Chapter Nine

Section 9.1

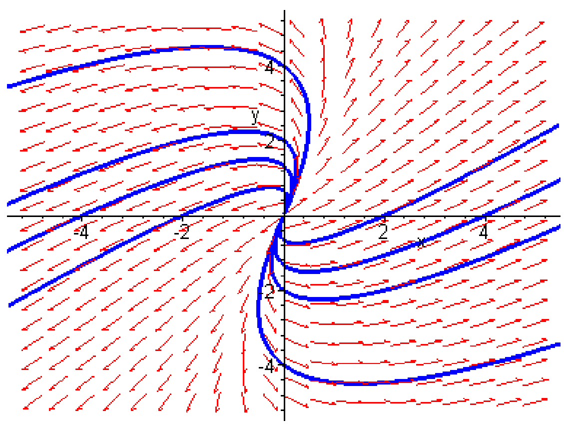
$2(a)$. Setting $\mathbf{x} = \boldsymbol{\xi}\, e^{rt}$ results in the algebraic equations

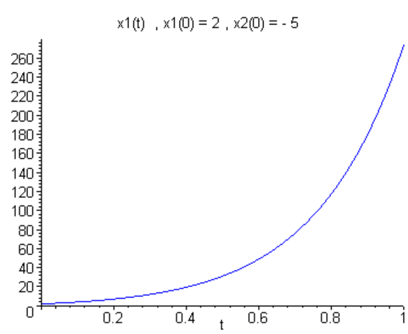
$$\begin{pmatrix} 5-r & -1 \\ 3 & 1-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

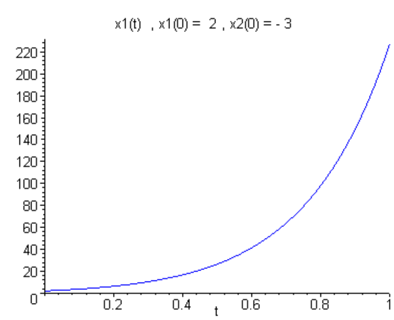
For a nonzero solution, we must have $det(\mathbf{A} - r\,\mathbf{I}) = r^2 - 6\,r + 8 = 0$. The roots of the characteristic equation are $r_1 = 2$ and $r_2 = 4$. For $r = 2$, the system of equations reduces to $3\xi_1 = \xi_2$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1\,,3)^T$. Substitution of $r = 4$ results in the single equation $\xi_1 = \xi_2$. A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (1\,,1)^T$.

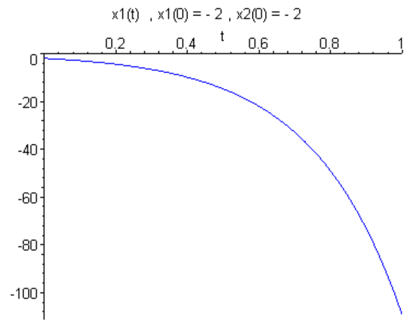
(b). The eigenvalues are *real* and *positive*, hence the critical point is an *unstable node*.

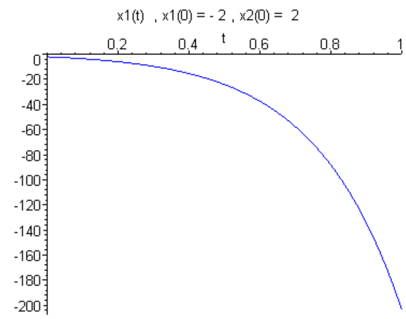
(c, d).











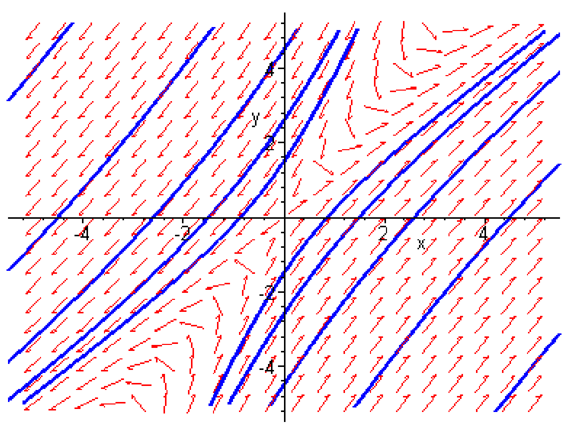
$3(a)$. Solution of the ODE requires analysis of the algebraic equations

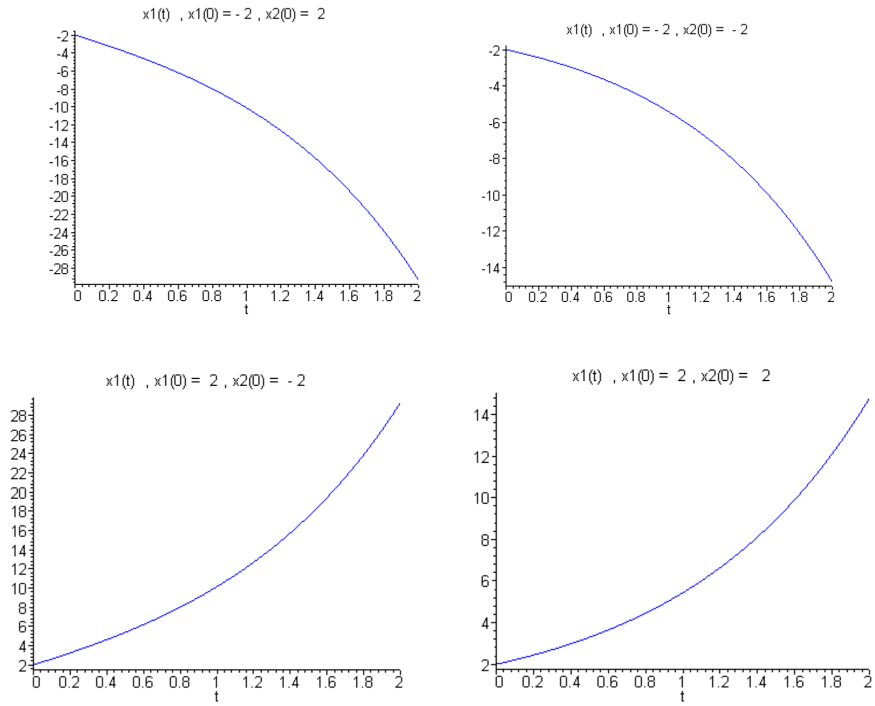
$$\begin{pmatrix} 2-r & -1 \\ 3 & -2-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $det(\mathbf{A} - r\,\mathbf{I}) = r^2 - 1 = 0$. The roots of the characteristic equation are $r_1 = 1$ and $r_2 = -1$. For $r = 1$, the system of equations reduces to $\xi_1 = \xi_2$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1\,,1)^T$. Substitution of $r = -1$ results in the single equation $3\,\xi_1 - \xi_2 = 0$. A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (1\,,3)^T$.

(b). The eigenvalues are *real*, with $r_1\,r_2 < 0$. Hence the critical point is a *saddle*.

(c, d).





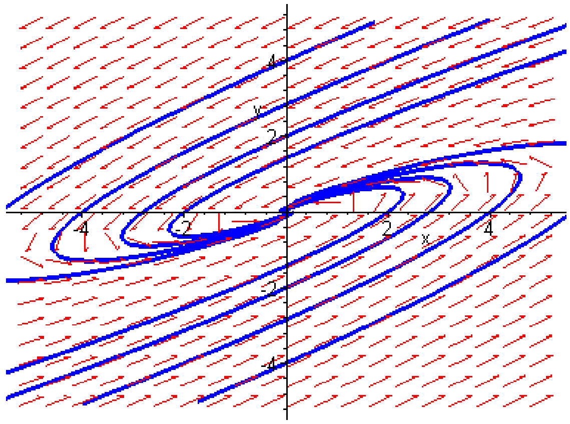
$5(a)$. The characteristic equation is given by

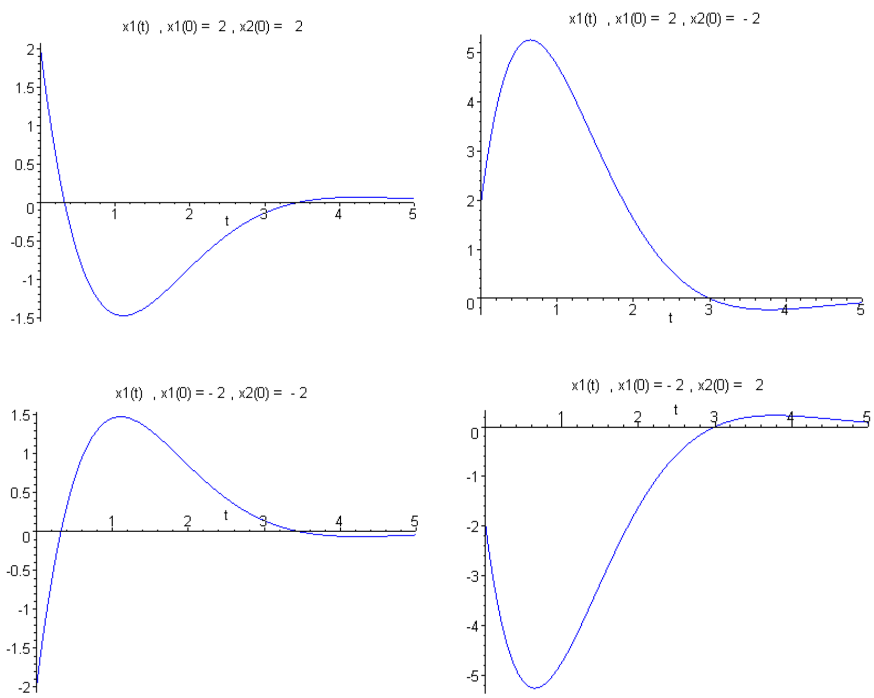
$$\begin{vmatrix} 1-r & -5 \\ 1 & -3-r \end{vmatrix} = r^2 + 2r + 2 = 0\,.$$

The equation has *complex* roots $r_1 = -1+i$ and $r_2 = -1-i$. For $r = -1+i$, the components of the solution vector must satisfy $\xi_1 - (2+i)\xi_2 = 0$. Thus the corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (2+i\,,1)^T$. Substitution of $r = -1-i$ results in the single equation $\xi_1 - (2-i)\xi_2 = 0$. A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (2-i\,,1)^T$.

(b). The eigenvalues are *complex conjugates*, with negative real part. Hence the origin is a *stable spiral*.

(c, d).





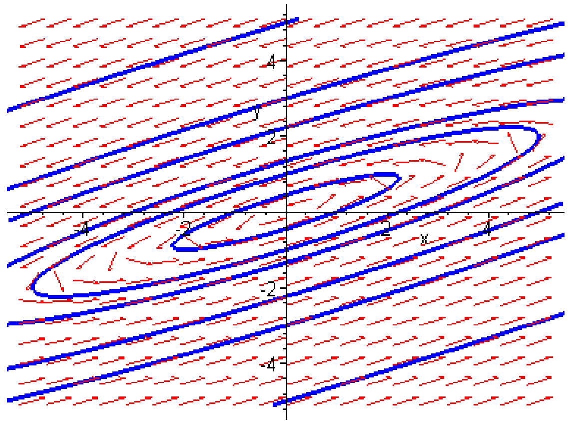
$6(a)$. Solution of the ODEs is based on the analysis of the algebraic equations

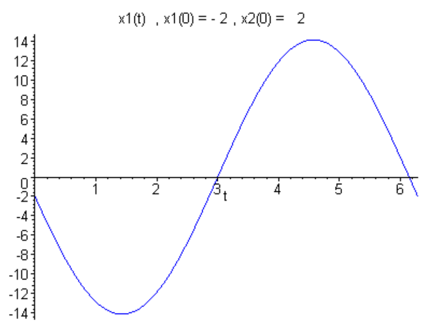
$$\begin{pmatrix} 2-r & -5 \\ 1 & -2-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

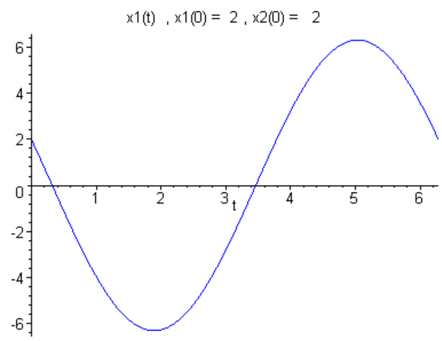
For a nonzero solution, we require that $det(\mathbf{A} - r\,\mathbf{I}) = r^2 + 1 = 0$. The roots of the characteristic equation are $r = \pm i$. Setting $r = i$, the equations are equivalent to $\xi_1 - (2+i)\xi_2 = 0$. The eigenvectors are $\boldsymbol{\xi}^{(1)} = (2+i\,,1)^T$ and $\boldsymbol{\xi}^{(2)} = (2-i\,,1)^T$.

(b). The eigenvalues are *purely imaginary*. Hence the critical point is a *center*.

(c, d).







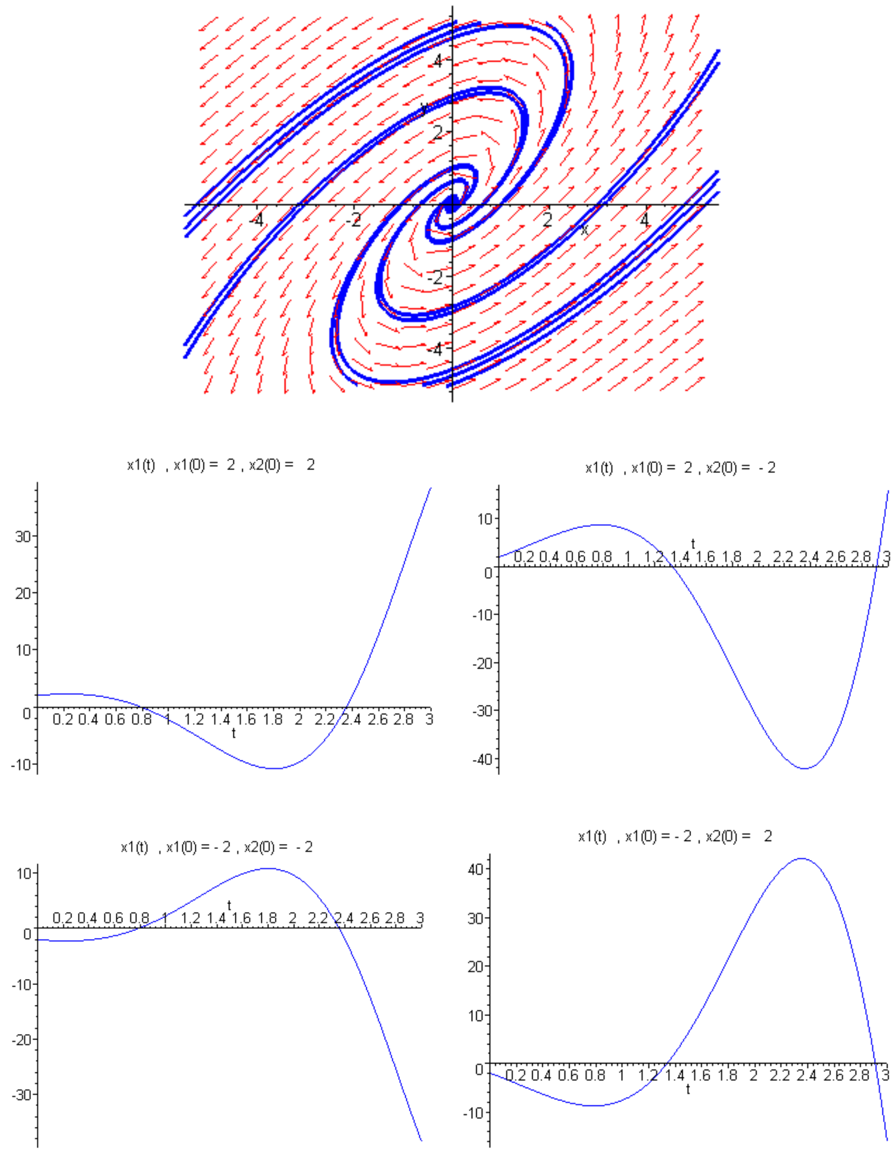
$7(a)$. Setting $\mathbf{x} = \boldsymbol{\xi}\, e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} 3-r & -2 \\ 4 & -1-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we require that $det(\mathbf{A} - r\,\mathbf{I}) = r^2 - 2r + 5 = 0$. The roots of the characteristic equation are $r = 1 \pm 2i$. Substituting $r = 1 - 2i$, the two equations reduce to $(1+i)\xi_1 - \xi_2 = 0$. The two eigenvectors are $\boldsymbol{\xi}^{(1)} = (1\,, 1+i)^T$ and $\boldsymbol{\xi}^{(2)} = (1\,, 1-i)^T$.

(b). The eigenvalues are *complex conjugates*, with positive real part. Hence the origin is an *unstable spiral*.

(c, d).



