx + \int_{L/4}^{3L/4} sin\frac{n\pi x}{L} dx + \int_{3L/4}^{L} \frac{4L-4x}{L} sin\frac{n\pi x}{L} dx\right] \\ &= 8\,\frac{sin\, n\pi/4 + sin\, 3n\pi/4}{n^2\pi^2}\,. \end{aligned}$$

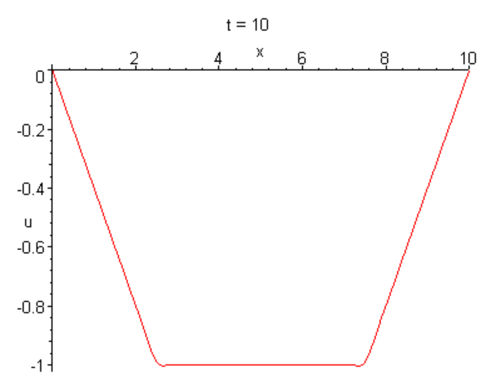
Therefore the displacement of the string is given by

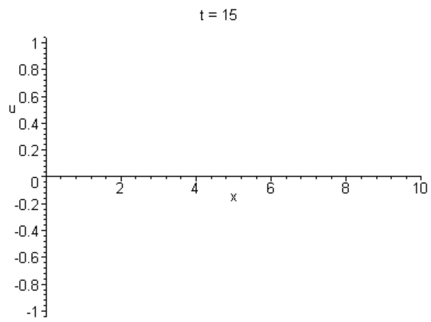
$$u(x\,,t) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\left[sin\frac{n\pi}{4} + sin\frac{3n\pi}{4}\right] sin\frac{n\pi x}{L}\, cos\frac{n\pi a\, t}{L}\,.$$

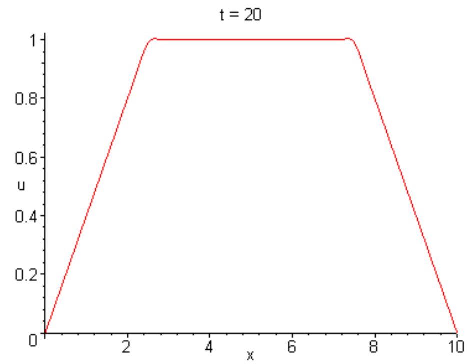
(b). With $a = 1$ and $L = 10$,

$$u(x\,,t) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\left[sin\frac{n\pi}{4} + sin\frac{3n\pi}{4}\right] sin\frac{n\pi x}{10}\, cos\frac{n\pi t}{10}\,.$$

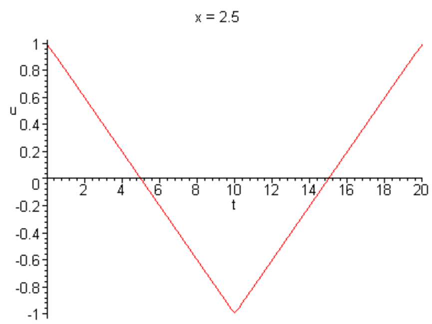


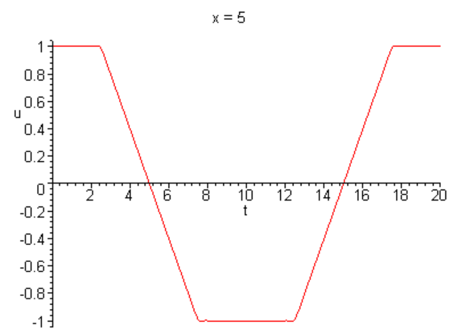


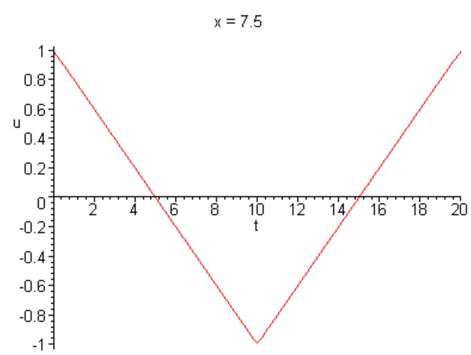




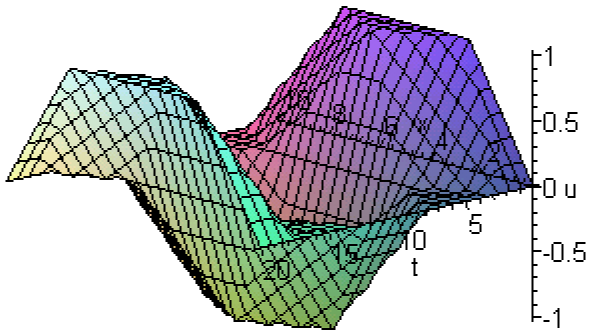
(c).

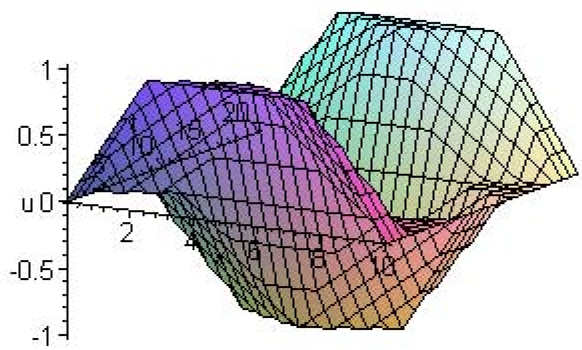






(d).





$3(a)$. The initial velocity is *zero*. As given by Eq. (20), the solution is

$$u(x\,,t)=\sum_{n=1}^{\infty}c_n\,sin\frac{n\pi x}{L}\,cos\frac{n\pi a\,t}{L}\,,$$

in which the coefficients are the Fourier *sine* coefficients of $f(x)$. That is,

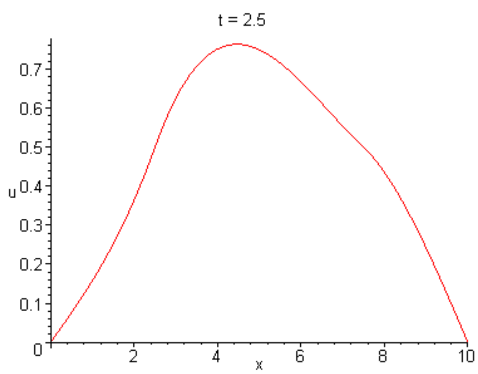
$$
\begin{aligned}
c_n &= \frac{2}{L}\int_0^L f(x) sin \frac{n\pi x}{L} dx \\
&= \frac{2}{L}\int_0^L \frac{8x(L-x)^2}{L^3} sin \frac{n\pi x}{L} dx \\
&= 32\, \frac{2 + cos\, n\pi}{n^3 \pi^3}\,.
\end{aligned}
$$

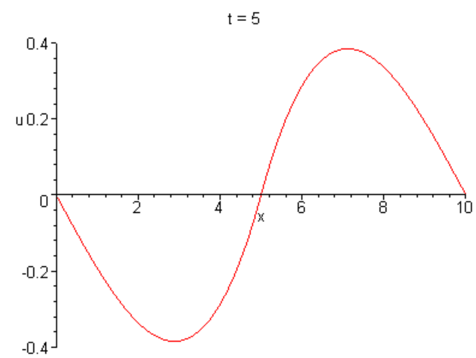
Therefore the displacement of the string is given by

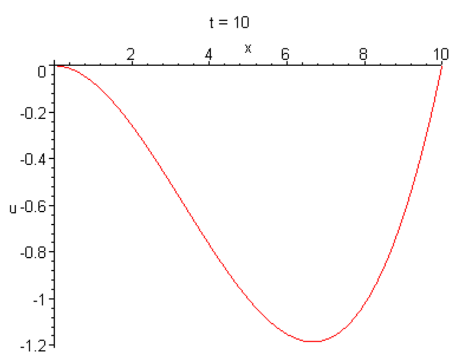
$$u(x\,,t) = \frac{32}{\pi^3}\sum_{n=1}^{\infty} \frac{2 + cos\, n\pi}{n^3} sin \frac{n\pi x}{L}\, cos \frac{n\pi a\, t}{L}\,.$$