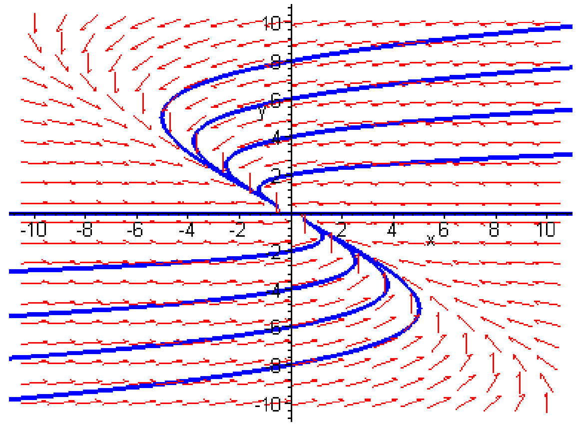
$8(a)$. The characteristic equation is given by

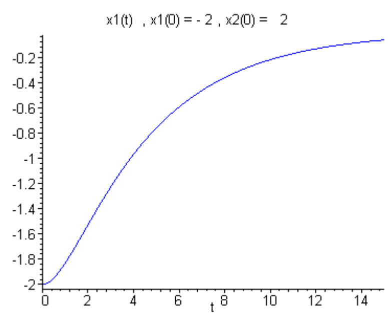
$$\begin{vmatrix} 1-r & -5 \\ 1 & -3-r \end{vmatrix} = (r+1)(r+0.25) = 0\,,$$

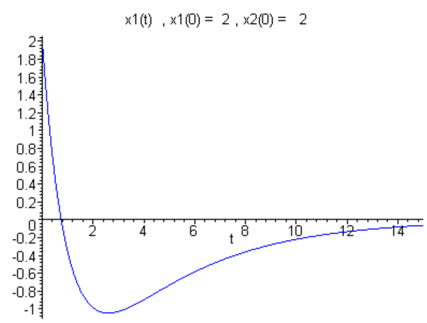
with roots $r_1 = -1$ and $r_2 = -0.25$. For $r = -1$, the components of the solution vector must satisfy $\xi_2 = 0$. Thus the corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1\,, 0)^T$. Substitution of $r = -0.25$ results in the single equation $0.75\,\xi_1 + \xi_2 = 0$. A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (4\,, -3)^T$.

(b). The eigenvalues are *real* and both *negative*. Hence the critical point is a *stable* *node*.

(c, d).







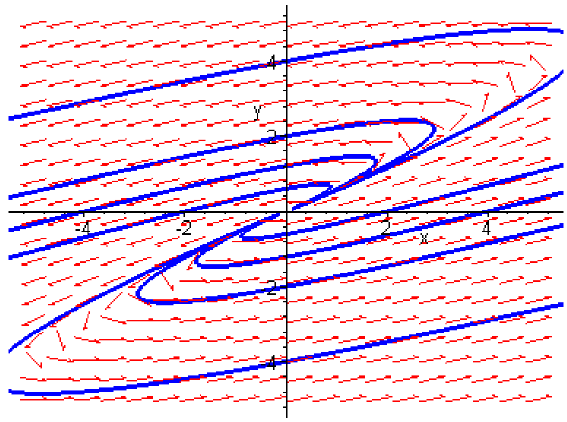
$9(a)$. Solution of the ODEs is based on the analysis of the algebraic equations

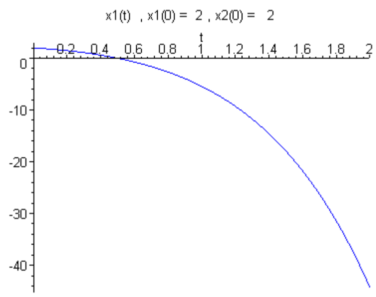
$$\begin{pmatrix} 3-r & -4 \\ 1 & -1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

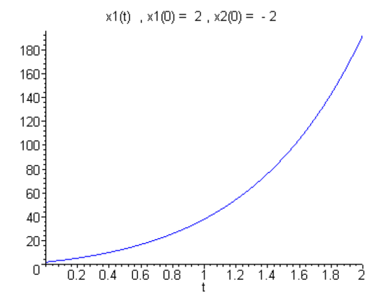
For a nonzero solution, we require that $det(\mathbf{A} - r\,\mathbf{I}) = r^2 - 2\,r + 1 = 0$. The single root of the characteristic equation is $r = 1$. Setting $r = 1$, the components of the solution vector must satisfy $\xi_1 - 2\,\xi_2 = 0$. A corresponding eigenvector is $\boldsymbol{\xi} = (2\,,1)^T$.

(b). Since there is only one linearly independent eigenvector, the critical point is an $unstable$, *improper node*.

(c, d).







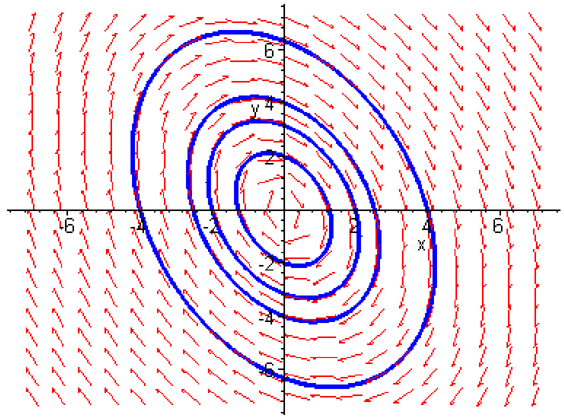
$10(a)$. The characteristic equation is given by

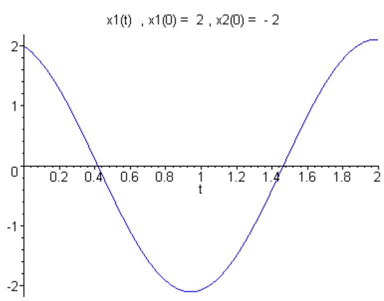
$$\begin{vmatrix} 1-r & 2 \\ -5 & -1-r \end{vmatrix} = r^2 + 9 = 0 \, .$$

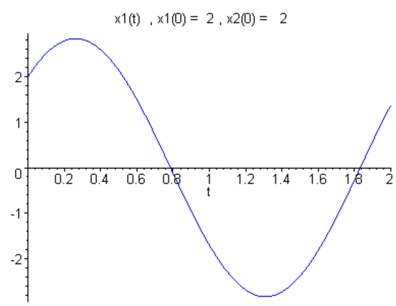
The equation has *complex* roots $r_{1,2} = \pm 3i$. For $r = -3i$, the components of the solution vector must satisfy $5\,\xi_1 + (1-3i)\xi_2 = 0$. Thus the corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1-3i\,,\,-5)^T$. Substitution of $r = 3i$ results in $5\,\xi_1 + (1+3i)\xi_2 = 0$. A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (1+3i\,,\,-5)^T$.

(b). The eigenvalues are *purely imaginary*, hence the critical point is a *center*.

(c, d).



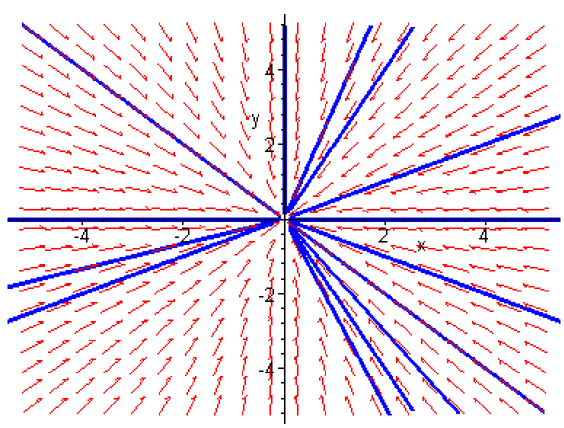


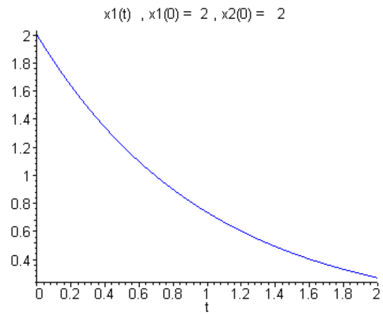


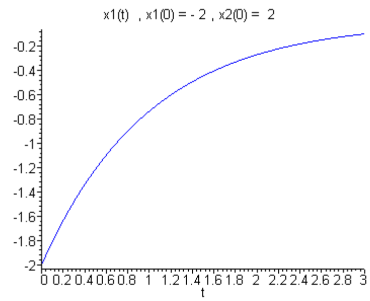
$11(a)$. The characteristic equation is $(r+1)^2 = 0$, with double root $r = -1$. It is easy to see that the two linearly independent eigenvectors are $\boldsymbol{\xi}^{(1)} = (1\,,0)^T$ and $\boldsymbol{\xi}^{(2)} = (0\,,1)^T$.

(b). Since there are two linearly independent eigenvectors, the critical point is a *stable proper node*.

(c, d).







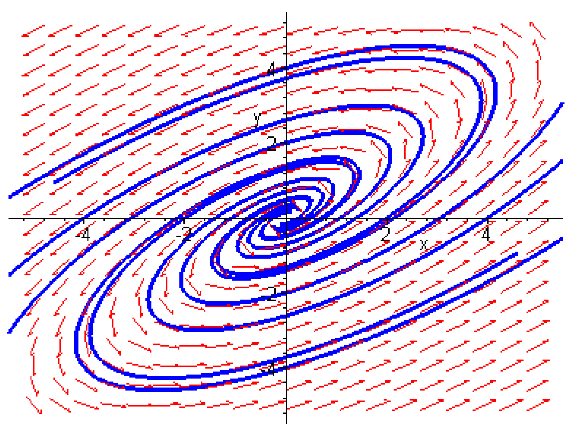
12(a). Setting $\mathbf{x} = \boldsymbol{\xi}\, e^{rt}$ results in the algebraic equations

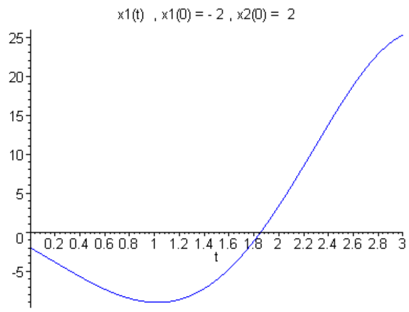
$$\begin{pmatrix} 2-r & -5/2 \\ 9/5 & -1-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

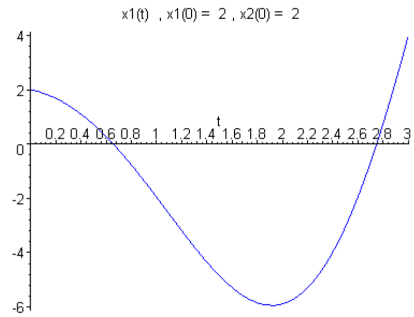
For a nonzero solution, we require that $det(\mathbf{A} - r\,\mathbf{I}) = r^2 - r + 5/2 = 0$. The roots of the characteristic equation are $r = 1/2 \pm 3i/2$. Substituting $r = 1/2 - 3i/2$, the equations reduce to $(3+3i)\xi_1 - 5\,\xi_2 = 0$. Therefore the two eigenvectors are $\boldsymbol{\xi}^{(1)} = (5\,, 3+3i)^T$ and $\boldsymbol{\xi}^{(2)} = (5\,, 3-3i)^T$.

(b). Since the eigenvalues are *complex*, with *positive* real part, the critical point is an *unstable spiral*.

(c, d).







14. Setting $\mathbf{x}' = \mathbf{0}$, that is,

$$\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 2 \\ -1 \end{pmatrix},$$

we find that the critical point is $\mathbf{x}^0 = (-1, 0)^T$. With the change of dependent variable, $\mathbf{x} = \mathbf{x}^0 + \mathbf{u}$, the differential equation can be written as

$$\frac{d\mathbf{u}}{dt} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\mathbf{u}\,.$$

The critical point for the transformed equation is the origin. Setting $\mathbf{u} = \boldsymbol{\xi}\, e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} -2-r & 1 \\ 1 & -2-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we require that $det(\mathbf{A} - r\,\mathbf{I}) = r^2 + 4r + 3 = 0$. The roots of the characteristic equation are $r = -3\,,\ -1$. Hence the critical point is a *stable node*.

15. Setting $\mathbf{x}' = \mathbf{0}$, that is,

$$\begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 1 \\ -5 \end{pmatrix},$$

we find that the critical point is $\mathbf{x}^0 = (-2, 1)^T$. With the change of dependent variable, $\mathbf{x} = \mathbf{x}^0 + \mathbf{u}$, the differential equation can be written as

$$\frac{d\mathbf{u}}{dt} = \begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix}\mathbf{u}\,.$$

The characteristic equation is $det(\mathbf{A} - r\,\mathbf{I}) = r^2 + 2r + 3 = 0$, with complex conjugate roots $r = -1 \pm i\sqrt{2}$. Since the real parts of the eigenvalues are *negative*, the critical point is a *stable spiral*.

16. The critical point $\mathbf{x}^0$ satisfies the system of equations

$$\begin{pmatrix} 0 & -\beta \\ \delta & 0 \end{pmatrix}\mathbf{x} = \begin{pmatrix} -\alpha \\ \gamma \end{pmatrix}.$$

It follows that $x^0 = \gamma/\delta$ and $y^0 = \alpha/\beta$. Using the transformation, $\mathbf{x} = \mathbf{x}^0 + \mathbf{u}$, the differential equation can be written as

$$\frac{d\mathbf{u}}{dt} = \begin{pmatrix} 0 & -\beta \\ \delta & 0 \end{pmatrix}\mathbf{u}.$$

The characteristic equation is $det(\mathbf{A} - r\,\mathbf{I}) = r^2 + \beta\,\delta = 0$. Since $\beta\,\delta > 0$, the roots are purely imaginary, with $r = \pm\, i\sqrt{\beta\delta}$. Hence the critical point is a *center*.

20. The system of ODEs can be written as

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\mathbf{x}.$$

The characteristic equation is $r^2 - p\,r + q = 0$. The roots are given by

$$r_{1,2} = \frac{p \pm \sqrt{p^2 - 4q}}{2} = \frac{p \pm \sqrt{\Delta}}{2}.$$

The results can be verified using Table 9.1.1 .

21(a). If $q > 0$ and $p < 0$, then the roots are either complex conjugates with negative real parts, or both real and negative.

(b). If $q > 0$ and $p = 0$, then the roots are purely imaginary.

(c). If $q < 0$, then the roots are real, with $r_1 \cdot r_2 > 0$. If $p > 0$, then either the roots are real, with $r_1 \cdot r_2 \geq 0$ or the roots are complex conjugates with positive real parts.

Section 9.2

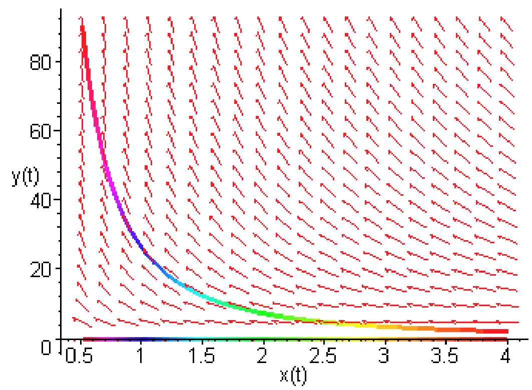
2. The differential equations can be combined to obtain a related ODE

$$\frac{dy}{dx} = -\frac{2y}{x}.$$

The equation is *separable*, with

$$\frac{dy}{y} = -\frac{2\,dx}{x}.$$

The solution is given by $y = C\,x^{-2}$. Note that the system is *uncoupled*, and hence we also have $x = x_0 e^{-t}$ and $y = y_0 e^{2t}$.



In order to determine the direction of motion along the trajectories, observe that for *positive* initial conditions, x will *decrease*, whereas y will *increase.*

4. The trajectories of the system satisfy the ODE

$$\frac{dy}{dx} = -\frac{bx}{ay}.$$

The equation is *separable*, with

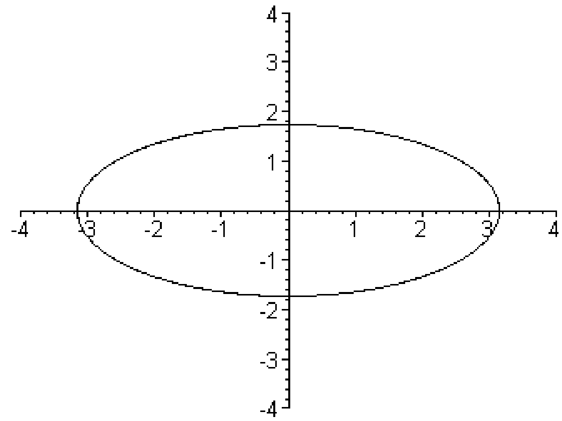
$$ay\,dy = -\,bx\,dx\,.$$

Hence the trajectories are given by $b\,x^2 + a\,y^2 = C^2$, in which C is arbitrary. Evidently, the trajectories are *ellipses*. Invoking the initial condition, we find that $C^2 = ab$. The system of ODEs can also be written as

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 0 & a \\ -\,b & 0 \end{pmatrix}\mathbf{x}\,.$$

Using the methods in Chapter 7, it is easy to show that

$$x = \sqrt{a}\,cos\,\sqrt{ab}\,t$$
$$y = -\,\sqrt{b}\,sin\,\sqrt{ab}\,t\,.$$



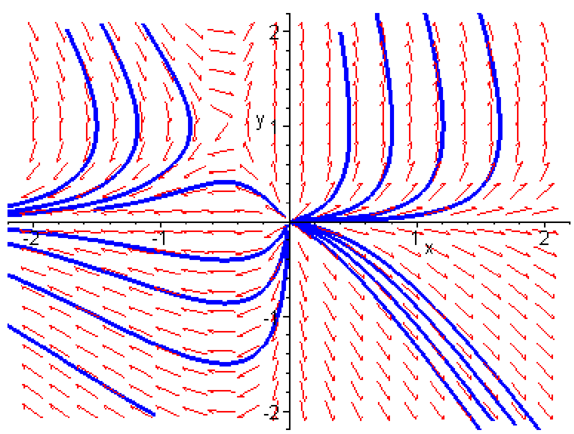
Note that for *positive* initial conditions, x will *increase*, whereas y will *decrease.*

5(a). The critical points are given by the solution set of the equations

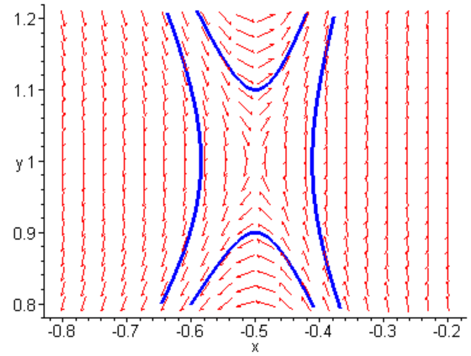
$$x(1-y)=0$$
$$y(1+2x)=0\,.$$

Clearly, $(0\,,0)$ is a solution. If $x\neq 0$, then $y=1$ and $x=\,-1/2$. Hence the critical points are $(0\,,0)$ and $(\,-1/2\,,1)$.