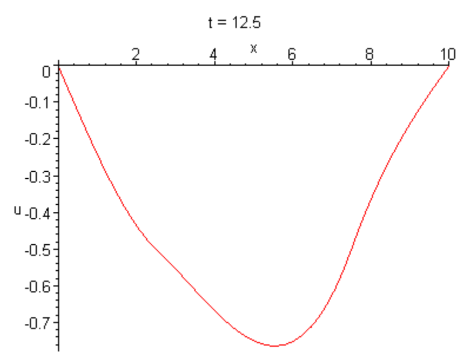
(b). With $a = 1$ and $L = 10$,

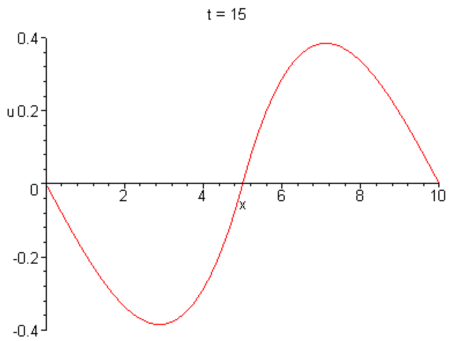
$$u(x\,,t) = \frac{32}{\pi^3}\sum_{n=1}^{\infty} \frac{2 + cos\, n\pi}{n^3} sin \frac{n\pi x}{10}\, cos \frac{n\pi t}{10}\,.$$

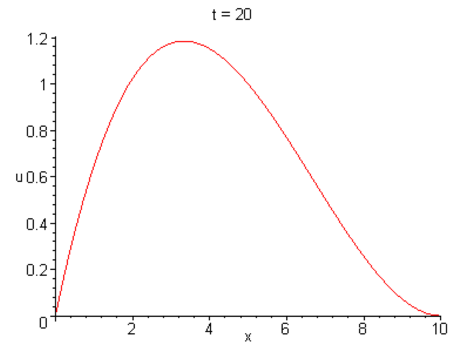




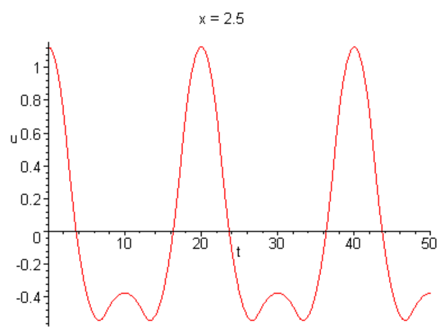


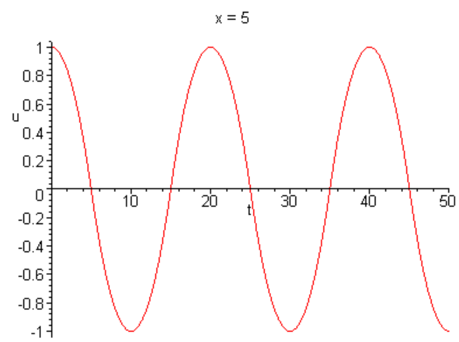


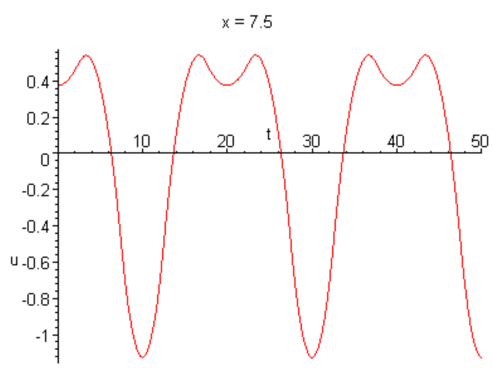

t = 15
u
x
0.4
0.2
0
-0.2
-0.4
2
4
6
8
10


t = 20
u
x
1.2
1
0.8
0.6
0.4
0.2
0
2
4
6
8
10

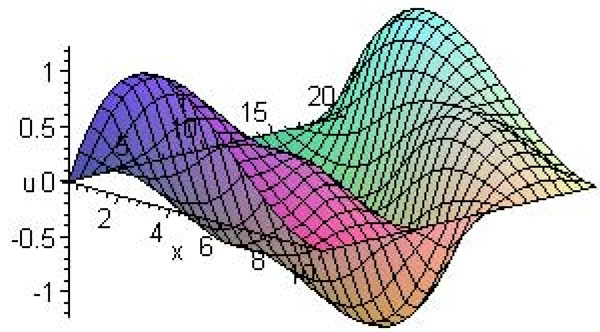
(c).

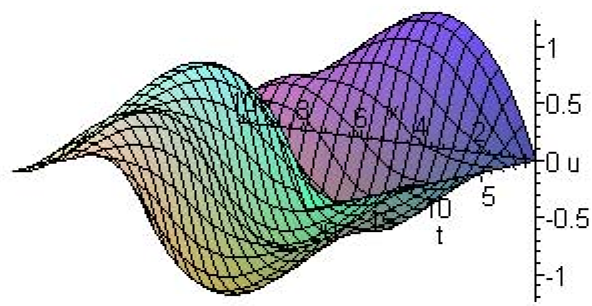

x = 2.5
u
t
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
10
20
30
40
50


x = 5
u
t
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
10
20
30
40
50


x = 7.5
u
t
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
10
20
30
40
50

(d).





$4(a)$. As given by Eq. (20), the solution is

$$u(x\,,t)=\sum_{n=1}^{\infty}c_n\,sin\frac{n\pi x}{L}\,cos\frac{n\pi a\,t}{L}\,,$$

in which the coefficients are the Fourier *sine* coefficients of $f(x)$. That is,

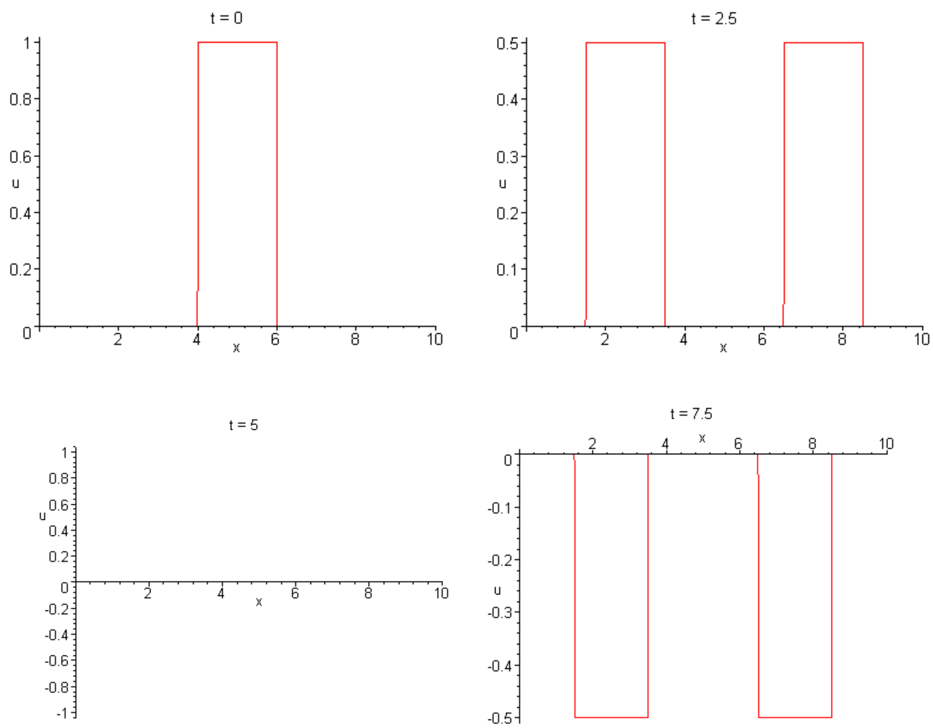
$$\begin{aligned} c_n &= \frac{2}{L}\int_0^L f(x) sin \frac{n\pi x}{L} dx \\ &= \frac{2}{L}\int_{L/2-1}^{L/2+1} sin \frac{n\pi x}{L} dx \\ &= 4\,\frac{sin \frac{n\pi}{2}\, sin \frac{n\pi}{L}}{n\pi}\,. \end{aligned}$$

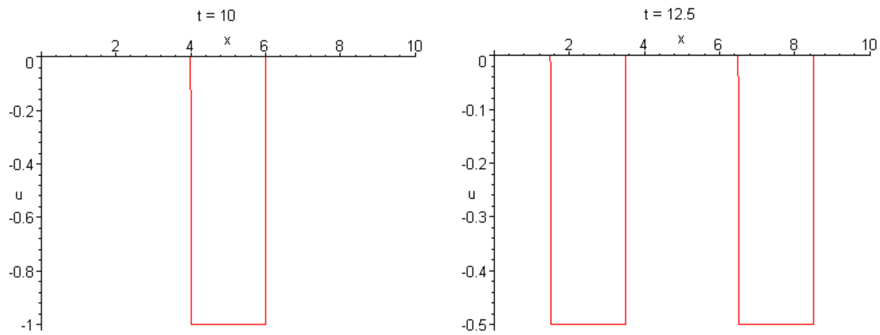
Therefore the displacement of the string is given by

$$u(x\,,t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[sin\frac{n\pi}{2}\; sin\frac{n\pi}{L}\right] sin\frac{n\pi x}{L}\; cos\frac{n\pi a\, t}{L}\,.$$

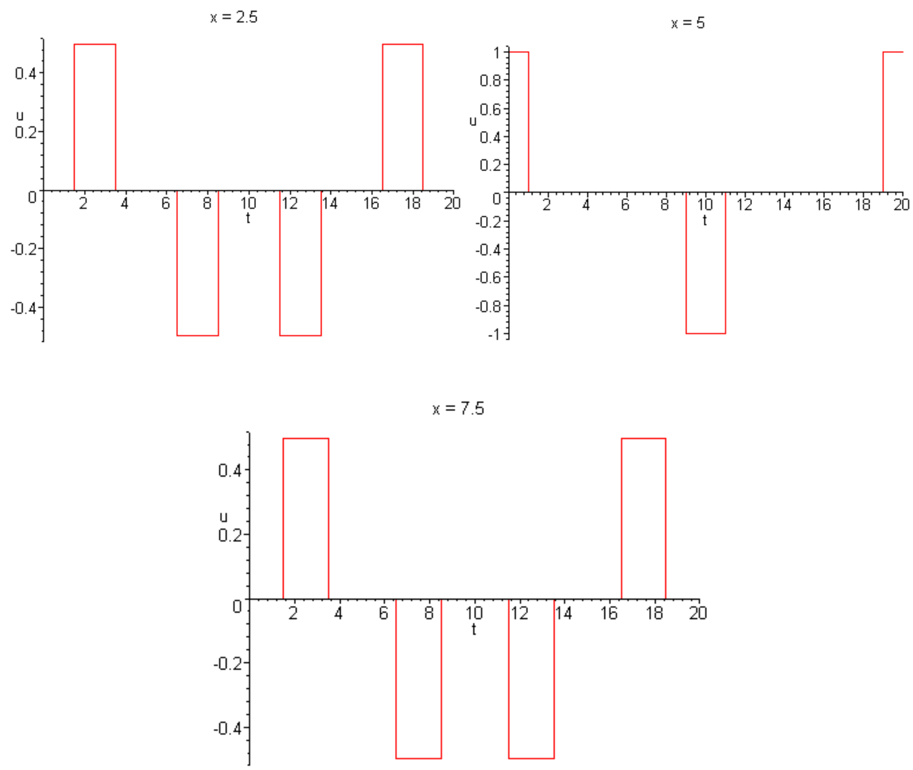
(b). With $a = 1$ and $L = 10$,

$$u(x\,,t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[sin\frac{n\pi}{2}\; sin\frac{n\pi}{10}\right] sin\frac{n\pi x}{10}\; cos\frac{n\pi t}{10}\,.$$

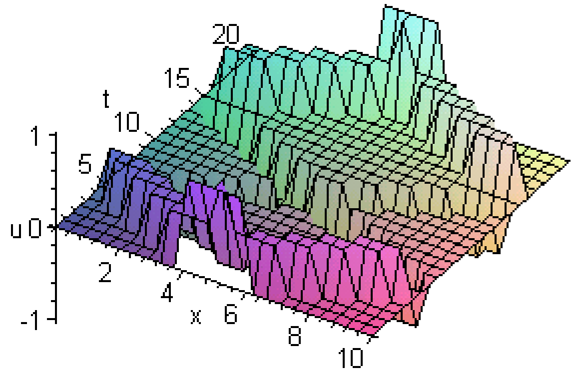


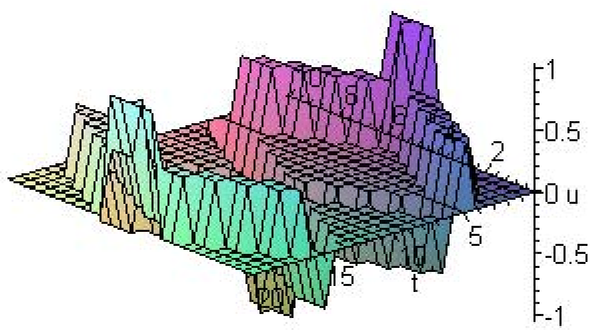


(c).



(d).





$5(a)$. The initial displacement is *zero.* Therefore the solution, as given by Eq. (34), is

$$u(x\,,t) = \sum_{n=1}^{\infty} k_n \, sin\frac{n\pi x}{L} \, sin\frac{n\pi a\, t}{L}\,,$$

in which the coefficients are the Fourier *sine* coefficients of $u_t(x\,,0) = g(x)$. It follows that

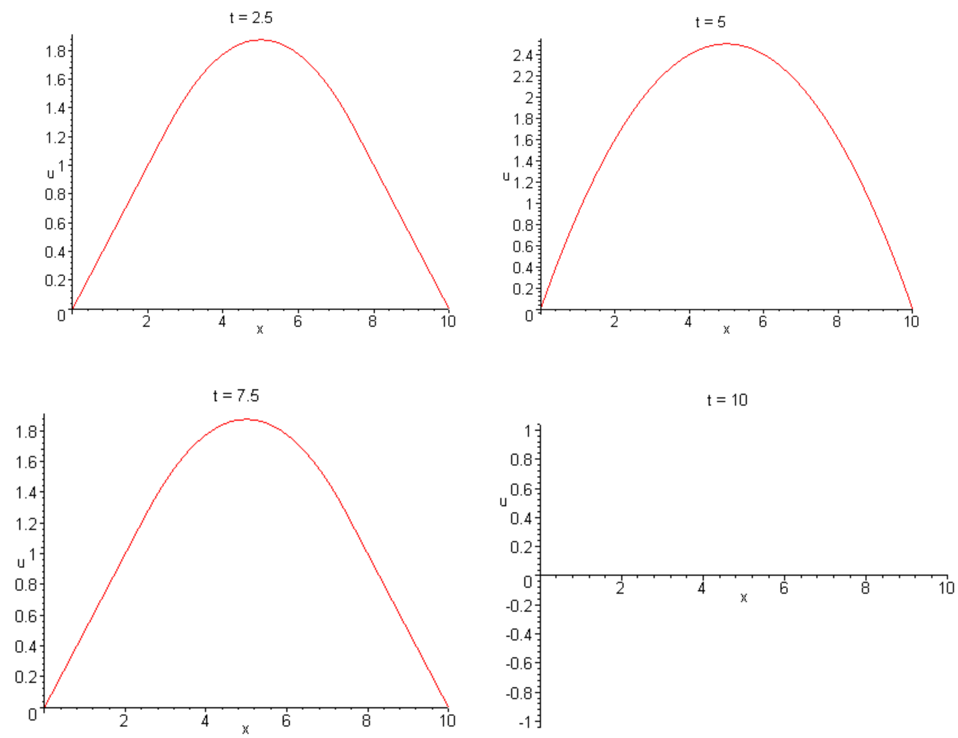
$$
\begin{aligned}
k_n &= \frac{2}{n\pi a}\int_0^L g(x) sin\frac{n\pi x}{L}dx \\
&= \frac{2}{n\pi a}\left[\int_0^{L/2}\frac{2x}{L}sin\frac{n\pi x}{L}dx + \int_{L/2}^{L}\frac{2(L-x)}{L}sin\frac{n\pi x}{L}dx\right] \\
&= 8L\,\frac{sin\, n\pi/2}{n^3\pi^3 a}\,.
\end{aligned}
$$