(b).



(c). Based on the phase portrait, all trajectories starting near the origin *diverge*. Hence the critical point $(0\,,0)$ is *unstable*. Examining the phase curves near the critical point $(\,-1/2\,,1)$,



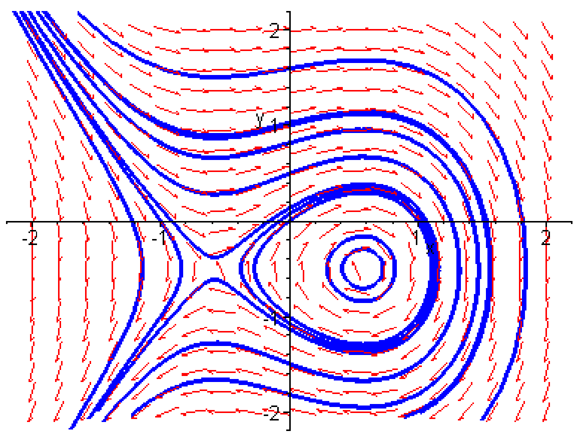
the equilibrium point has the properties of a *saddle*, and hence it is *unstable*.

6(a). The critical points are solutions of the equations

$$\begin{gathered} 1 + 2y = 0 \\ 1 - 3x^2 = 0\,. \end{gathered}$$

There are two equilibrium points, $\left(-1/\sqrt{3}\,,\,-1/2\right)$ and $\left(1/\sqrt{3}\,,\,-1/2\right)$.

(b).



(c). Locally, the trajectories near the point $\left(-1/\sqrt{3}\,,\,-1/2\right)$ resemble the behavior near a *saddle*. Hence the critical point is *unstable*. Near the point $\left(1/\sqrt{3}\,,\,-1/2\right)$, the solutions are *periodic*. Therefore the second critical point is *stable*.

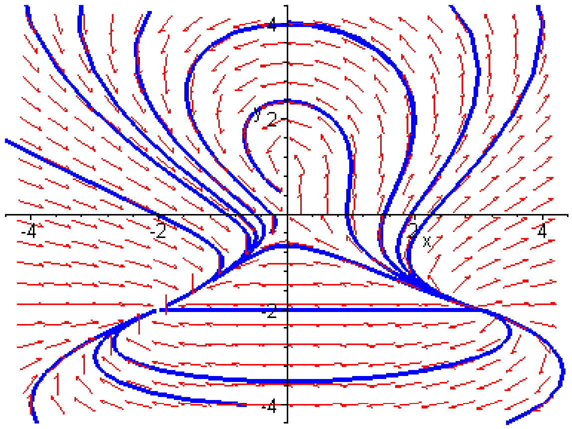
8(a). The critical points are solutions of the equations

$$\begin{aligned} -(x-y)(1-x-y) &= 0 \\ x(2+y) &= 0\,. \end{aligned}$$

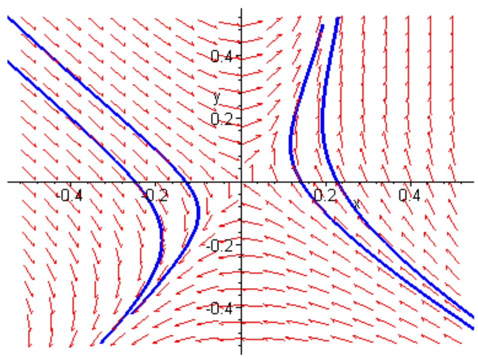
If $x = y$, then $x = y = 0$ or $x = y = -2$. If $x = 1 - y$, then $x = 0$ and $y = 1$, or $x = 3$ and $y = -2$. It follows that the critical points are $(0\,,0)$, $(-2\,,-2)$, $(0\,,1)$

and $(3\,,\,-2)$.

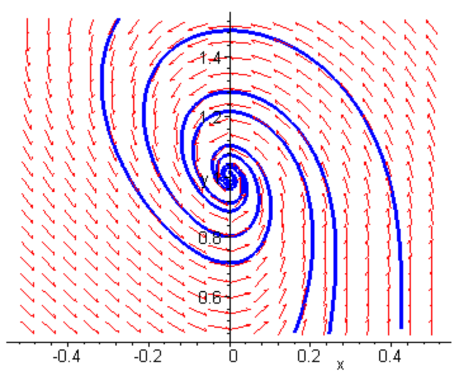
(b).



(c). Near the origin, the trajectories resemble those of a *saddle*, and hence it is *unstable*.



Near the critical point $(0\,,1)$, the trajectories resemble those of a stable *spiral*. Hence the equilibrium point is *asymptotically stable*.



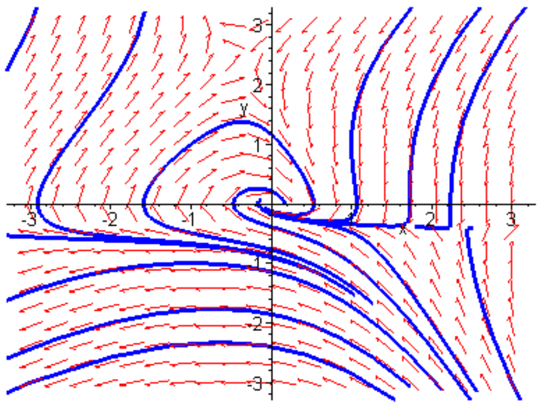
Based on the global phase portrait, it is evident that the other critical points are *nodes.* Closer examination reveals that the point $(-2\,,\,-2)$ is *asymptotically stable*, whereas the point $(3\,,\,-2)$ is *unstable*.

9(a). The critical points are given by the solution set of the equations

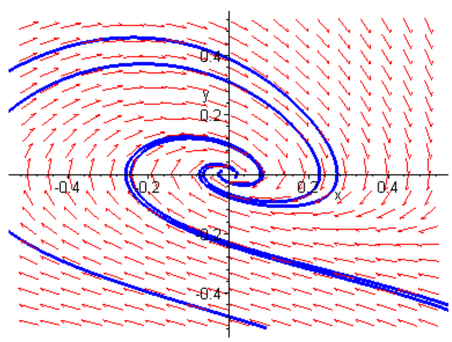
$$y(2-x-y)=0$$
$$-x-y-2xy=0\,.$$

Clearly, $(0\,,0)$ is a critical point. If $x=2-y$, then it follows that $y(y-2)=1$. The additional critical points are $\left(1-\sqrt{2}\,,1+\sqrt{2}\right)$ and $\left(1+\sqrt{2}\,,1-\sqrt{2}\right)$.

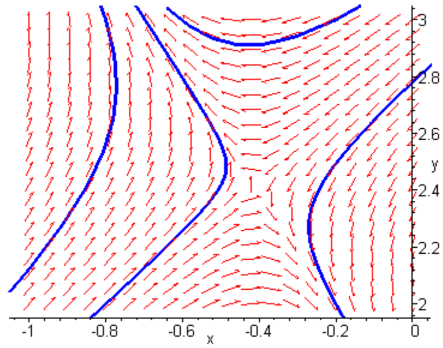
(b).



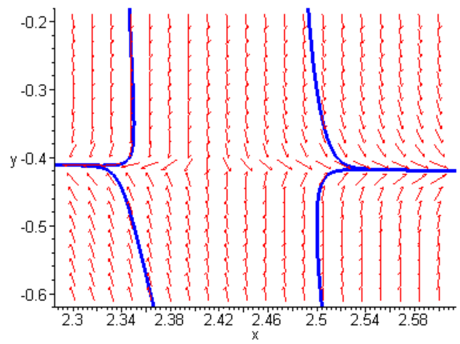
(c). The behavior near the origin is that of a *stable spiral*. Hence the point $(0\,,0)$ is *asymptotically stable*.



At the critical point $\left(1-\sqrt{2}\,,1+\sqrt{2}\right)$, the trajectories resemble those near a *saddle*. Hence the critical point is *unstable*.



Near the point $\left(1+\sqrt{2}\,,1-\sqrt{2}\right)$, the trajectories resemble those near a *saddle*. Hence the critical point is also *unstable*.

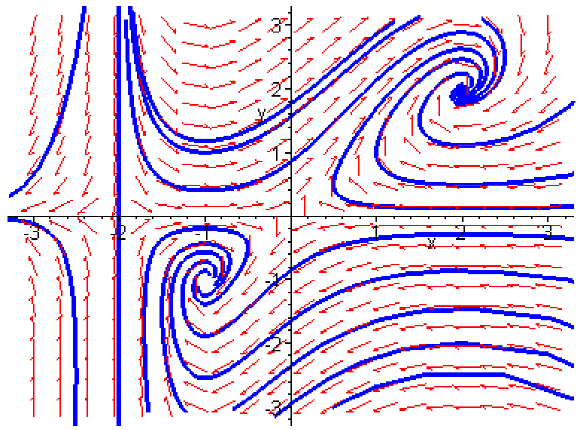


$10(a)$. The critical points are solutions of the equations

$$(2+x)(y-x)=0$$
$$y\left(2+x-x^2\right)=0\,.$$

The origin is evidently a critical point. If $x=\,-2$, then $y=0$. If $x=y$, then either $y=0$ or $x=y=\,-1$ or $x=y=2$. Hence the other critical points are $(\,-2\,,0)$, $(\,-1\,,\,-1)$ and $(2\,,2)$.

(b).



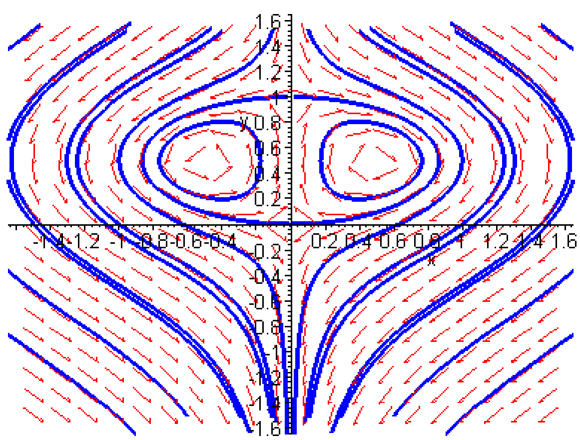
(c). Based on the global phase portrait, the critical points $(0\,,0)$ and $(\,-\,2\,,0)$ have the characteristics of a *saddle*. Hence these points are *unstable*. The behavior near the remaining two critical points resembles those near a *stable spiral*. Hence the critical points $(\,-\,1\,,\,-\,1)$ and $(2\,,2)$ are *asymptotically stable*.

$11(a)$. The critical points are given by the solution set of the equations

$$x(1-2y)=0$$
$$y-x^2-y^2=0\,.$$

If $x=0$, then either $y=0$ or $y=1$. If $y=1/2$, then $x=\pm 1/2$. Hence the critical points are at $(0\,,0)$, $(0\,,1)$, $(\,-\,1/2\,,1/2)$ and $(1/2\,,1/2)$.

(b).



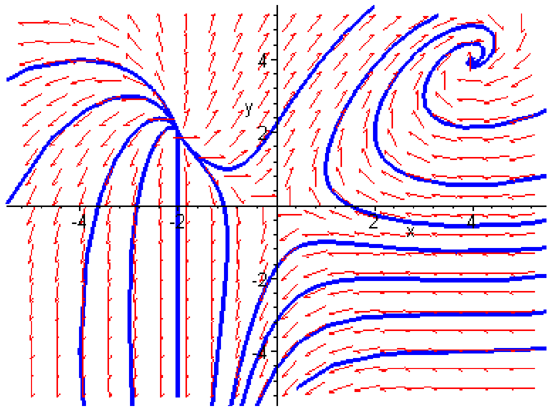
(c). The trajectories near the critical points $(\,-\,1/2\,,1/2)$ and $(1/2\,,1/2)$ are closed curves. Hence the critical points have the characteristics of a *center*, which is *stable*. The trajectories near the critical points $(0\,,0)$ and $(0,1)$ resemble those near a *saddle*. Hence these critical points are *unstable*.

$13(a)$. The critical points are solutions of the equations

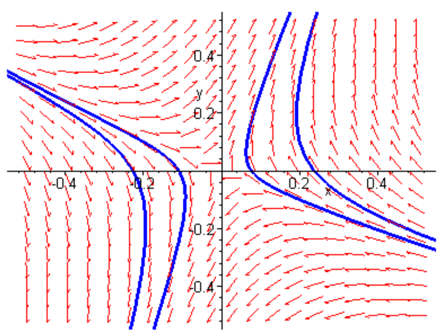
$$(2+x)(y-x)=0$$
$$(4-x)(y+x)=0\,.$$

If $y=x$, then either $x=y=0$ or $x=y=4$. If $x=-2$, then $y=2$. If $x=-y$, then $y=2$ or $y=0$. Hence the critical points are at $(0\,,0)$, $(4\,,4)$ and $(-2\,,2)$.

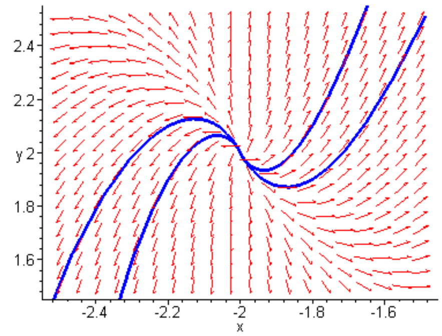
(b).



(c). The critical point at $(4\,,4)$ is evidently a *stable spiral*, which is *asymptotically stable*. Closer examination of the critical point at $(0\,,0)$ reveals that it is a *saddle*, which is *unstable*.



The trajectories near the critical point $(-2\,,2)$ resemble those near an *unstable node*.

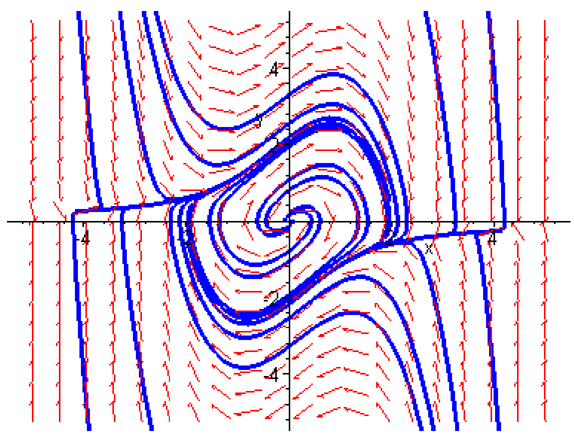


$14(a)$. The critical points consist of the solution set of the equations

$$y = 0$$
$$(1 - x^2)y - x = 0\,.$$

It is easy to see that the only critical point is at $(0\,,0)$.

(b).



(c). The origin is an *unstable spiral*.

$16(a)$. The trajectories are solutions of the differential equation

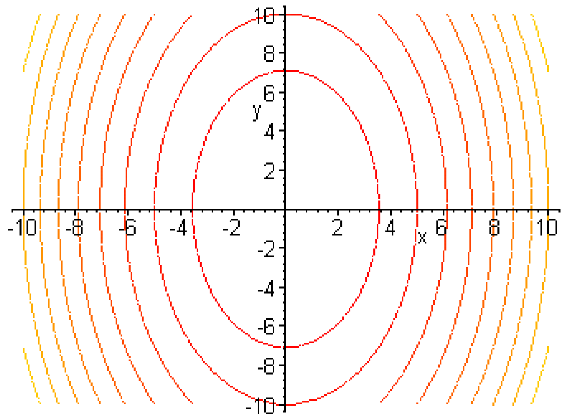
$$\frac{dy}{dx} = -\frac{4x}{y},$$

which can also be written as $4x\,dx + y\,dy = 0$. Integrating, we obtain

$$4x^2 + y^2 = C^2.$$

Hence the trajectories are ellipses.

(b).



Based on the differential equations, the direction of motion on each trajectory is *clockwise*.

$17(a)$. The trajectories of the system satisfy the ODE

$$\frac{dy}{dx} = \frac{2x+y}{y}\,,$$

which can also be written as $(2x+y)dx - ydy = 0$. This differential equation is *homogeneous*. Setting $y = x\,v(x)$, we obtain

$$v + x\frac{dv}{dx} = \frac{2}{v} + 1\,,$$

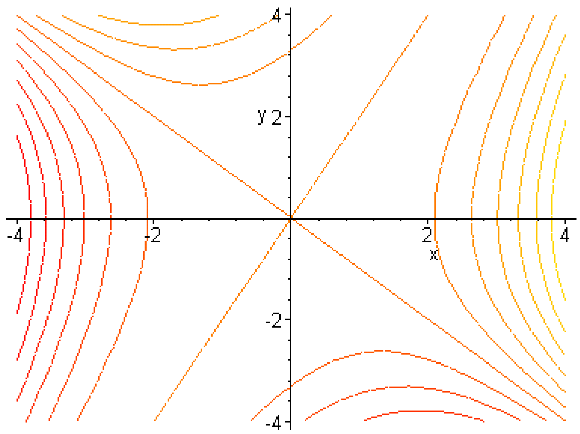
that is,

$$x\frac{dv}{dx} = \frac{2+v-v^2}{v}\,.$$

The resulting ODE is *separable*, with solution $x^3(v+1)(v-2)^2 = C$. Reverting back to the original variables, the trajectories are level curves of

$$H(x\,,y) = (x+y)(y-2x)^2.$$

(b).