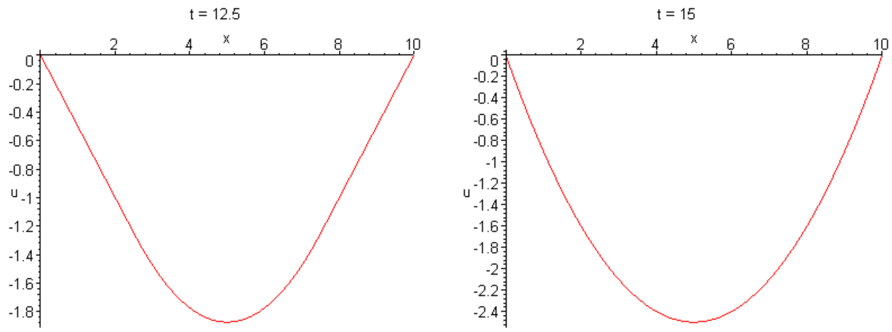
Therefore the displacement of the string is given by

$$
u(x\,,t) = \frac{8L}{a\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}sin\frac{n\pi}{2}\;sin\frac{n\pi x}{L}\;sin\frac{n\pi a\,t}{L}\,.
$$

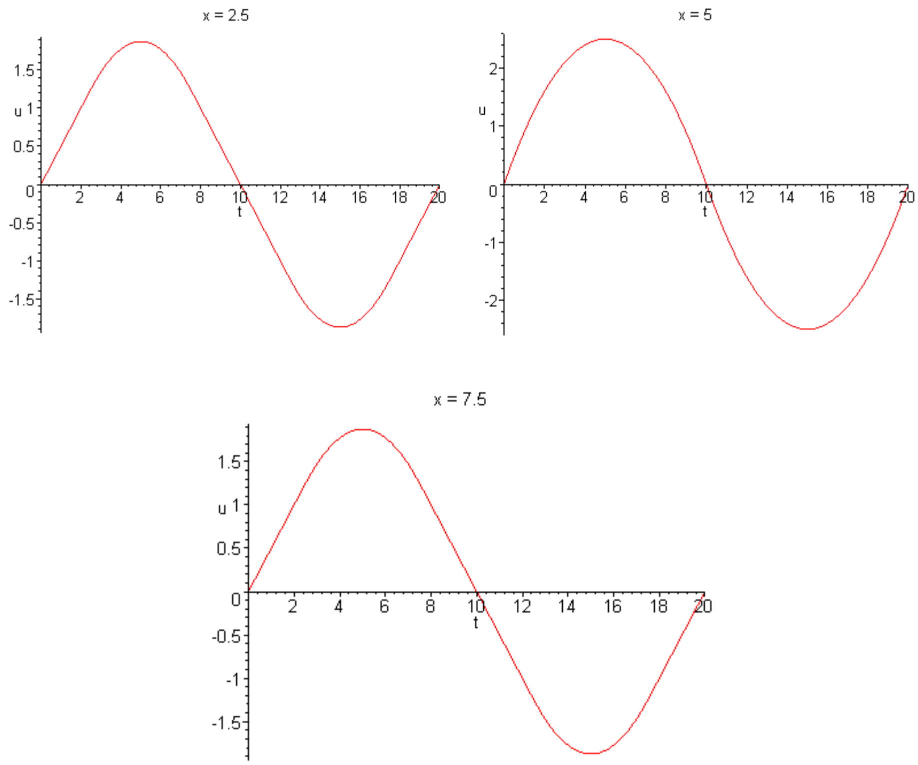
(b). With $a = 1$ and $L = 10$,

$$
u(x\,,t) = \frac{80}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}sin\frac{n\pi}{2}\;sin\frac{n\pi x}{10}\;sin\frac{n\pi t}{10}\,.
$$

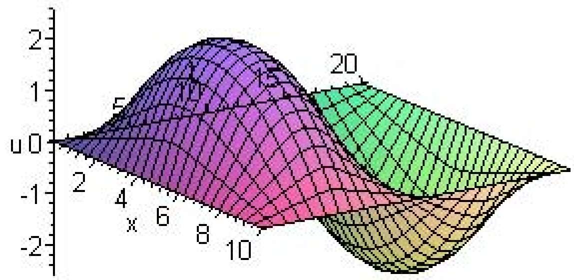


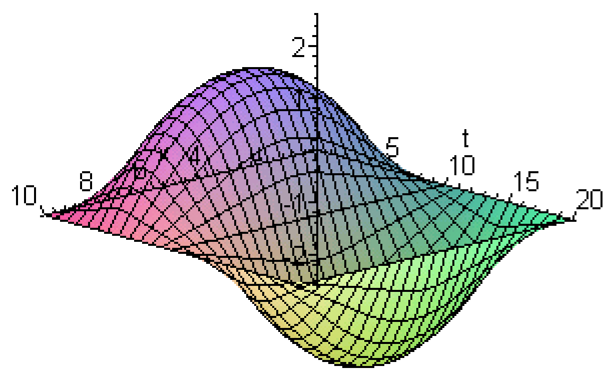

t = 12.5
x
u
2
4
6
8
10
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.4
-1.6
-1.8
t = 15
x
u
2
4
6
8
10
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.4
-1.6
-1.8
-2
-2.2
-2.4

(c).


x = 2.5
u
t
1.5
1
0.5
0
-0.5
-1
-1.5
2
4
6
8
10
12
14
16
18
20
x = 5
u
t
2
1
0
-1
-2
x = 7.5
u
t
1.5
1
0.5
0
-0.5
-1
-1.5

(d).





$7(a)$. The initial displacement is *zero.* As given by Eq. (34), the solution is

$$u(x\,,t)=\sum_{n=1}^{\infty}k_{\,n}\,sin\frac{n\pi x}{L}\,sin\frac{n\pi a\,t}{L}\,,$$

in which the coefficients are the Fourier *sine* coefficients of $u_t(x\,,0)=g(x)$. It follows that

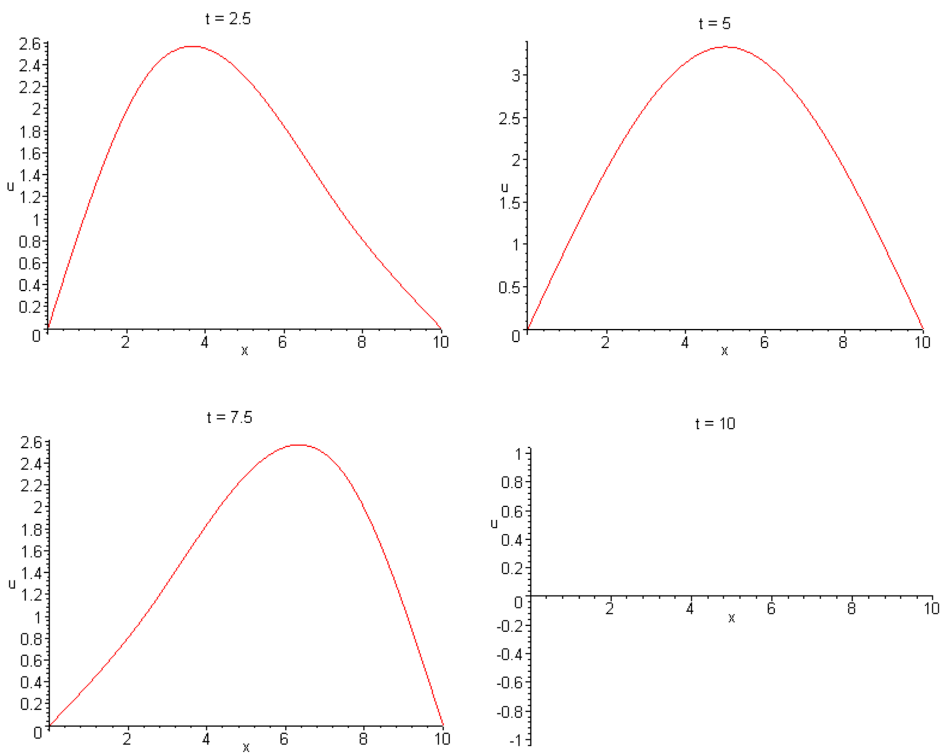
$$\begin{aligned} k_n &= \frac{2}{n\pi a}\int_0^L g(x) sin\frac{n\pi x}{L}dx \\ &= \frac{2}{n\pi a}\int_0^L \frac{8x(L-x)^2}{L^3} sin\frac{n\pi x}{L}dx \\ &= 32L\,\frac{2+cos\,n\pi}{n^4\pi^4 a}\,. \end{aligned}$$