The origin is a *saddle*. Along the line $y = 2x$, solutions increase without bound. Along the line $y = -x$, solutions converge toward the origin.

$18(a)$. The trajectories are solutions of the differential equation

$$\frac{dy}{dx} = \frac{x+y}{x-y},$$

which is *homogeneous*. Setting $y = x\,v(x)$, we obtain

$$v + x\frac{dv}{dx} = \frac{x+xv}{x-xv},$$

that is,

$$x\frac{dv}{dx} = \frac{1+v^2}{1-v}.$$

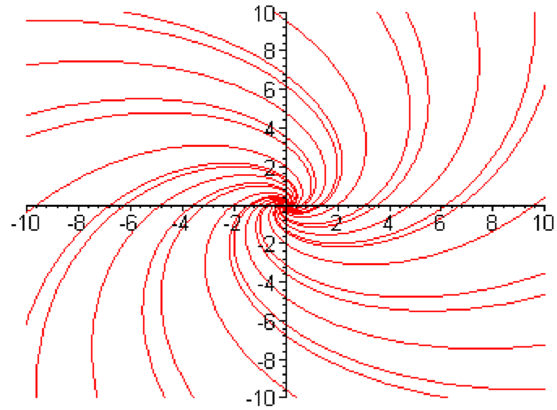
The resulting ODE is *separable*, with solution

$$arctan(v) = ln|x|\sqrt{1+v^2}.$$

Reverting back to the original variables, the trajectories are level curves of

$$H(x\,,y) = arctan(y/x) - ln\sqrt{x^2+y^2}.$$

(b).



The origin is a *stable spiral*.

$20(a)$. The trajectories are solutions of the differential equation

$$\frac{dy}{dx} = \frac{-2xy^2 + 6\,xy}{2x^2y - 3x^2 - 4y},$$

which can also be written as $(2xy^2 - 6\,xy)dx + (2x^2y - 3x^2 - 4y)dy = 0$. The resulting ODE is *exact*, with

$$\frac{\partial H}{\partial x} = 2xy^2 - 6\,xy \text{ and } \frac{\partial H}{\partial y} = 2x^2y - 3x^2 - 4y\,.$$

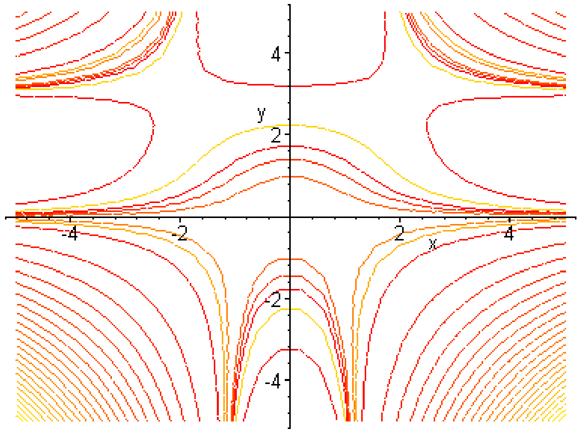
Integrating the first equation, we find that $H(x\,,y) = x^2y^2 - 3x^2y + f(y)$. It follows that

$$\frac{\partial H}{\partial y} = 2x^2y - 3x^2 + f'(y).$$

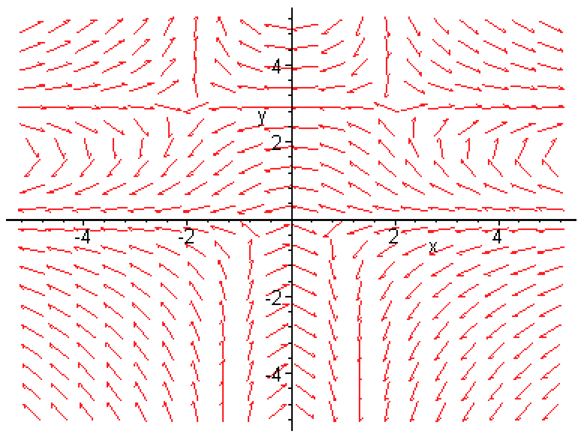
Comparing the two partial derivatives, we obtain $f(y) = -2y^2 + c$. Hence

$$H(x\,,y) = x^2y^2 - 3x^2y - 2y^2.$$

(b).



The associated direction field shows the direction of motion along the trajectories.



$22(a)$. The trajectories are solutions of the differential equation

$$\frac{dy}{dx} = \frac{-6\,x + x^3}{6\,y},$$

which can also be written as $(6\,x - x^3)dx + 6\,ydy = 0$. The resulting ODE is *exact*, with

$$\frac{\partial H}{\partial x} = 6\,x - x^3 \text{ and } \frac{\partial H}{\partial y} = 6\,y\,.$$

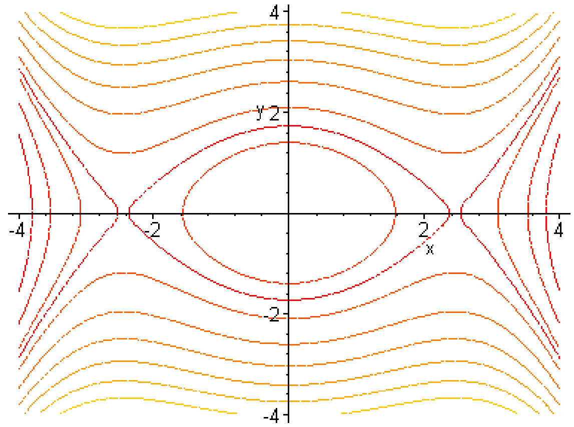
Integrating the first equation, we have $H(x\,,y) = 3x^2 - x^4/4 + f(y)$. It follows that

$$\frac{\partial H}{\partial y} = f'(y).$$

Comparing the two partial derivatives, we conclude that $f(y) = 3y^2 + c$. Hence

$$H(x\,,y) = 3x^2 - \frac{x^4}{4} + 3y^2.$$

(b).



Section 9.3

1. Write the system in the form $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(\mathbf{x})$. In this case, it is evident that

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -y^2 \\ x^2 \end{pmatrix}.$$

That is, $\mathbf{g}(\mathbf{x}) = (-y^2, x^2)^T$. Using polar coordinates, $\|\mathbf{g}(\mathbf{x})\| = r^2\sqrt{sin^4\theta + cos^4\theta}$ and $\|\mathbf{x}\| = r$. Hence

$$\lim_{r\to 0}\frac{\|\mathbf{g}(\mathbf{x})\|}{\|\mathbf{x}\|} = \lim_{r\to 0} r\sqrt{sin^4\theta + cos^4\theta} = 0\,,$$

and the system is *almost linear*. The origin is an isolated critical point of the linear system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

The characteristic equation of the coefficient matrix is $r^2 + r - 2 = 0$, with roots $r_1 = 1$ and $r_2 = -2$. Hence the critical point is a *saddle*, which is *unstable*.

2. The system can be written as

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -4 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix}.$$

Following the discussion in Example 3, we note that $F(x\,,y) = -x + y + 2xy$ and $G(x\,,y) = -4x - y + x^2 - y^2$. Both of the functions F and G are *twice differentiable*, hence the system is *almost linear*. Furthermore,

$$F_x = -1 + 2y\,,\ F_y = 1 + 2x\,,\ G_x = -4 + 2x\,,\ G_y = -1 - 2y\,.$$

The origin is an isolated critical point, with

$$\begin{pmatrix} F_x(0\,,0) & F_y(0\,,0) \\ G_x(0\,,0) & G_y(0\,,0) \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -4 & -1 \end{pmatrix}.$$

The characteristic equation of the associated linear system is $r^2 + 2\,r + 5 = 0$, with complex conjugate roots $r_{1,2} = -1 \pm 2i$. The origin is a *stable spiral*, which is *asymptotically stable*.

5(a). The critical points consist of the solution set of the equations

$$(2 + x)(y - x) = 0$$
$$(4 - x)(y + x) = 0\,.$$

As shown in Prob. 13 of Section 9.2, the only critical points are at $(0\,,0)$, $(4\,,4)$ and $(-2\,,2)$.

(b,c). First note that $F(x\,,y) = (2+x)(y-x)$ and $G(x\,,y) = (4-x)(y+x)$. The *Jacobian* matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} F_x(x\,,y) & F_y(x\,,y) \\ G_x(x\,,y) & G_y(x\,,y) \end{pmatrix} = \begin{pmatrix} -2-2x+y & 2+x \\ 4-y-2x & 4-x \end{pmatrix}.$$

At the origin, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,0) = \begin{pmatrix} -2 & 2 \\ 4 & 4 \end{pmatrix},$$

with eigenvalues $r_1 = 1-\sqrt{17}$ and $r_2 = 1+\sqrt{17}$. The eigenvalues are real, with opposite sign. Hence the critical point is a *saddle*, which is *unstable*. At the equilibrium point $(-2\,,2)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(-2\,,2) = \begin{pmatrix} 4 & 0 \\ 6 & 6 \end{pmatrix},$$

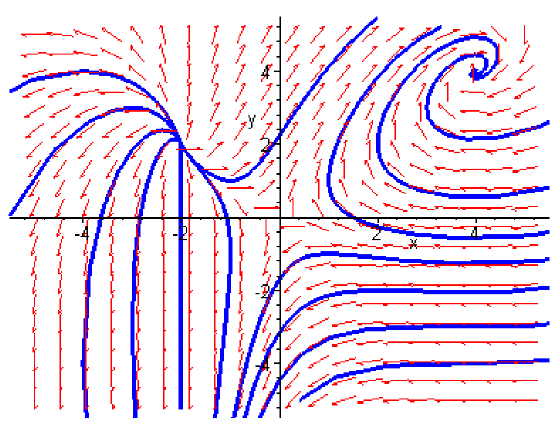
with eigenvalues $r_1 = 4$ and $r_2 = 6$. The eigenvalues are real, unequal and positive, hence the critical point is an *unstable node*. At the point $(4\,,4)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(4\,,4) = \begin{pmatrix} -6 & 6 \\ -8 & 0 \end{pmatrix},$$

with complex conjugate eigenvalues $r_{1,2} = -3 \pm i\sqrt{39}$. The critical point is a *stable spiral*, which is *asymptotically stable*.

Based on Table 9.3.1, the nonlinear terms do not affect the stability and type of each critical point.

(d).



$7(a)$. The critical points are solutions of the equations

$$1 - y = 0$$
$$(x - y)(x + y) = 0\,.$$

The first equation requires that $y = 1$. Based on the second equation, $x = \pm 1$. Hence the critical points are $(-1\,,1)$ and $(1\,,1)$.

(b, c). $F(x\,,y) = 1 - y$ and $G(x\,,y) = x^2 - y^2$. The *Jacobian* matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} F_x(x\,,y) & F_y(x\,,y) \\ G_x(x\,,y) & G_y(x\,,y) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 2x & -2y \end{pmatrix}.$$

At the critical point $(-1\,,1)$, the coefficient matrix of the linearized system is

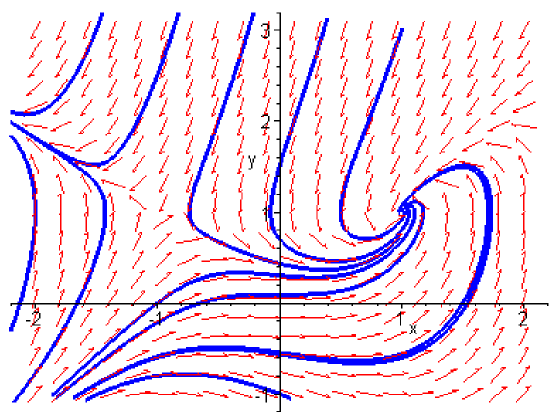
$$\mathbf{J}(-1\,,1) = \begin{pmatrix} 0 & -1 \\ -2 & -2 \end{pmatrix},$$

with eigenvalues $r_1 = -1 - \sqrt{3}$ and $r_2 = -1 + \sqrt{3}$. The eigenvalues are real, with opposite sign. Hence the critical point is a *saddle*, which is *unstable*. At the equilibrium point $(1\,,1)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(1\,,1) = \begin{pmatrix} 0 & -1 \\ 2 & -2 \end{pmatrix},$$

with complex conjugate eigenvalues $r_{1,2} = -1 \pm i$. The critical point is a *stable spiral*, which is *asymptotically stable*.

(d).



Based on Table 9.3.1, the nonlinear terms do not affect the stability and type of each critical point.

$8(a)$. The critical points are given by the solution set of the equations

$$x(1-x-y)=0$$
$$y(2-y-3x)=0\,.$$

If $x=0$, then either $y=0$ or $y=2$. If $y=0$, then $x=0$ or $x=1$. If $y=1-x$, then either $x=1/2$ or $x=1$. If $y=2-3x$, then $x=0$ or $x=1/2$. Hence the critical points are at $(0\,,0)$, $(0\,,2)$, $(1\,,0)$ and $(1/2\,,1/2)$.

(b,c). Note that $F(x\,,y)=x-x^2-xy$ and $G(x\,,y)=(2y-y^2-3xy)/4$. The *Jacobian* matrix of the vector field is

$$\mathbf{J}=\begin{pmatrix} F_x(x\,,y) & F_y(x\,,y) \\ G_x(x\,,y) & G_y(x\,,y)\end{pmatrix}=\begin{pmatrix} 1-2x-y & -x \\ -3y/4 & 1/2-y/2-3x/4\end{pmatrix}.$$

At the origin, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,0)=\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2}\end{pmatrix},$$

with eigenvalues $r_1=1$ and $r_2=1/2$. The eigenvalues are real and both positive. Hence the critical point is an *unstable node*. At the equilibrium point $(0\,,2)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,2)=\begin{pmatrix} -1 & 0 \\ -\frac{3}{2} & -\frac{1}{2}\end{pmatrix},$$

with eigenvalues $r_1=-1$ and $r_2=-1/2$. The eigenvalues are both negative, hence the critical point is a *stable node*. At the point $(1\,,0)$, the coefficient matrix of the linearized system is

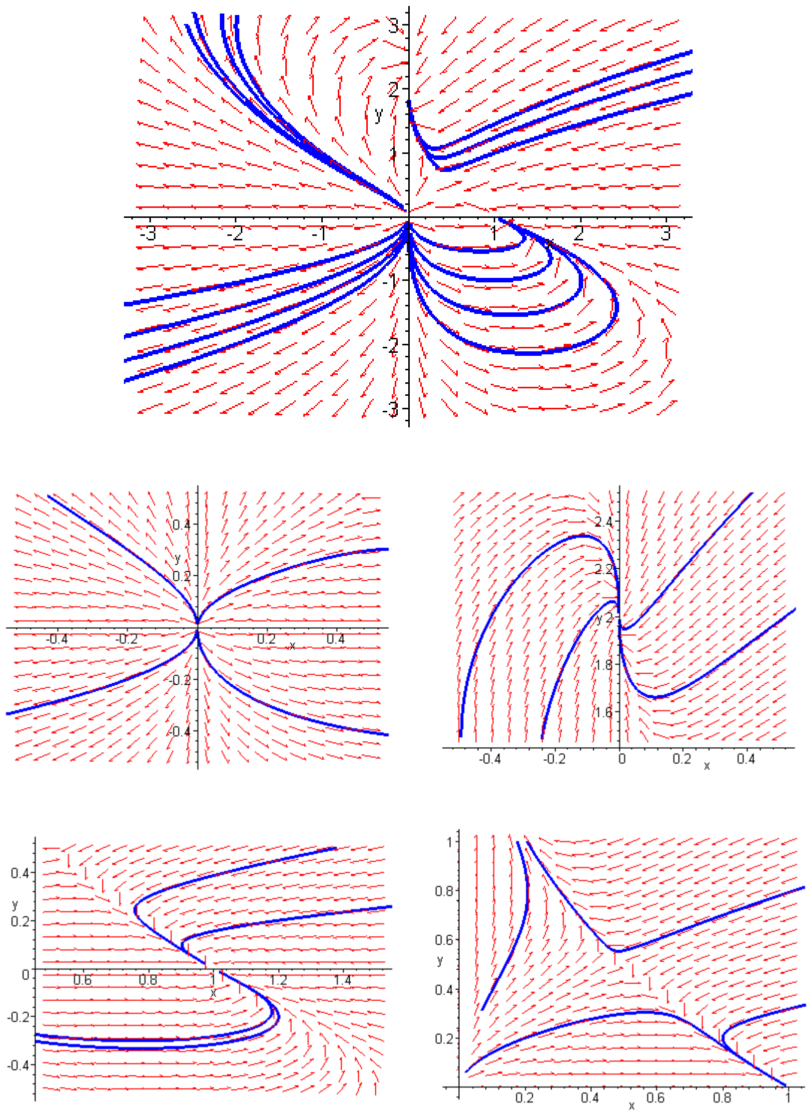
$$\mathbf{J}(1\,,0)=\begin{pmatrix} -1 & -1 \\ 0 & -\frac{1}{4}\end{pmatrix},$$

with eigenvalues $r_1=-1$ and $r_2=-1/4$. Both of the eigenvalues are negative, and hence the critical point is a *stable node*. At the critical point $(1/2\,,1/2)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(1/2\,,1/2)=\begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{3}{8} & -\frac{1}{8}\end{pmatrix},$$

with eigenvalues $r_1=-5/16-\sqrt{57}/16$ and $r_2=-5/16+\sqrt{57}/16$. The eigenvalues are real, with opposite sign. Hence the critical point is a *saddle*, which is *unstable*.

(d).



Based on Table $9.3.1$, the nonlinear terms do not affect the stability and type of each critical point.

$9(a)$. Based on Prob. 8, in Section 9.2, the critical points are at $(0\,,0)$, $(\,-2\,,\,-2)$, $(0\,,1)$ and $(3\,,\,-2)$.

(b,c). First note that $F(x\,,y) = -(x-y)(1-x-y)$ and $G(x\,,y) = x(2+y)$. The *Jacobian* matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 2x-1 & 1-2y \\ 2+y & x \end{pmatrix}.$$

At the origin, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,0) = \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix},$$

with eigenvalues $r_1 = 1$ and $r_2 = -2$. The eigenvalues are real, with opposite sign. Hence the critical point is a *saddle*, which is *unstable*. At the critical point $(0\,,1)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,1) = \begin{pmatrix} -1 & -1 \\ 3 & 0 \end{pmatrix},$$

with complex conjugate eigenvalues $r_{1,2} = -1/2 \pm i\sqrt{11}/2$. The critical point is a *stable spiral*, which is *asymptotically stable*. At the point $(-2\,,-2)$, the coefficient matrix of the linearized system is

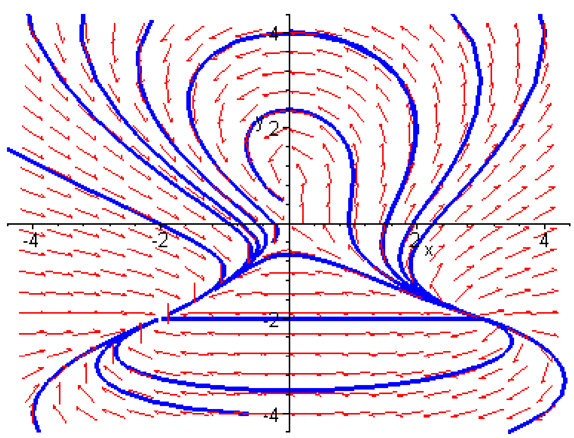
$$\mathbf{J}(-2\,,-2) = \begin{pmatrix} -5 & 5 \\ 0 & -2 \end{pmatrix},$$

with eigenvalues $r_1 = -2$ and $r_2 = -5$. The eigenvalues are unequal and negative, hence the critical point is a *stable node*. At the point $(3\,,-2)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(3\,,-2) = \begin{pmatrix} 5 & 5 \\ 0 & 3 \end{pmatrix},$$

with eigenvalues $r_1 = 3$ and $r_2 = 5$. The eigenvalues are unequal and positive, hence the critical point is an *unstable node*.

(d).



Based on Table 9.3.1 , the nonlinear terms do not affect the stability and type of each critical point.

$11(a)$. The critical points are solutions of the equations

$$\begin{aligned} 2x + y + xy^3 &= 0 \\ x - 2y - xy &= 0\,. \end{aligned}$$

Substitution of $y = x/(x+2)$ into the first equation results in

$$3x^4 + 13x^3 + 28x^2 + 20x = 0\,.$$

One root of the resulting equation is $x = 0$. The only other real root of the equation is

$$x = \frac{1}{9}\left[\left(287 + 18\sqrt{2019}\right)^{1/3} - 83\left(287 + 18\sqrt{2019}\right)^{-1/3} - 13\right].$$

Hence the critical points are $(0\,,0)$ and $(\,-\,1.19345...\,,1.4797...)$.

(b, c). $F(x\,,y) = x - x^2 - xy$ and $G(x\,,y) = (2y - y^2 - 3xy)/4$. The *Jacobian* matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} F_x(x\,,y) & F_y(x\,,y) \\ G_x(x\,,y) & G_y(x\,,y) \end{pmatrix} = \begin{pmatrix} 2 + y^3 & 1 + 3xy^2 \\ 1 - y & -2 - x \end{pmatrix}.$$

At the origin, the coefficient matrix of the linearized system is

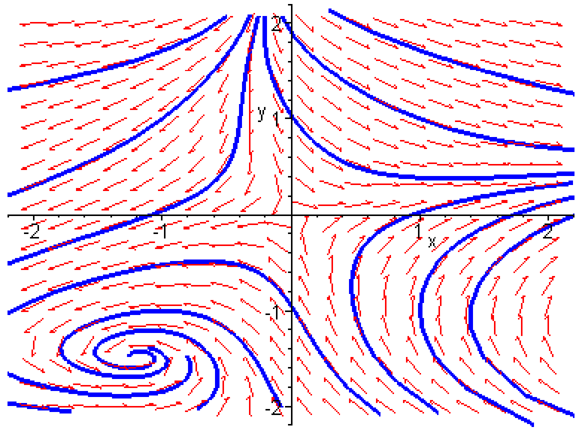
$$\mathbf{J}(0\,,0) = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix},$$

with eigenvalues $r_1 = \sqrt{5}$ and $r_2 = -\sqrt{5}$. The eigenvalues are real and of opposite sign. Hence the critical point is a *saddle*, which is *unstable*. At the equilibrium point $(\,-\,1.19345...\,,1.4797...)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(\,-\,1.19345\,,1.4797) = \begin{pmatrix} -\,1.2399 & -\,6.8393 \\ -\,2.4797 & -\,0.8065 \end{pmatrix},$$

with complex conjugate eigenvalues $r_{1,2} = -\,1.0232 \pm 4.1125\,i$. The critical point is a *stable spiral*, which is *asymptotically stable*.

(d).



In both cases, the nonlinear terms do not affect the stability and type of the critical point.

$12(a)$. The critical points are given by the solution set of the equations

$$(1+x)sin\,y = 0$$
$$1 - x - cos\,y = 0\,.$$

If $x = -1$, then we must have $cos\,y = 2$, which is impossible. Therefore $sin\,y = 0$, which implies that $y = n\pi\,, n = 0\,, \pm 1\,, 2\,, \ldots$. Based on the second equation,

$$x = 1 - cos\,n\pi\,.$$

It follows that the critical points are located at $(0\,, 2k\pi)$ and $(2\,, (2k+1)\pi)$, where $k = 0\,, \pm 1\,, 2\,, \ldots$.

(b, c). Given that $F(x\,, y) = (1+x)sin\,y$ and $G(x\,, y) = 1 - x - cos\,y$, the *Jacobian* matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} sin\,y & (1+x)cos\,y \\ -1 & sin\,y \end{pmatrix}.$$

At the critical points $(0\,, 2k\pi)$, the coefficient matrix of the linearized system is

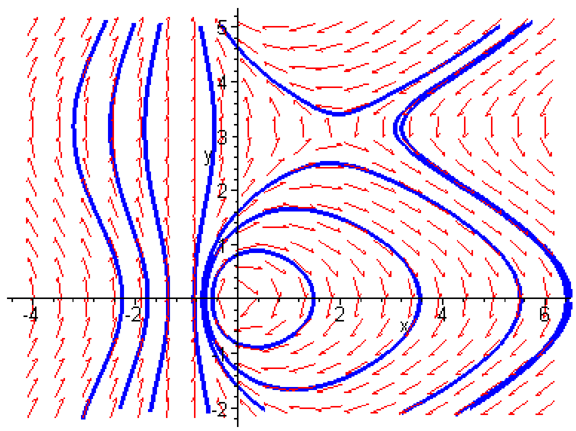
$$\mathbf{J}(0\,, 2k\pi) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

with purely complex eigenvalues $r_{1,2} = \pm\, i$. The critical points of the associated linear systems are *centers*, which are *stable*. Note that Theorem 9.3.2 does *not* provide a definite conclusion regarding the relation between the nature of the critical points of the nonlinear systems and their corresponding linearizations. At the points $(2\,, (2k+1)\pi)$, the coefficient matrix of the linearized system is

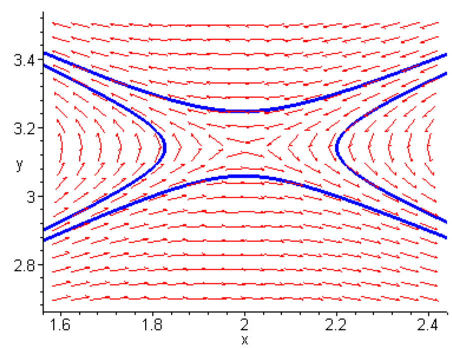
$$\mathbf{J}[2\,, (2k+1)\pi] = \begin{pmatrix} 0 & -3 \\ -1 & 0 \end{pmatrix},$$

with eigenvalues $r_1 = \sqrt{3}$ and $r_2 = -\sqrt{3}$. The eigenvalues are real, with opposite sign. Hence the critical points of the associated linear systems are *saddles*, which are *unstable*.

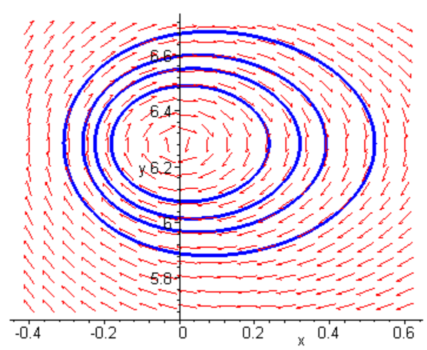
(d).



As asserted in Theorem 9.3.2 , the trajectories near the critical points $(2\,,(2k+1)\pi)$ resemble those near a saddle.



Upon closer examination, the critical points $(0\,,2k\pi)$ are indeed centers.



$13(a)$. The critical points are solutions of the equations

$$x - y^2 = 0$$
$$y - x^2 = 0\,.$$

Substitution of $y = x^2$ into the first equation results in

$$x - x^4 = 0\,,$$

with real roots $x = 0\,,1\,.$ Hence the critical points are at $(0\,,0)$ and $(1\,,1)$.

(b,c). In this problem, $F(x\,,y) = x - y^2$ and $G(x\,,y) = y - x^2\,.$ The *Jacobian* matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 1 & -\,2y \\ -\,2x & 1 \end{pmatrix}.$$

At the origin, the coefficient matrix of the linearized system is

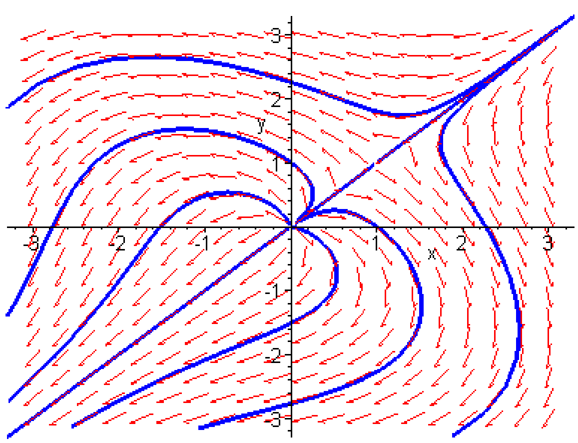
$$\mathbf{J}(0\,,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

with *repeated* eigenvalues $r_1 = 1$ and $r_2 = 1\,.$ It is easy to see that the corresponding eigenvectors are linearly independent. Hence the critical point is an *unstable proper node*. Theorem 9.3.2 does *not* provide a definite conclusion regarding the relation between the nature of the critical point of the nonlinear system and the corresponding linearization. At the critical point $(1\,,1)$, the coefficient matrix of the linearized system is

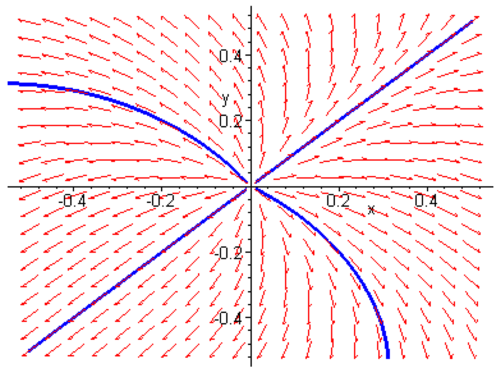
$$\mathbf{J}(1\,,1) = \begin{pmatrix} 1 & -\,2 \\ -\,2 & 1 \end{pmatrix},$$

with eigenvalues $r_1 = 3$ and $r_2 = \,-\,1\,.$ The eigenvalues are real, with opposite sign. Hence the critical point is a *saddle*, which is *unstable*.

(d).



Closer examination reveals that the critical point at the origin is indeed a proper node.



$14(a)$. The critical points are given by the solution set of the equations

$$1 - xy = 0$$
$$x - y^3 = 0\,.$$

After multiplying the second equation by y, it follows that $y = \pm 1$. Hence the critical points of the system are at $(1\,,1)$ and $(\,-1\,,\,-1)$.

(b,c). Note that $F(x\,,y) = 1 - xy$ and $G(x\,,y) = x - y^3$. The *Jacobian* matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} -\,y & -\,x \\ 1 & -\,3y^2 \end{pmatrix}.$$

At the critical point $(1\,,1)$, the coefficient matrix of the linearized system is

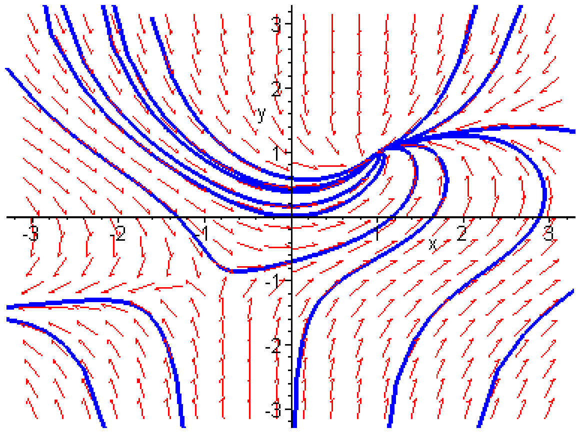
$$\mathbf{J}(1\,,1) = \begin{pmatrix} -\,1 & -\,1 \\ 1 & -\,3 \end{pmatrix},$$

with eigenvalues $r_1 = \,-\,2$ and $r_2 = \,-\,2$. The eigenvalues are real and *equal*. It is easy to show that there is only *one* linearly independent eigenvector. Hence the critical point is a *stable improper node*. Theorem 9.3.2 does *not* provide a definite conclusion regarding the relation between the nature of the critical point of the nonlinear system and the corresponding linearization. At the point $(\,-\,1\,,\,-\,1)$, the coefficient matrix of the linearized system is

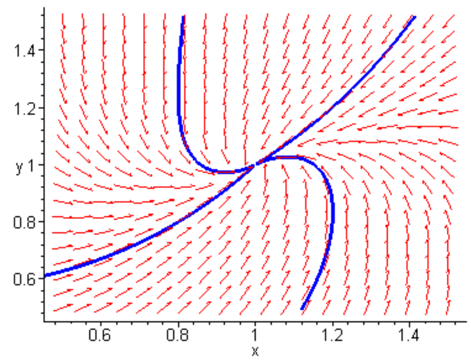
$$\mathbf{J}(\,-\,1\,,\,-\,1) = \begin{pmatrix} 1 & 1 \\ 1 & -\,3 \end{pmatrix},$$

with eigenvalues $r_1 = \,-\,1 + \sqrt{5}$ and $r_2 = \,-\,1 - \sqrt{5}$. The eigenvalues are real, with opposite sign. Hence the critical point of the associated linear system is a *saddle*, which is *unstable*.

(d).



Closer examination reveals that the critical point at $(1\,,1)$ is indeed a *stable* improper node, which is asymptotically stable.



$15(a)$. The critical points are given by the solution set of the equations

$$\begin{aligned} -2x - y - x\left(x^2 + y^2\right) &= 0 \\ x - y + y\left(x^2 + y^2\right) &= 0\,. \end{aligned}$$

It is clear that the origin is a critical point. Solving the *first* equation for y, we find that

$$y = \frac{-1 \pm \sqrt{1 - 8x^2 - 4x^4}}{2x}\,.$$

Substitution of these relations into the *second* equation results in two equations of the form $f_1(x) = 0$ and $f_2(x) = 0$. Plotting these functions, we note that only $f_1(x) = 0$ has real roots given by $x \approx \pm\, 0.33076$. It follows that the additional critical points are at $(-\,0.33076\,, 1.0924)$ and $(0.33076\,, -\,1.0924)$.