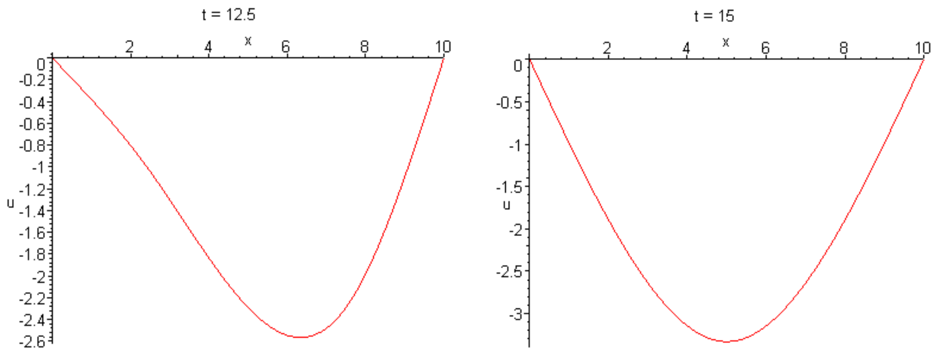
Therefore the displacement of the string is given by

$$u(x\,,t) = \frac{32L}{a\pi^4}\sum_{n=1}^{\infty}\frac{2+cos\,n\pi}{n^4}\,sin\frac{n\pi x}{L}\,sin\frac{n\pi a\,t}{L}\,.$$

(b). With $a=1$ and $L=10$,

$$u(x\,,t) = \frac{320}{\pi^4}\sum_{n=1}^{\infty}\frac{2+cos\,n\pi}{n^4}\,sin\frac{n\pi x}{10}\,sin\frac{n\pi t}{10}\,.$$





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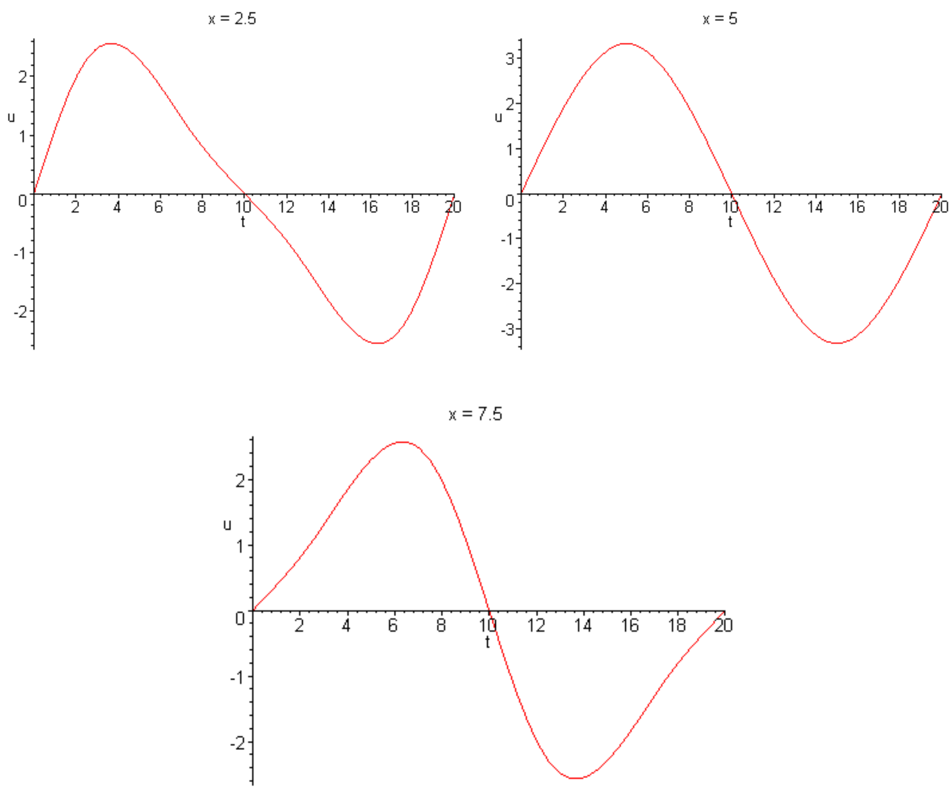

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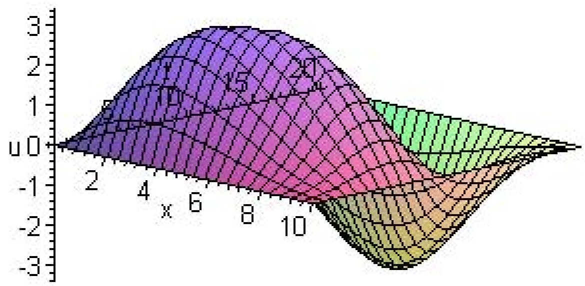
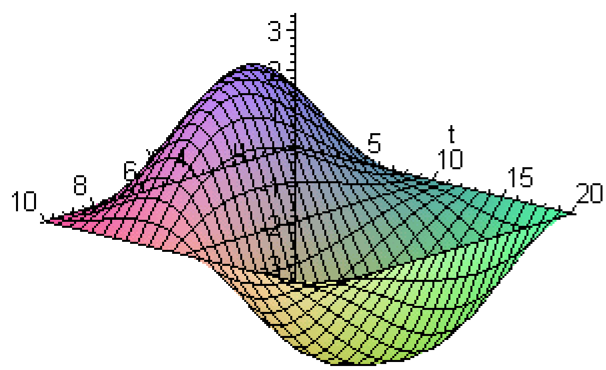




$8(a)$. As given by Eq. (34), the solution is

$$u(x\,,t)=\sum_{n=1}^{\infty}k_n\,sin\frac{n\pi x}{L}\,sin\frac{n\pi a\,t}{L}\,,$$

in which the coefficients are the Fourier *sine* coefficients of $u_t(x\,,0)=g(x)$. It follows that

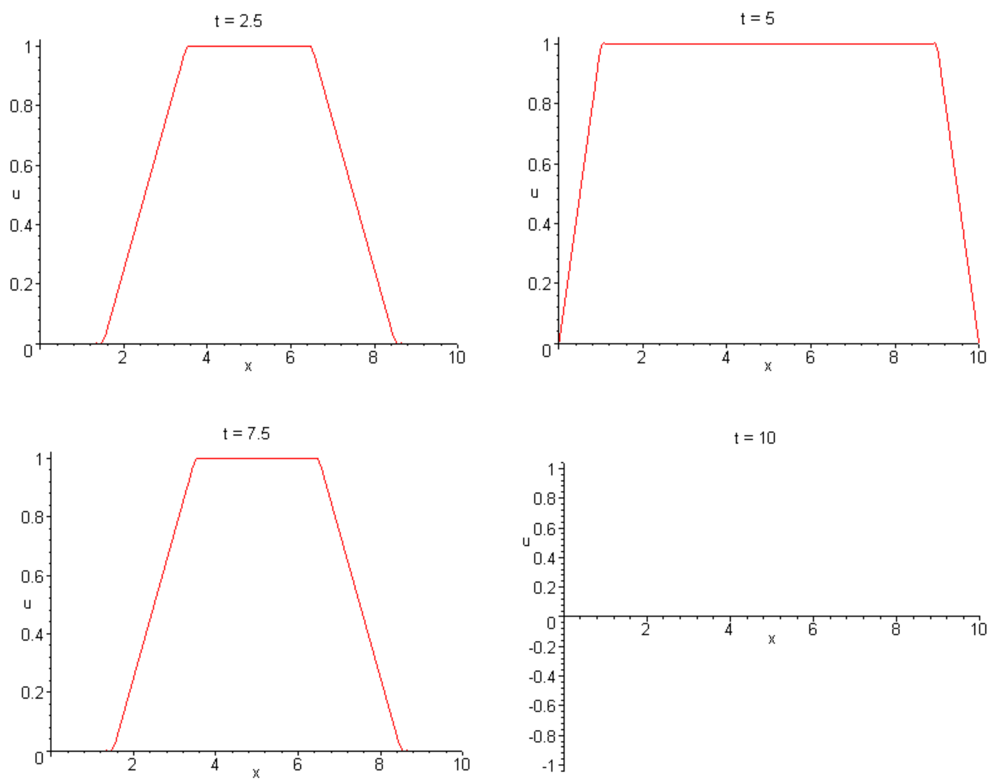
$$\begin{aligned} k_n &= \frac{2}{n\pi a}\int_0^L g(x) sin\frac{n\pi x}{L}dx \\ &= \frac{2}{n\pi a}\int_{L/2-1}^{L/2+1} sin\frac{n\pi x}{L}dx \\ &= 4L\,\frac{sin\frac{n\pi}{2}\, sin\frac{n\pi}{L}}{n^2\pi^2 a}\,. \end{aligned}$$