(b,c). Given that

$$F(x\,,y) = -2x - y - x\left(x^2 + y^2\right)$$
$$G(x\,,y) = x - y + y\left(x^2 + y^2\right),$$

the *Jacobian* matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} -2 - 3x^2 - y^2 & -1 - 2xy \\ 1 + 2xy & -1 + x^2 + 3y^2 \end{pmatrix}.$$

At the critical point $(0\,,0)$, the coefficient matrix of the linearized system is

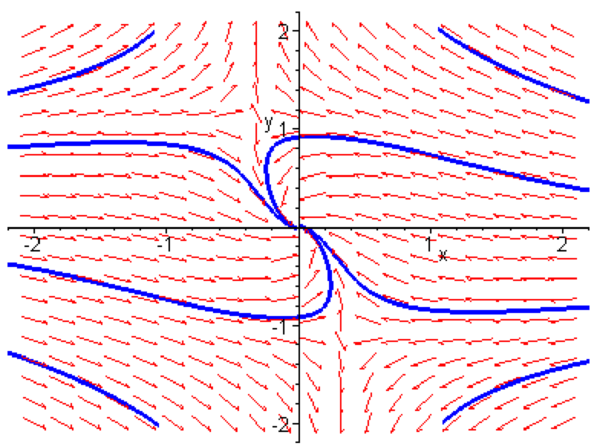
$$\mathbf{J}(0\,,0) = \begin{pmatrix} -2 & -1 \\ 1 & -1 \end{pmatrix},$$

with complex conjugate eigenvalues $r_{1,2} = \left(-3 \pm i\sqrt{3}\right)/2$. Hence the critical point is a *stable spiral*, which is *asymptotically stable*. At the point $(-\ 0.33076\,, 1.0924)$, the coefficient matrix of the linearized system is

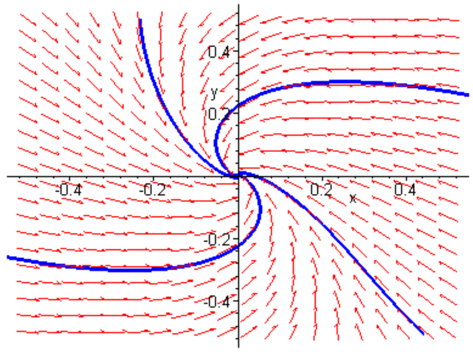
$$\mathbf{J}(-\ 0.33076\,, 1.0924) = \begin{pmatrix} -3.5216 & -0.27735 \\ 0.27735 & 2.6895 \end{pmatrix},$$

with eigenvalues $r_1 = -3.5092$ and $r_2 = 2.6771$. The eigenvalues are real, with opposite sign. Hence the critical point of the associated linear system is a *saddle*, which is *unstable*. Identical results hold for the point at $(0.33076\,, -1.0924)$.

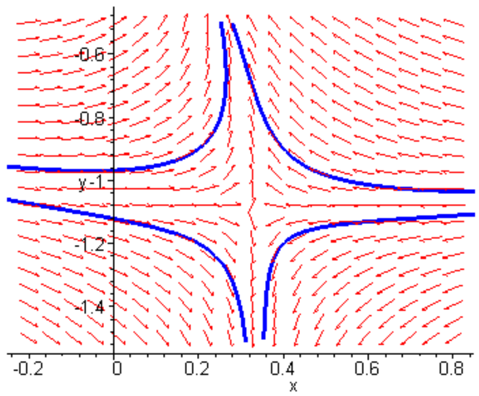
(d).



A closer look at the origin reveals a spiral:



Near the point $(0.33076\,,\ -1.0924)$ the nature of the critical point is evident:



Based on Table 9.3.1 , the nonlinear terms do not affect the stability and type of each critical point.

$16(a)$. The critical points are solutions of the equations

$$y + x\left(1 - x^2 - y^2\right) = 0$$
$$-x + y\left(1 - x^2 - y^2\right) = 0\,.$$

Multiply the *first* equation by y and the *second* equation by x. The difference of the two equations gives $x^2 + y^2 = 0$. Hence the only critical point is at the origin.

(b, c). With $F(x\,,y) = y + x(1 - x^2 - y^2)$ and $G(x\,,y) = -x + y(1 - x^2 - y^2)$, the *Jacobian* matrix of the vector field is

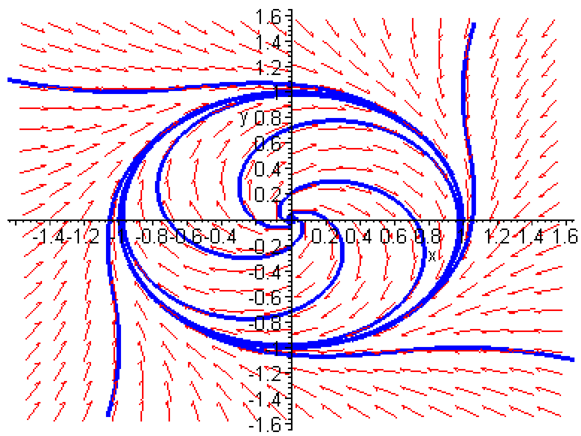
$$\mathbf{J} = \begin{pmatrix} 1 - 3x^2 - y^2 & 1 - 2xy \\ -1 - 2xy & 1 - x^2 - 3y^2 \end{pmatrix}.$$

At the origin, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,0) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

with complex conjugate eigenvalues $r_{1,2} = 1 \pm i$. Hence the origin is an *unstable* *spiral*.

(d).



17(a). The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ 1+6x^2 & 0 \end{pmatrix}.$$

At the origin, the coefficient matrix of the linearized system is

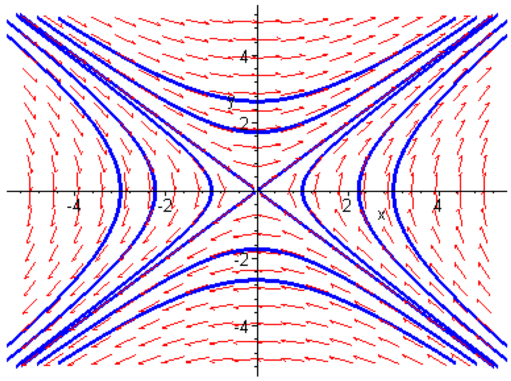
$$\mathbf{J}(0\,,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

with eigenvalues $r_1 = 1$ and $r_2 = -1$. The eigenvalues are real, with opposite sign. Hence the critical point is a *saddle point*.

(b). The trajectories of the *linearized* system are solutions of the differential equation

$$\frac{dy}{dx} = \frac{x}{y},$$

which is separable. Integrating both sides of the equation $x\,dx - y\,dy = 0$, the solution is $x^2 - y^2 = C$. The trajectories consist of a family of hyperbolas.



It is easy to show that the general solution is given by $x(t) = c_1 e^t + c_2 e^{-t}$ and $y(t) = c_1 e^t - c_2 e^{-t}$. The only *bounded* solutions consist of those for which $c_1 = 0$. In that case, $x(t) = c_2 e^{-t} = -y(t)$.

(c). The trajectories of the given system are solutions of the differential equation

$$\frac{dy}{dx} = \frac{x + 2x^3}{y},$$

which can also be written as $(x + 2x^3)dx - y\,dy = 0$. The resulting ODE is *exact*, with

$$\frac{\partial H}{\partial x} = x + 2x^3 \text{ and } \frac{\partial H}{\partial y} = -y\,.$$

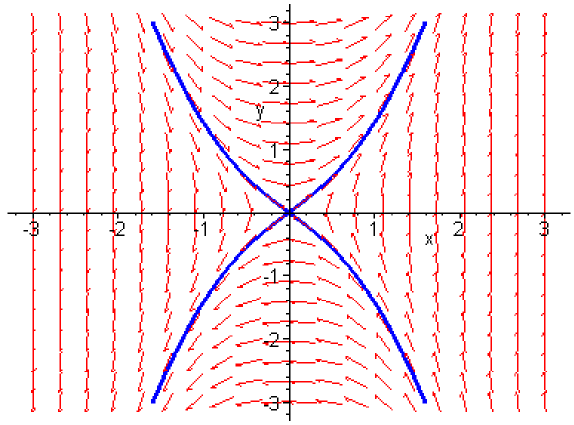
Integrating the first equation, we find that $H(x\,,y) = x^2/2 + x^4/2 + f(y)$. It follows that

$$\frac{\partial H}{\partial y} = f'(y).$$

Comparing the partial derivatives, we obtain $f(y) = -y^2/2 + c$. Hence the solutions are level curves of the function

$$H(x\,,y) = x^2/2 + x^4/2 - y^2/2\,.$$

The trajectories *approaching* to, or *diverging* from, the origin are no longer straight lines.



$19(a)$. The solutions of the system of equations

$$y = 0$$
$$-\,\omega^2 sin\, x = 0$$

consist of the points $(\pm\, n\pi\,, 0)\,, n = 0\,, 1\,, 2\,, \cdots$. The functions $F(x\,, y) = y$ and $G(x\,, y) = \,-\,\omega^2 sin\, x$ are *analytic* on the entire plane. It follows that the system is almost linear near each of the critical points.

(b). The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ -\,\omega^2 cos\, x & 0 \end{pmatrix}.$$

At the origin, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,, 0) = \begin{pmatrix} 0 & 1 \\ -\,\omega^2 & 0 \end{pmatrix},$$

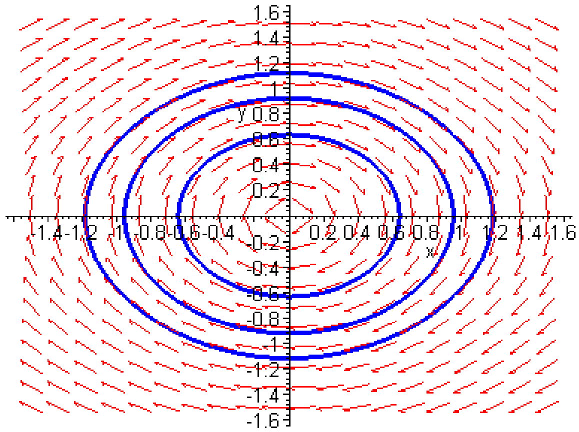
with purely complex eigenvalues $r_{1,2} = \,\pm\, i\omega$. Hence the origin is a *center*. Since the eigenvalues are purely complex, Theorem 9.3.2 gives no definite conclusion about the critical point of the nonlinear system. Physically, the critical point corresponds to the state $\theta = 0\,, \theta' = 0$. That is, the rest configuration of the pendulum.

(c). At the critical point $(\pi\,, 0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(\pi\,, 0) = \begin{pmatrix} 0 & 1 \\ \omega^2 & 0 \end{pmatrix},$$

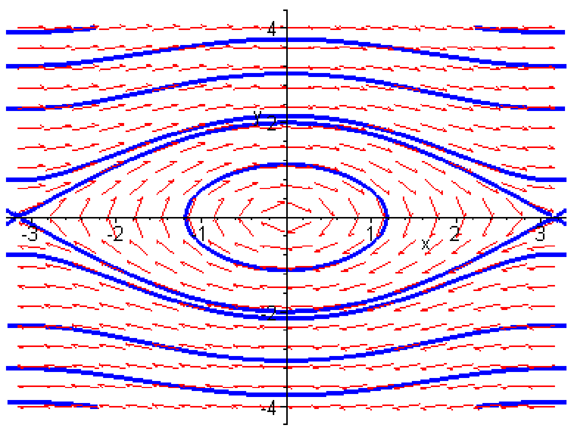
with eigenvalues $r_{1,2} = \,\pm\, \omega$. The eigenvalues are real and of opposite sign. Hence the critical point is a *saddle*. Theorem 9.3.2 asserts that the critical point for the nonlinear system is also a saddle, which is unstable. This critical point corresponds to the state $\theta = \pi\,, \theta' = 0$. That is, the *upright* rest configuration.

(d). Let $\omega^2 = 1$. The following is a plot of the phase curves near $(0\,,0)$.



The local phase portrait shows that the origin is indeed a center.

(e).



It should be noted that the phase portrait has a periodic pattern, since $\theta = x \; mod \, 2\pi$.

$20(a)$. The trajectories of the system in Problem 19 are solutions of the differential equation

$$\frac{dy}{dx} = \frac{-\,\omega^2 sin\,x}{y},$$

which can also be written as $\omega^2 sin\,x\,dx + y\,dy = 0$. The resulting ODE is *exact*, with

$$\frac{\partial H}{\partial x} = \omega^2 sin\,x \;\text{ and }\; \frac{\partial H}{\partial y} = y\,.$$

Integrating the first equation, we find that $H(x\,,y) = -\,\omega^2 cos\,x + f(y)$. It follows that

$$\frac{\partial H}{\partial y} = f'(y).$$

Comparing the partial derivatives, we obtain $f(y) = y^2/2 + C$. Hence the solutions are level curves of the function

$$H(x\,,y) = \,-\,\omega^2 cos\,x + y^2/2\,.$$

Adding an arbitrary constant, say ω^2, to the function $H(x\,,y)$ does not change the nature of the level curves. Hence the trajectories are can be written as

$$\frac{1}{2}y^2 + \omega^2(1 - cos\,x) = c\,,$$

in which c is an arbitrary constant.

(b). Multiplying by mL^2 and reverting to the original physical variables, we obtain

$$\frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2 + mL^2\omega^2(1 - cos\,\theta) = mL^2c\,.$$

Since $\omega^2 = g/L$, the equation can be written as

$$\frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2 + mgL(1 - cos\,\theta) = E\,,$$

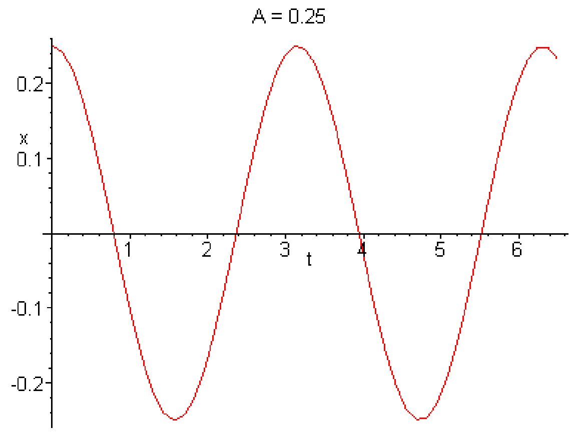
in which $E = mL^2c$.

(c). The *absolute velocity* of the point mass is given by $v = L\,d\theta/dt$. The kinetic energy of the mass is $T = mv^2/2$. Choosing the rest position as the *datum*, that is, the level of *zero potential energy*, the gravitational potential energy of the point mass is

$$V = mgL(1 - cos\,\theta).$$

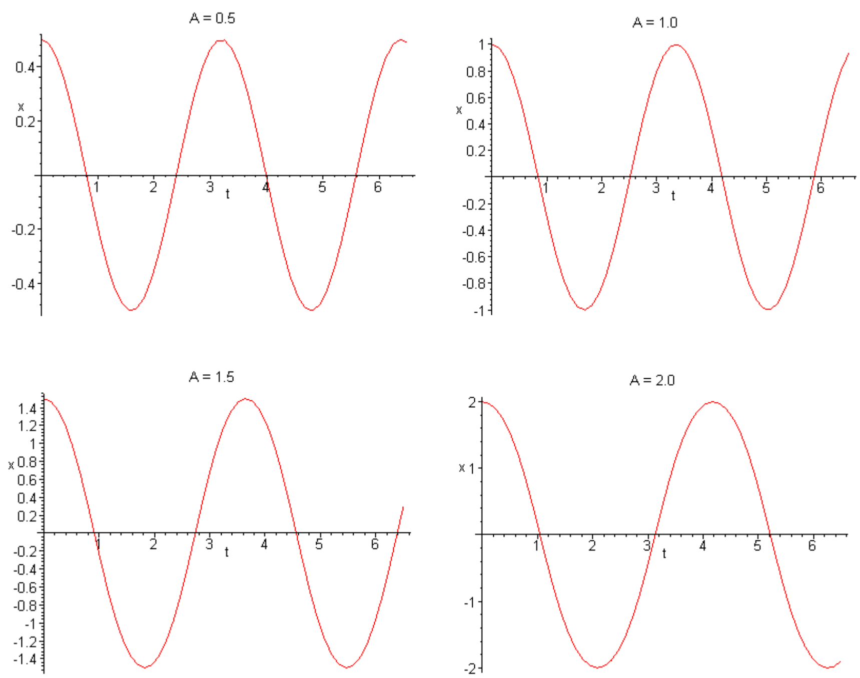
It follows that the total energy, $T + V$, is *constant* along the trajectories.

$21(a)$. $A = 0.25$



Since the system is *undamped*, and $y(0) = 0$, the amplitude is 0.25. The period is estimated at $\tau \approx 3.16$.

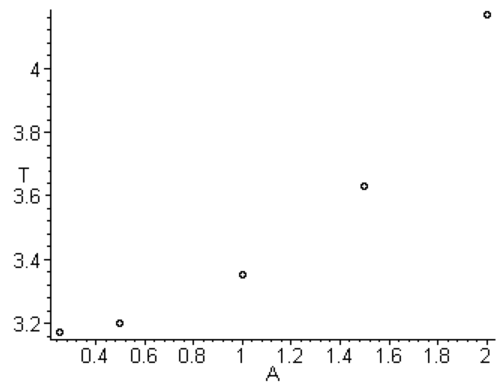
(b).



	R	τ
$A = 0.5$	0.5	3.20
$A = 1.0$	1.0	3.35
$A = 1.5$	1.5	3.63
$A = 2.0$	2.0	4.17

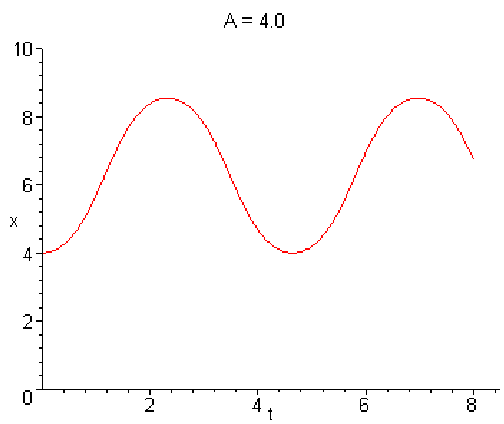
(c). Since the system is conservative, the amplitude is equal to the initial amplitude. On

the other hand, the period of the pendulum is a *monotone increasing* function of the initial
position A.



It appears that as $A \to 0$, the period approaches π, the period of the corresponding *linear* pendulum $(2\pi/\omega)$.

(d).



The pendulum is released from rest, at an inclination of $4 - \pi$ *radians* from the vertical. Based on *conservation of energy*, the pendulum will swing past the lower equilibrium position $(\theta = 2\pi)$ and come to rest, momentarily, at a maximum rotational displacement of $\theta_{max} = 3\pi - (4 - \pi) = 4\pi - 4$. The transition between the two dynamics occurs at $A = \pi$, that is, once the pendulum is released *beyond* the upright configuration.

24(a). It is evident that the origin is a critical point of each system. Furthermore, it is easy to see that the corresponding linear system, in each case, is given by

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -x.$$

The eigenvalues of the coefficient matrix are $r_{1,2} = \pm i$. Hence the critical point of the linearized system is a *center*.

(b). Using polar coordinates, it is also easy to show that

$$\lim_{r\to 0}\frac{\|\mathbf{g}(\mathbf{x})\|}{\|\mathbf{x}\|}=0\,.$$

Alternatively, the nonlinear terms are *analytic* in the entire plane. Hence both systems are
almost linear near the origin.

(c). For system (ii), note that

$$x\frac{dx}{dt}+y\frac{dy}{dt}=xy-x^2\left(x^2+y^2\right)-xy-y^2\left(x^2+y^2\right).$$

Converting to polar coordinates, and differentiating the equation $r^2=x^2+y^2$ with respect to t, we find that

$$r\frac{dr}{dt}=x\frac{dx}{dt}+y\frac{dy}{dt}=\,-\left(x^2+y^2\right)^2=\,-r^4.$$

That is, $r'=\,-r^3$. It follows that $r^2=1/(2t+c)$, where $c=1/r_0^2$. Since $r\to 0$ as $t\to 0$, regardless of the value of r_0, the origin is an $asymptotically\ stable$ equilibrium point.

On the other hand, for system (i),

$$r\frac{dr}{dt}=x\frac{dx}{dt}+y\frac{dy}{dt}=\left(x^2+y^2\right)^2=r^4.$$

That is, $r'=r^3$. Solving the differential equation results in

$$r^2=\frac{c-2t}{(2t-c)^2}\,.$$

Imposing the initial condition $r(0)=r_0$, we obtain a specific solution

$$r^2=\,-\frac{r_0^2}{2\,r_0^2\,t-1}\,.$$

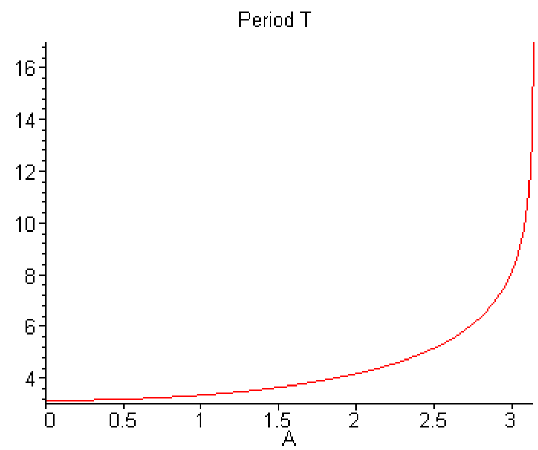
Since the solution becomes *unbounded* as $t\to 1/2r_0^2$, the critical point is *unstable*.

25. The characteristic equation of the coefficient matrix is $r^2+1=0$, with complex roots $r_{1,2}=\pm i$. Hence the critical point at the origin is a *center*. The characteristic equation of the perturbed matrix is $r^2-2\,\epsilon\,r+1+\epsilon^2=0$, with complex conjugate roots $r_{1,2}=\epsilon\pm i$. As long as $\epsilon\neq 0$, the critical point of the perturbed system is a *spiral point*. Its stability depends on the sign of ϵ .

26. The characteristic equation of the coefficient matrix is $(r+1)^2=0$, with roots

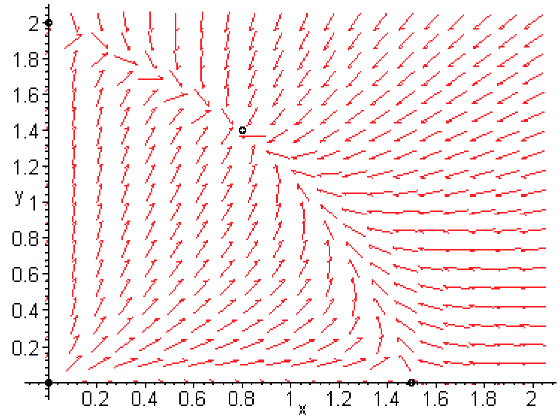
$r_1 = r_2 = -1$. Hence the critical point is an *asymptotically stable node*. On the other hand, the characteristic equation of the perturbed system is $r^2 + 2r + 1 + \epsilon = 0$, with roots $r_{1,2} = -1 \pm \sqrt{-\epsilon}$. If $\epsilon > 0$, then $r_{1,2} = -1 \pm i\sqrt{\epsilon}$ are complex roots. The critical point is a *stable spiral*. If $\epsilon < 0$, then $r_{1,2} = -1 \pm \sqrt{|\epsilon|}$ are real and both negative ($|\epsilon| \ll 1$). The critical point remains a *stable node*.

27(d). Set $k = sin(\alpha/2) = sin(A/2)$ and $g/L = 4$.



Section 9.4

$1(a)$.



(b). The critical points are solutions of the system of equations

$$x(1.5 - x - 0.5\,y) = 0$$
$$y(2 - y - 0.75\,x) = 0\,.$$

The four critical points are $(0\,,0)$, $(0\,,2)$, $(1.5\,,0)$ and $(0.8\,,1.4)$.

(c). The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 3/2 - 2x - y/2 & -\,x/2 \\ -\,3y/4 & 2 - 3x/4 - 2y \end{pmatrix}.$$

At the critical point $(0\,,0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,0) = \begin{pmatrix} 3/2 & 0 \\ 0 & 2 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = 3/2\,,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\,;\ \ r_2 = 2\,,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The eigenvalues are positive, hence the origin is an *unstable node*.

At the critical point $(0\,,2)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,2) = \begin{pmatrix} 1/2 & 0 \\ -\,3/2 & -\,2 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = 1/2\,,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ -0.6 \end{pmatrix};\ \ r_2 = -2\,,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The eigenvalues are of opposite sign. Hence the critical point is a *saddle*, which is *unstable*.

At the critical point $(1.5\,,0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(1.5\,,0) = \begin{pmatrix} -1.5 & -0.75 \\ 0 & 0.875 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = -1.5\,,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix};\ \ r_2 = 0.875\,,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} -0.31579 \\ 1 \end{pmatrix}.$$

The eigenvalues are of opposite sign. Hence the critical point is also a *saddle*, which is *unstable*.

At the critical point $(0.8\,,1.4)$, the coefficient matrix of the linearized system is

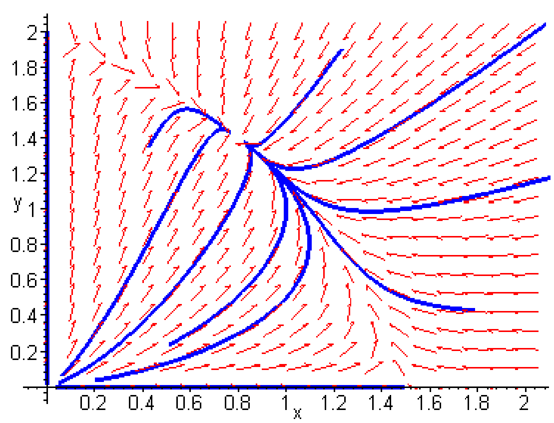
$$\mathbf{J}(0.8\,,1.4) = \begin{pmatrix} -0.8 & -0.4 \\ -1.05 & -1.4 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = -\frac{11}{10} + \frac{\sqrt{51}}{10}\,,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ \frac{3-\sqrt{51}}{4} \end{pmatrix};\ \ r_2 = -\frac{11}{10} - \frac{\sqrt{51}}{10}\,,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ \frac{3+\sqrt{51}}{4} \end{pmatrix}.$$

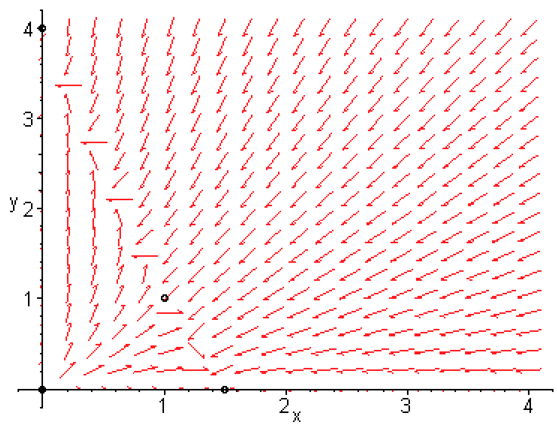
The eigenvalues are both negative. Hence the critical point is a *stable node*, which is *asymptotically stable*.

(d, e).



(f). Except for initial conditions lying on the coordinate axes, almost all trajectories converge to the stable node at $(0.8\,,1.4)$.

$2(a)$.



(b). The critical points are the solution set of the system of equations

$$x(1.5 - x - 0.5\,y) = 0$$
$$y(2 - 0.5\,y - 1.5\,x) = 0\,.$$

The four critical points are $(0\,,0)$, $(0\,,4)$, $(1.5\,,0)$ and $(1\,,1)$.

(c). The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 3/2 - 2x - y/2 & -\,x/2 \\ -\,3y/2 & 2 - 3x/2 - y \end{pmatrix}.$$

At the origin, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,0) = \begin{pmatrix} 3/2 & 0 \\ 0 & 2 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = 3/2\,,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\ ;\ \ r_2 = 2\,,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The eigenvalues are positive, hence the origin is an *unstable node*.

At the critical point $(0\,,4)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,4) = \begin{pmatrix} -\,1/2 & 0 \\ -\,6 & -\,2 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = -1/2\,,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ -4 \end{pmatrix};\ \ r_2 = -2\,,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The eigenvalues are both negative, hence the critical point $(0\,,4)$ is a *stable node*, which is *asymptotically stable*.

At the critical point $(3/2\,,0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(3/2\,,0) = \begin{pmatrix} -3/2 & -3/4 \\ 0 & -1/4 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = -3/2\,,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix};\ \ r_2 = -1/4\,,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}.$$

The eigenvalues are both negative, hence the critical point is a *stable node*, which is *asymptotically stable*.

At the critical point $(1, 1)$, the coefficient matrix of the linearized system is

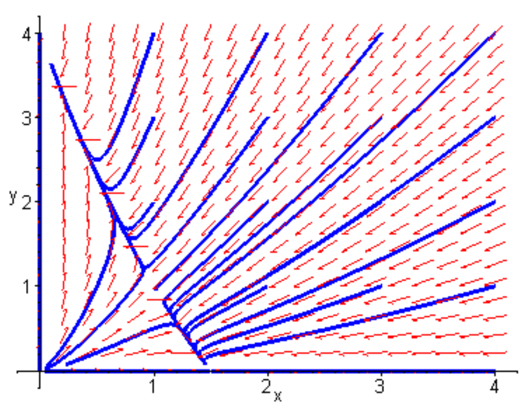
$$\mathbf{J}(1\,,1) = \begin{pmatrix} -1 & -1/2 \\ -3/2 & -1/2 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = \frac{-3+\sqrt{13}}{4}\,,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ -\frac{1+\sqrt{13}}{2} \end{pmatrix};\ \ r_2 = -\frac{3+\sqrt{13}}{4}\,,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ \frac{-1+\sqrt{13}}{2} \end{pmatrix}.$$

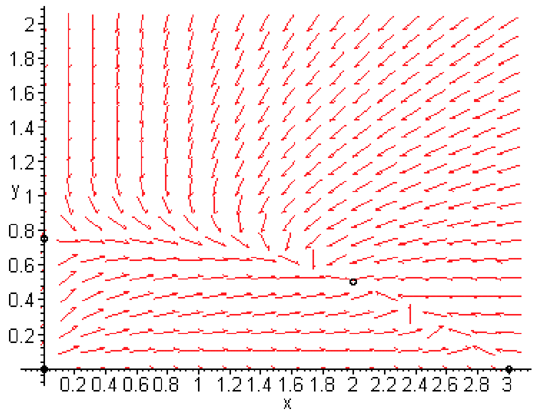
The eigenvalues are of opposite sign, hence $(1, 1)$ is a *saddle*, which is *unstable*.

(d, e).



(f). Trajectories *approaching* the critical point $(1\,,1)$ form a *separatrix*. Solutions on either side of the separatrix approach either $(0\,,4)$ or $(1.5\,,0)$.

$4(a)$.



(b). The critical points are solutions of the system of equations

$$x(1.5 - 0.5\,x - y) = 0$$
$$y(0.75 - y - 0.125\,x) = 0\,.$$

The four critical points are $(0\,,0)$, $(0\,,3/4)$, $(3\,,0)$ and $(2\,,1/2)$.

(c). The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 3/2 - x - y & -\,x \\ -\,y/8 & 3/4 - x/8 - 2y \end{pmatrix}.$$

At the origin, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,0) = \begin{pmatrix} 3/2 & 0 \\ 0 & 3/4 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = 3/2\,,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix};\ \ r_2 = 3/4\,,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The eigenvalues are positive, hence the origin is an *unstable node*.

At the critical point $(0\,,3/4)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,3/4) = \begin{pmatrix} 3/4 & 0 \\ -\,3/32 & -\,3/4 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = 3/4\,,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} -\,16 \\ 1 \end{pmatrix};\ \ r_2 = -\,3/4\,,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The eigenvalues are of opposite sign, hence the critical point $(0\,,3/4)$ is a *saddle*, which is *unstable*.

At the critical point $(3\,,0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(3\,,0) = \begin{pmatrix} -3/2 & -3 \\ 0 & 3/8 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = -3/2\,,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\ ;\ \ r_2 = 3/8\,,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} -8 \\ 5 \end{pmatrix}.$$

The eigenvalues are of opposite sign, hence the critical point $(0\,,3/4)$ is a *saddle*, which is *unstable*.

At the critical point $(2\,,1/2)$, the coefficient matrix of the linearized system is

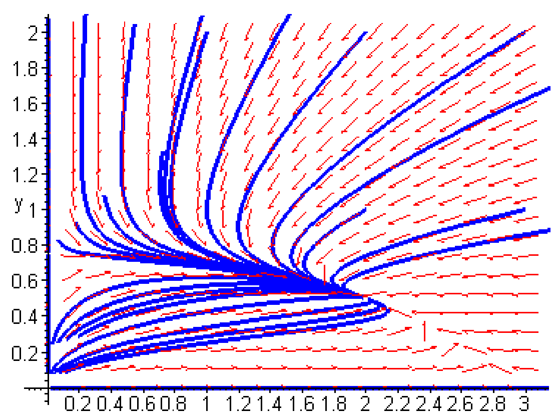
$$\mathbf{J}(2\,,1/2) = \begin{pmatrix} -1 & -2 \\ -1/16 & -1/2 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = \frac{-3+\sqrt{3}}{4}\,,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ -\frac{1+\sqrt{3}}{8} \end{pmatrix}\ ;\ r_2 = -\frac{3+\sqrt{3}}{4}\,,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ \frac{-1+\sqrt{3}}{8} \end{pmatrix}.$$

The eigenvalues are negative, hence the critical point $(2\,,1/2)$ is a *stable node*, which is *asymptotically stable*.

(d,e).



(f). Except for initial conditions along the coordinate axes, almost all solutions converge
to the stable node $(2\,,1/2)$.

7. It follows immediately that

$$\begin{aligned}(\sigma_1 X+\sigma_2 Y)^2-4\sigma_1\sigma_2 XY &= \sigma_1^2X^2+2\sigma_1\sigma_2XY+\sigma_2^2Y^2-4\sigma_1\sigma_2XY\\ &= (\sigma_1X-\sigma_2Y)^2.\end{aligned}$$

Since all parameters and variables are *positive*, it follows that

$$(\sigma_1X+\sigma_2Y)^2-4(\sigma_1\sigma_2-\alpha_1\alpha_2)XY\geq 0\,.$$

Hence the radicand in Eq.(39) is *nonnegative*.

$10(a)$. The critical points consist of the solution set of the equations

$$\begin{aligned}x(\epsilon_1-\sigma_1x-\alpha_1y)&=0\\ y(\epsilon_2-\sigma_2y-\alpha_2x)&=0\,.\end{aligned}$$

If $x=0$, then either $y=0$ or $y=\epsilon_2/\sigma_2$. If $\epsilon_1-\sigma_1x-\alpha_1y=0$, then solving for x results in $x=(\epsilon_1-\alpha_1y)/\sigma_1$. Substitution into the *second* equation yields

$$(\sigma_1\sigma_2-\alpha_1\alpha_2)y^2-(\sigma_1\epsilon_2-\epsilon_1\alpha_2)y=0\,.$$

Based on the hypothesis, it follows that $(\sigma_1\epsilon_2-\epsilon_1\alpha_2)y=0$. So if $\sigma_1\epsilon_2-\epsilon_1\alpha_2\neq 0$, then $y=0$, and the critical points are located at $(0\,,0)$, $(0\,,\epsilon_2/\sigma_2)$ and $(\epsilon_1/\sigma_1,0)$.

For the case $\sigma_1\epsilon_2-\epsilon_1\alpha_2=0$, y can be arbitrary. From the relation $x=(\epsilon_1-\alpha_1y)/\sigma_1$, we conclude that all points on the line $\sigma_1x+\alpha_1y=\epsilon_1$ are critical points, in addition to the point $(0\,,0)$.