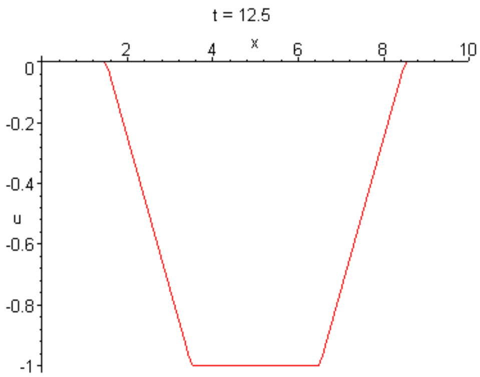
Therefore the displacement of the string is given by

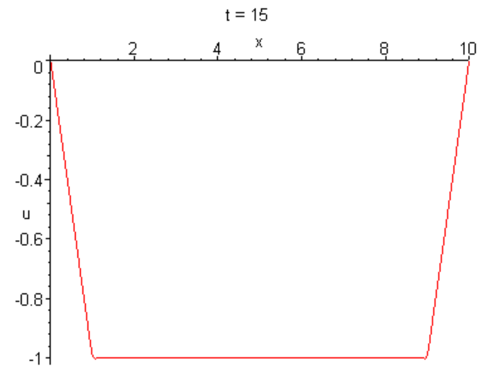
$$u(x\,,t) = \frac{4L}{a\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[sin\frac{n\pi}{2}\; sin\frac{n\pi}{L}\right] sin\frac{n\pi x}{L}\; sin\frac{n\pi a\,t}{L}\,.$$

(b). With $a = 1$ and $L = 10$,

$$u(x\,,t) = \frac{40}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[sin\frac{n\pi}{2}\; sin\frac{n\pi}{10}\right] sin\frac{n\pi x}{10}\; sin\frac{n\pi t}{10}\,.$$







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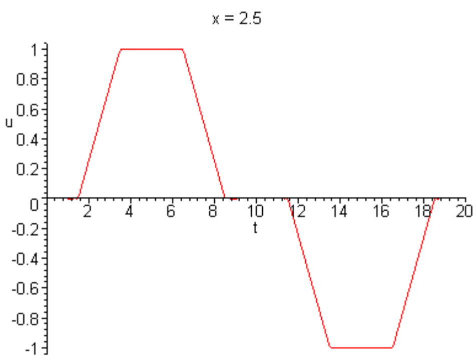

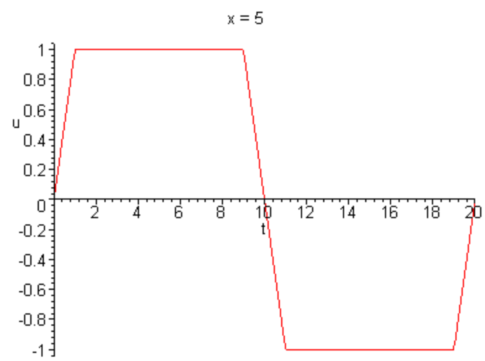

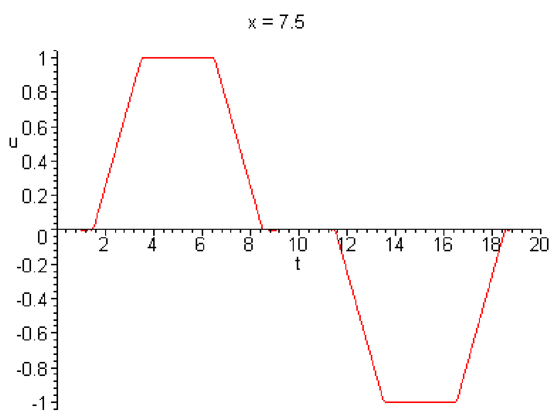

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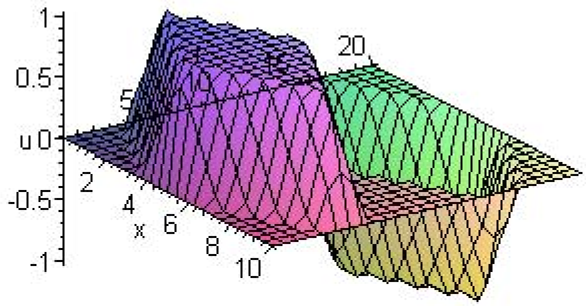
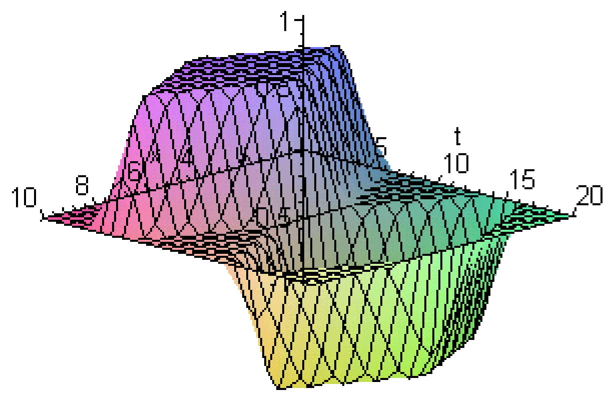




$11(a)$. As shown in Prob. 9, the solution is

$$u(x\,,t)=\sum_{n=1}^{\infty}c_n\,sin\frac{(2n-1)\pi x}{2L}\,cos\frac{(2n-1)\pi a\,t}{2L}\,,$$

in which the coefficients are the Fourier *sine* coefficients of $f(x)$. It follows that

$$\begin{aligned} c_n &= \frac{2}{L}\int_0^L f(x) sin \frac{(2n-1)\pi x}{2L} dx \\ &= \frac{2}{L}\int_0^L \frac{8x(L-x)^2}{L^3} sin \frac{(2n-1)\pi x}{2L} dx \\ &= 512\, \frac{3cos\, n\pi + (2n-1)\pi}{(2n-1)^4 \pi^4}\,. \end{aligned}$$

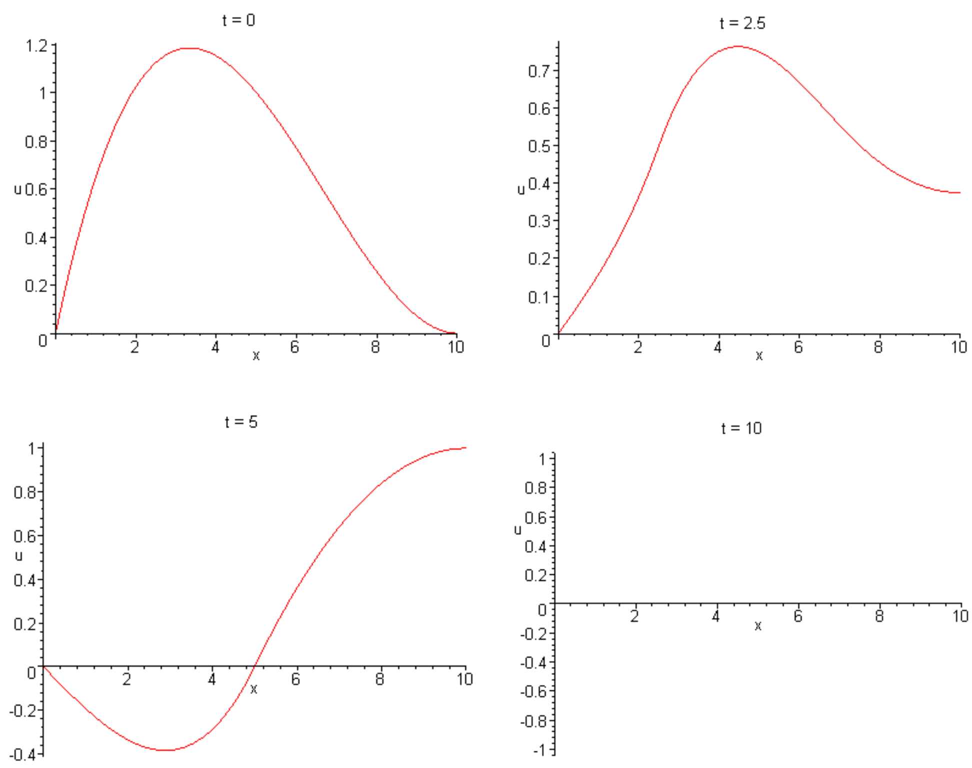
Therefore the displacement of the string is given by

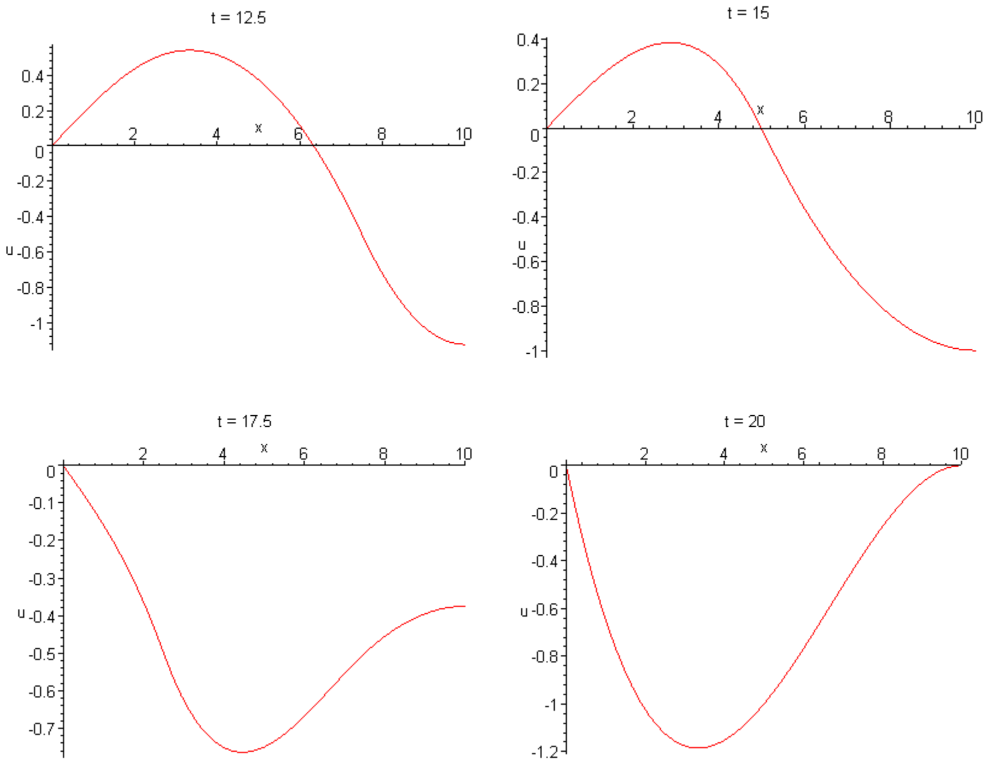
$$u(x\,,t) = \frac{512}{\pi^4}\sum_{n=1}^{\infty} \frac{3cos\, n\pi + (2n-1)\pi}{(2n-1)^4}\, sin \frac{(2n-1)\pi x}{2L}\, cos \frac{(2n-1)\pi a\, t}{2L}\,.$$

Note that the period is $T = 4L/a$.

(b). With $a = 1$ and $L = 10$,

$$u(x\,,t) = \frac{512}{\pi^4}\sum_{n=1}^{\infty} \frac{3cos\, n\pi + (2n-1)\pi}{(2n-1)^4}\, sin \frac{(2n-1)\pi x}{20}\, cos \frac{(2n-1)\pi t}{20}\,.$$





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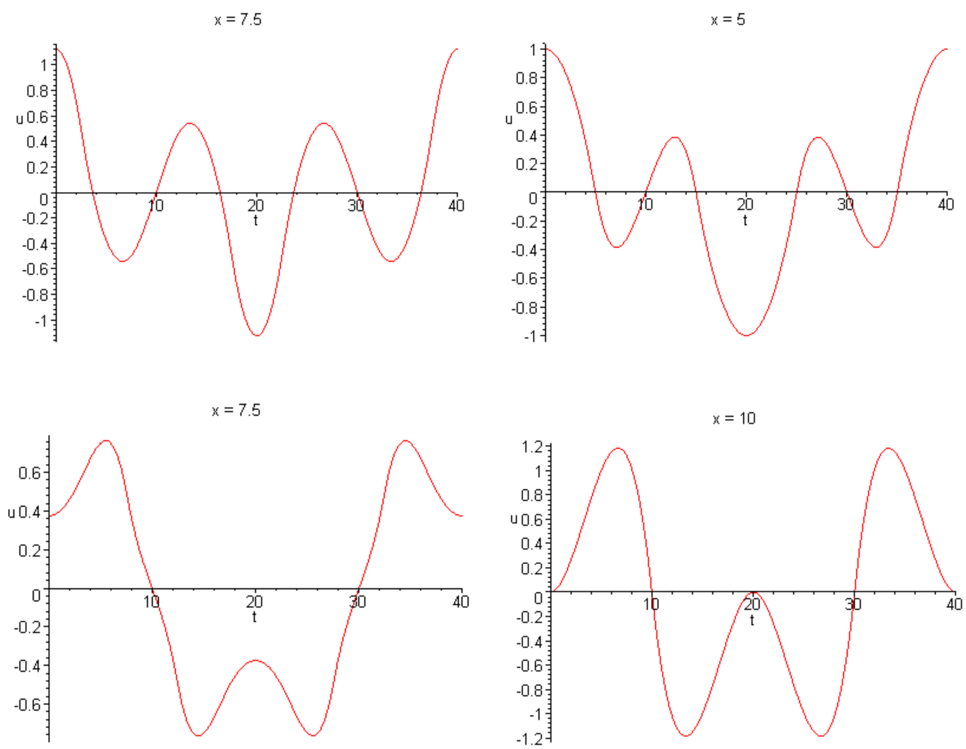

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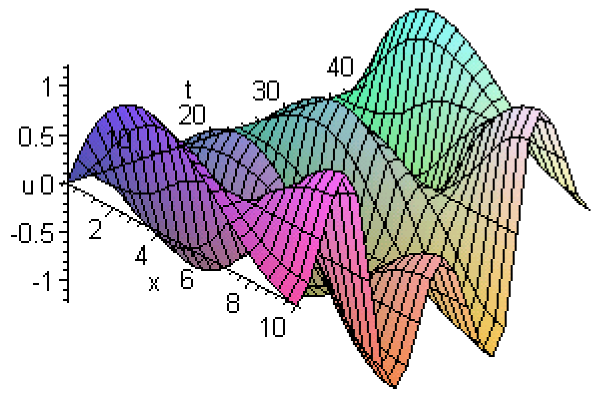

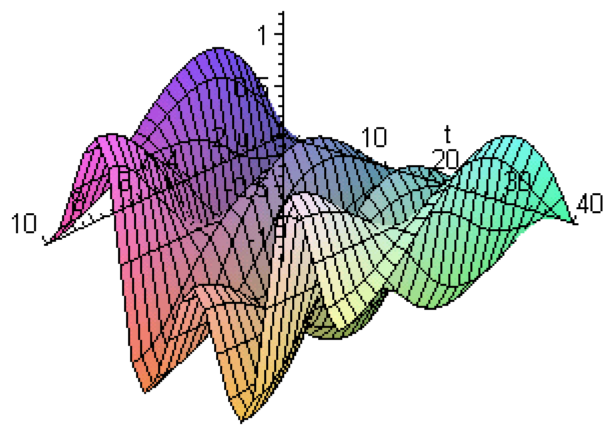

12. The *wave equation* is given by

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \,.$$

Setting $s = x/L$, we have

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial s}\frac{ds}{dx} = \frac{1}{L}\frac{\partial u}{\partial s} \,.$$

It follows that

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{L^2}\frac{\partial^2 u}{\partial s^2}\,.$$

Likewise, with $\tau = at/L$,

$$\frac{\partial u}{\partial t} = \frac{a}{L}\frac{\partial u}{\partial \tau} \quad \text{and} \quad \frac{\partial^2 u}{\partial t^2} = \frac{a^2}{L^2}\frac{\partial^2 u}{\partial \tau^2}\,.$$

Substitution into the original equation results in

$$\frac{\partial^2 u}{\partial s^2} = \frac{\partial^2 u}{\partial \tau^2}\,.$$

15. The given specifications are $L = 5\,ft$, $T = 50\,lb$, and *weight* per unit length $\gamma = 0.026$ *lb/ft*. It follows that $\rho = \gamma/32.2 = 80.75 \times 10^{-5}$ *slugs/ft*.

(a). The transverse waves propagate with a speed of $a = \sqrt{T/\rho} = 248$ *ft/sec*.

(b). The *natural frequencies* are $\omega_n = n\pi a/L = 49.8\,\pi n$ *rad/sec*.

(c). The new wave speed is $a = \sqrt{(T + \Delta T)/\rho}$. For a string with fixed ends, the natural modes are proportional to the functions

$$M_n(x) = sin\frac{n\pi x}{L}\,,$$

which are independent of a.

19. The solution of the wave equation

$$a^2 v_{xx} = v_{tt}$$

in an infinite one-dimensional medium subject to the initial conditions

$$v(x\,,0) = f(x)\,,\;\; v_t(x\,,0) = 0\,, \quad -\infty < x < \infty$$

is given by

$$v(x\,,t) = \frac{1}{2}[f(x - at) + f(x + at)].$$

The solution of the wave equation

$$a^2 w_{xx} = w_{tt}\,,$$

on the same domain, subject to the initial conditions

$$w(x\,,0) = 0\,,\;\; w_t(x\,,0) = g(x)\,, \quad -\infty < x < \infty$$

is given by

$$w(x\,,t) = \frac{1}{2a}\int_{x-at}^{x+at} g(\xi)d\xi\,.$$

Let $u(x\,,t) = v(x\,,t) + w(x\,,t)$. Since the PDE is *linear*, it is easy to see that $u(x\,,t)$ is a solution of the wave equation $a^2u_{xx} = u_{tt}$. Furthermore, we have

$$u(x\,,0) = v(x\,,0) + w(x\,,0) = f(x)$$

and

$$u_t(x\,,0) = v_t(x\,,0) + w_t(x\,,0) = g(x)\,.$$

Hence $u(x\,,t)$ is a solution of the general wave propagation problem.

20. The solution of the specified wave propagation problem is

$$u(x\,,t) = \sum_{n=1}^{\infty} c_n\, sin\frac{n\pi x}{L}\, cos\frac{n\pi a\,t}{L}\,.$$

Using a standard trigonometric identity,

$$\begin{aligned} sin\frac{n\pi x}{L}\, cos\frac{n\pi a\,t}{L} &= \frac{1}{2}\left[sin\left(\frac{n\pi x}{L} + \frac{n\pi a\,t}{L}\right) + sin\left(\frac{n\pi x}{L} - \frac{n\pi a\,t}{L}\right)\right] \\ &= \frac{1}{2}\left[sin\frac{n\pi}{L}(x+at) + sin\frac{n\pi}{L}(x-at)\right]. \end{aligned}$$

We can therefore also write the solution as

$$u(x\,,t) = \frac{1}{2}\sum_{n=1}^{\infty} c_n\left[sin\frac{n\pi}{L}(x+at) + sin\frac{n\pi}{L}(x-at)\right].$$

Assuming that the series can be split up,

$$u(x\,,t) = \frac{1}{2}\left[\sum_{n=1}^{\infty} c_n\, sin\frac{n\pi}{L}(x-at) + \sum_{n=1}^{\infty} c_n\, sin\frac{n\pi}{L}(x+at)\right].$$

Comparing the solution to the one given by Eq. (28), we can infer that

$$h(x) = \sum_{n=1}^{\infty} c_n\, sin\frac{n\pi x}{L}\,.$$

21. Let $h(\xi)$ be a $2L$-periodic function defined by

$$h(\xi) = \begin{cases} f(\xi)\,, & 0 \le \xi \le L\,; \\ -f(-\xi)\,, & -L \le \xi \le 0\,. \end{cases}$$

Set $u(x\,,t) = \frac{1}{2}[h(x-at) + h(x+at)]$. Assuming the appropriate differentiability conditions on h,

$$\frac{\partial u}{\partial x} = \frac{1}{2}[h'(x-at) + h'(x+at)]$$

and

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2}[h''(x-at) + h''(x+at)].$$

Likewise,

$$\frac{\partial^2 u}{\partial t^2} = \frac{a^2}{2}[h''(x-at) + h''(x+at)].$$

It follows immediately that

$$a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}\,.$$

Let $t \geq 0$. Checking the first boundary condition,

$$u(0\,,t) = \frac{1}{2}[h(-at) + h(at)] = \frac{1}{2}[-h(at) + h(at)] = 0\,.$$

Checking the other boundary condition,

$$\begin{aligned} u(L\,,t) &= \frac{1}{2}[h(L-at) + h(L+at)] \\ &= \frac{1}{2}[-h(at-L) + h(at+L)]. \end{aligned}$$

Since h is $2L$-periodic, $h(at-L) = h(at-L+2L)$. Therefore $u(L\,,t) = 0$. Furthermore, for $0 \leq x \leq L$,

$$u(x\,,0) = \frac{1}{2}[h(x) + h(x)] = h(x) = f(x).$$

Hence $u(x\,,t)$ is a solution of the problem.

23. Assuming that we can differentiate term-by-term,

$$\frac{\partial u}{\partial t} = -\pi a\sum_{n=1}^{\infty}\frac{c_n\,n}{L}sin\frac{n\pi x}{L}\,sin\frac{n\pi a\,t}{L}$$

and

$$\frac{\partial u}{\partial x} = \pi\sum_{n=1}^{\infty}\frac{c_n\,n}{L}cos\frac{n\pi x}{L}\,cos\frac{n\pi a\,t}{L}\,.$$

Formally,

$$\left(\frac{\partial u}{\partial t}\right)^2 = \pi^2 a^2 \sum_{n=1}^{\infty}\left(\frac{c_n\, n}{L}\right)^2 sin^2\,\frac{n\pi x}{L}\,sin^2\,\frac{n\pi a\,t}{L} + + \pi^2 a^2 \sum_{n\neq m}^{\infty} F_{nm}(x,t)$$

and

$$\left(\frac{\partial u}{\partial x}\right)^2 = \pi^2 \sum_{n=1}^{\infty}\left(\frac{c_n\, n}{L}\right)^2 cos^2\,\frac{n\pi x}{L}\,cos^2\,\frac{n\pi a\,t}{L} + + \pi^2 \sum_{n\neq m}^{\infty} G_{nm}(x,t)\,,$$

in which $F_{nm}(x,t)$ and $G_{nm}(x,t)$ contain *products* of the natural modes and their derivatives. Based on the *orthogonality* of the natural modes,

$$\int_0^L \left(\frac{\partial u}{\partial t}\right)^2 dx = \pi^2 a^2 \frac{L}{2}\sum_{n=1}^{\infty}\left(\frac{c_n\, n}{L}\right)^2 sin^2\,\frac{n\pi a\,t}{L}$$

and

$$\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx = \pi^2 \frac{L}{2}\sum_{n=1}^{\infty}\left(\frac{c_n\, n}{L}\right)^2 cos^2\,\frac{n\pi a\,t}{L}\,.$$

Recall that $a^2 = T/\rho$. It follows that

$$\int_0^L \left[\rho\left(\frac{\partial u}{\partial t}\right)^2 + T\left(\frac{\partial u}{\partial x}\right)^2\right] dx = \pi^2 \frac{TL}{2}\sum_{n=1}^{\infty}\left(\frac{c_n\, n}{L}\right)^2 sin^2\,\frac{n\pi a\,t}{L} + + \pi^2 \frac{TL}{2}\sum_{n=1}^{\infty}\left(\frac{c_n\, n}{L}\right)^2 cos^2\,\frac{n\pi a\,t}{L}\,.$$

Therefore,

$$\int_0^L \left[\frac{1}{2}\rho\left(\frac{\partial u}{\partial t}\right)^2 + \frac{1}{2}T\left(\frac{\partial u}{\partial x}\right)^2\right] dx = \pi^2 \frac{T}{4L}\sum_{n=1}^{\infty} n^2 c_n^2\,.$$