(b). The Jacobian matrix of the vector field is

$$\mathbf{J}=\begin{pmatrix}\epsilon_1-2\sigma_1x-\alpha_1y & -\alpha_1x\\ -\alpha_2y & \epsilon_2-2\sigma_2y-\alpha_2x\end{pmatrix}.$$

At the origin, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,0)=\begin{pmatrix}\epsilon_1 & 0\\ 0 & \epsilon_2\end{pmatrix},$$

with eigenvalues $r_1=\epsilon_1$ and $r_2=\epsilon_2$. Since both eigenvalues are *positive*, the origin is an *unstable node*.

At the point $(0\,,\epsilon_2/\sigma_2)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,\epsilon_2/\sigma_2)=\begin{pmatrix}(\epsilon_1\alpha_2-\sigma_1\epsilon_2)/\alpha_2 & 0\\ \epsilon_2\alpha_2/\sigma_2 & -\epsilon_2\end{pmatrix},$$

with eigenvalues $r_1=(\epsilon_1\alpha_2-\sigma_1\epsilon_2)/\alpha_2$ and $r_2=-\epsilon_2$. If $\sigma_1\epsilon_2-\epsilon_1\alpha_2>0$, then both eigenvalues are *negative*. Hence the point $(0\,,\epsilon_2/\sigma_2)$ is a *stable node*, which is *asymptotically stable*. If $\sigma_1\epsilon_2-\epsilon_1\alpha_2<0$, then the eigenvalues are of opposite sign. Hence the point $(0\,,\epsilon_2/\sigma_2)$ is a *saddle*, which is *unstable*.

At the point $(\epsilon_1/\sigma_1, 0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(\epsilon_1/\sigma_1, 0) = \begin{pmatrix} -\epsilon_1 & -\epsilon_1\alpha_1/\sigma_1 \\ 0 & (\sigma_1\epsilon_2 - \epsilon_1\alpha_2)/\sigma_1 \end{pmatrix},$$

with eigenvalues $r_1 = (\sigma_1\epsilon_2 - \epsilon_1\alpha_2)/\sigma_1$ and $r_2 = -\epsilon_1$. If $\sigma_1\epsilon_2 - \epsilon_1\alpha_2 > 0$, then the eigenvalues are of *opposite* sign. Hence the point $(\epsilon_1/\sigma_1, 0)$ is a *saddle*, which is *unstable.* If $\sigma_1\epsilon_2 - \epsilon_1\alpha_2 < 0$, then both eigenvalues are *negative*. In that case the point $(\epsilon_1/\sigma_1, 0)$ is a *stable node*, which is *asymptotically stable.*

(c). As shown in Part (a), when $\sigma_1\epsilon_2 - \epsilon_1\alpha_2 = 0$, the set of critical points consists of $(0\,,0)$ and all of the points on the straight line $\sigma_1 x + \alpha_1 y = \epsilon_1$. Based on Part (b), the origin is still an *unstable node*. Setting $y = (\epsilon_1 - \sigma_1 x)/\alpha_1$, the Jacobian matrix of the vector field, *along the given straight line*, is

$$\mathbf{J} = \begin{pmatrix} -\sigma_1 x & -\alpha_1 x \\ -\alpha_2(\epsilon_1 - \sigma_1 x)/\alpha_1 & \alpha_2 x - \epsilon_1\alpha_2/\sigma_1 \end{pmatrix}.$$

The characteristic equation of the matrix is

$$r^2 + \left[\frac{\epsilon_1\alpha_2 - \alpha_2\sigma_1 x + \sigma_1^2 x}{\sigma_1}\right] r = 0\,.$$

Using the given hypothesis, $(\epsilon_1\alpha_2 - \alpha_2\sigma_1 x + \sigma_1^2 x)/\sigma_1 = \epsilon_2 - \alpha_2 x + \sigma_1 x$. Hence the characteristic equation can be written as

$$r^2 + [\epsilon_2 - \alpha_2 x + \sigma_1 x] r = 0\,.$$

First note that $0 \leq x \leq \epsilon_1/\sigma_1$. Since the coefficient in the quadratic equation is *linear*, and

$$\epsilon_2 - \alpha_2 x + \sigma_1 x = \begin{cases} \epsilon_2 \ \text{ at } \ x = 0 \\ \epsilon_1 \ \text{ at } \ x = \epsilon_1/\sigma_1\,, \end{cases}$$

it follows that the coefficient is *positive* for $0 \leq x \leq \epsilon_1/\sigma_1$. Therefore, along the straight line $\sigma_1 x + \alpha_1 y = \epsilon_1$, one eigenvalue is *zero* and the other one is *negative*. Hence the continuum of critical points consists of *stable nodes*, which are *asymptotically stable*.

$11(a)$. The critical points are solutions of the system of equations

$$x(1 - x - y) + \delta a = 0$$
$$y(0.75 - y - 0.5\,x) + \delta b = 0\,.$$

Assume solutions of the form

$$x = x_0 + x_1\delta + x_2\delta^2 + \cdots$$
$$y = y_0 + y_1\delta + y_2\delta^2 + \cdots.$$

Substitution of the series expansions results in

$$x_0(1 - x_0 - y_0) + (x_1 - 2x_1x_0 - x_0y_1 - x_1y_0 + a)\delta + \cdots = 0$$
$$y_0(0.75 - y_0 - 0.5\,x_0\,) + (0.75\,y_1 - 2y_0y_1 - x_1y_0/2 - x_0y_1/2 + b)\delta + \cdots = 0\,.$$

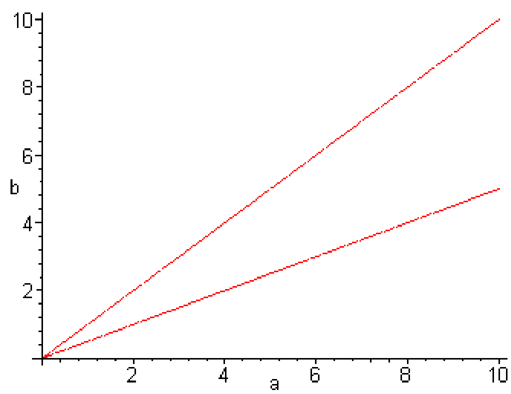
(b). Taking a limit as $\delta \to 0$, the equations reduce to the original system of equations. It follows that $x_0 = y_0 = 0.5$.

(c). Setting the coefficients of the *linear* terms equal to zero, we find that

$$-\,y_1/2 - x_1/2 + a = 0$$
$$-\,x_1/4 - y_1/2 + b = 0\,,$$

with solution $x_1 = 4a - 4b$ and $y_1 = -\,2a + 4b$.

(d). Consider the ab - parameter space. The collection of points for which $b < a$ represents an *increase* in the level of species 1. At points where $b > a$, $x_1\delta < 0$. Likewise, the collection of points for which $2b > a$ represents an *increase* in the level of species 2. At points where $2b < a$, $y_1\delta < 0$.



It follows that if $b < a < 2b$, the level of *both* species will *increase*. This condition is represented by the wedge-shaped region on the graph. Otherwise, the level of one species
will increase, whereas the level of the other species will simultaneously decrease. Only for $a = b = 0$ will both populations remain the same.

$13(a)$. The critical points consist of the solution set of the equations

$$-\,y = 0$$
$$-\,\gamma\,y - x(x - 0.15)(x - 2) = 0\,.$$

Setting $y = 0$, the second equation becomes $x(x - 0.15)(x - 2) = 0$, with roots $x = 0$, 0.15 and 2. Hence the critical points are located at $(0\,,0)$, $(0.15\,,0)$ and $(2\,,0)$. The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 0 & -1 \\ -3x^2 + 4.3\,x - 0.3 & -\gamma \end{pmatrix}.$$

At the origin, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,0) = \begin{pmatrix} 0 & -1 \\ -0.3 & -\gamma \end{pmatrix},$$

with eigenvalues

$$r_{1,2} = -\frac{\gamma}{2} \pm \frac{1}{10}\sqrt{25\,\gamma^2 + 30}\,.$$

Regardless of the value of γ, the eigenvalues are real and of opposite sign. Hence $(0\,,0)$ is a *saddle*, which is *unstable*.

At the critical point $(0.15\,,0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0.15\,,0) = \begin{pmatrix} 0 & -1 \\ 0.2775 & -\gamma \end{pmatrix},$$

with eigenvalues

$$r_{1,2} = -\frac{\gamma}{2} \pm \frac{1}{20}\sqrt{100\,\gamma^2 - 111}\,.$$

If $100\,\gamma^2 - 111 \geq 0$, then the eigenvalues are real. Furthermore, since $r_1 r_2 = 0.2775$, both eigenvalues will have the same sign. Therefore the critical point is a *node*, with its stability dependent on the *sign* of γ. If $100\,\gamma^2 - 111 < 0$, the eigenvalues are complex conjugates. In that case the critical point $(0.15\,,0)$ is a *spiral*, with its stability dependent on the *sign* of γ.

At the critical point $(2\,,0)$, the coefficient matrix of the linearized system is

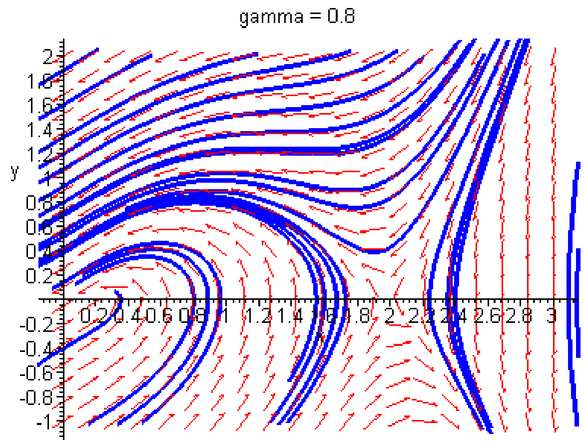
$$\mathbf{J}(2\,,0) = \begin{pmatrix} 0 & -1 \\ -3.7 & -\gamma \end{pmatrix},$$

with eigenvalues

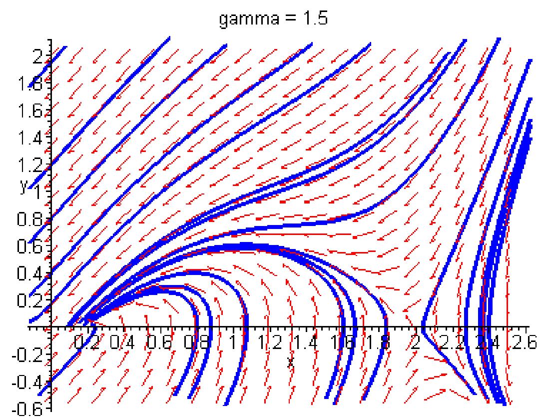
$$r_{1,2} = -\frac{\gamma}{2} \pm \frac{1}{10}\sqrt{25\,\gamma^2 + 370}\,.$$

Regardless of the value of γ, the eigenvalues are real and of opposite sign. Hence $(2\,,0)$ is a *saddle*, which is *unstable*.

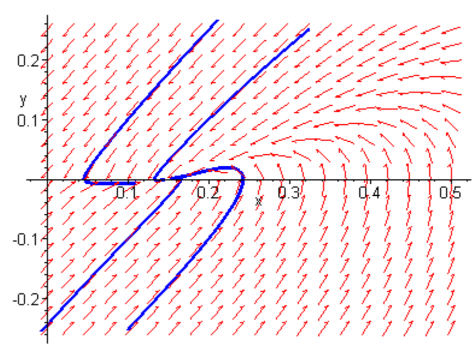
(b).



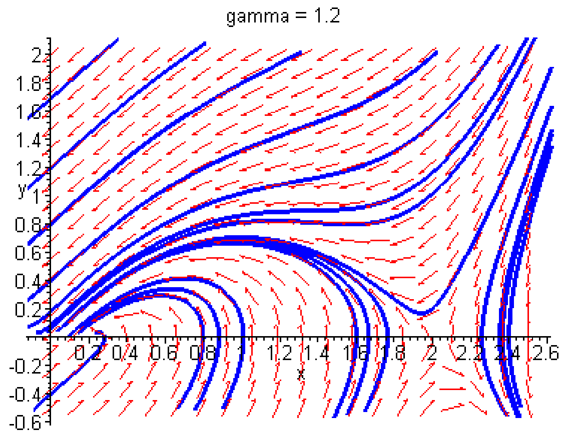
It is evident that for $\gamma = 0.8$, the critical point $(0.15\,,0)$ is a *stable spiral*.

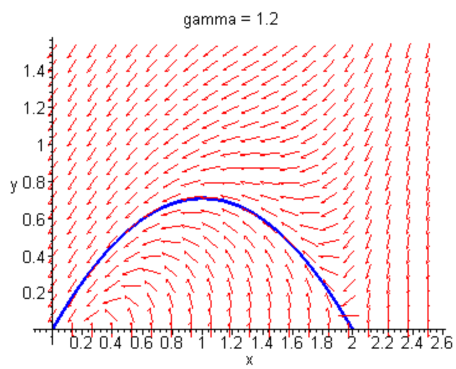


Closer examination shows that for $\gamma = 1.5$, the critical point $(0.15\,,0)$ is a *stable node*.



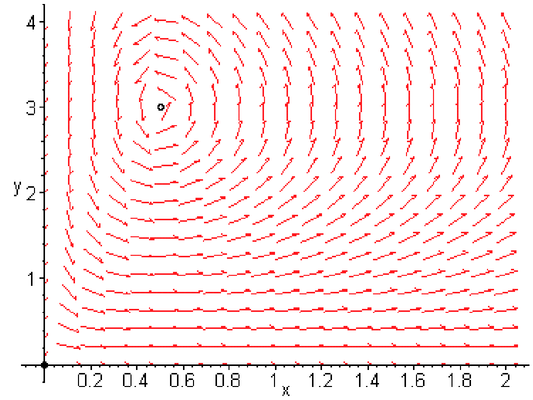
(c). Based on the phase portraits in Part (b), it is apparent that the required value of γ satisfies $0.8 < \gamma < 1.5$. Using the initial condition $x(0) = 2$ and $y(0) = 0.01$, it is possible to solve the initial value problem for various values of γ. A reasonable first guess is $\gamma = \sqrt{1.11}$. This value marks the change in qualitative behavior of the critical point $(0.15\,,0)$. Numerical experiments show that the solution remains positive for $\gamma \approx 1.20\,$.





Section 9.5

$1(a)$.



(b). The critical points are solutions of the system of equations

$$x(1.5 - 0.5\,y) = 0$$
$$y(-0.5 + x) = 0\,.$$

The two critical points are $(0\,,0)$ and $(0.5\,,3)$.

(c). The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 3/2 - y/2 & -x/2 \\ y & -1/2 + x \end{pmatrix}.$$

At the critical point $(0\,,0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,0) = \begin{pmatrix} 3/2 & 0 \\ 0 & -1/2 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = 3/2\,,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\ ;\ \ r_2 = -1/2\,,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The eigenvalues are of opposite sign, hence the origin is a *saddle*, which is *unstable*.

At the critical point $(0.5\,,3)$, the coefficient matrix of the linearized system is

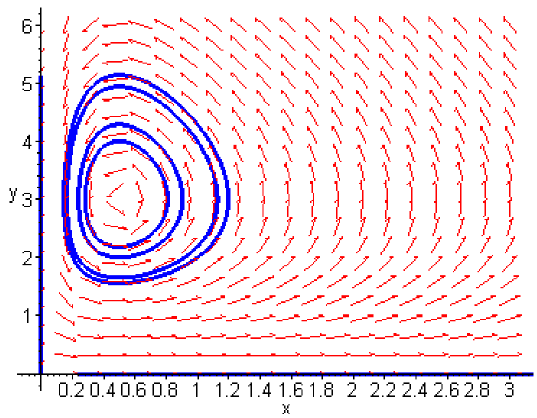
$$\mathbf{J}(0.5\,,3) = \begin{pmatrix} 0 & -1/4 \\ 3 & 0 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = i\,\frac{\sqrt{3}}{2}\,,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ -2\,i\sqrt{3} \end{pmatrix}\ ;\ \ r_2 = -i\,\frac{\sqrt{3}}{2}\,,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 2\,i\sqrt{3} \end{pmatrix}.$$

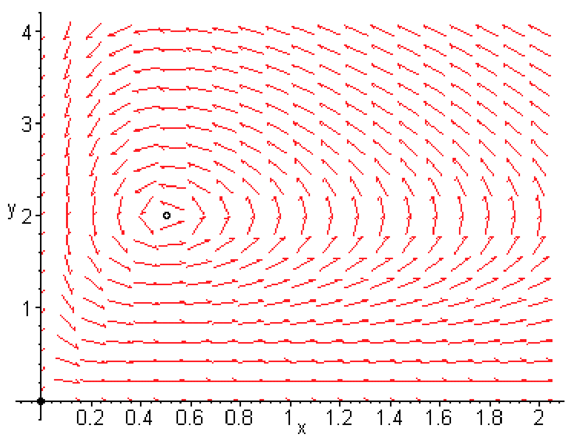
The eigenvalues are purely imaginary. Hence the critical point is a *center*, which is *stable*.

(d, e).



(f). Except for solutions along the coordinate axes, almost all trajectories are closed curves about the critical point $(0.5\,,3)$.

$2(a)$.



(b). The critical points are the solution set of the system of equations

$$x(1 - 0.5\,y) = 0$$
$$y(-0.25 + 0.5\,x) = 0\,.$$

The two critical points are $(0\,,0)$ and $(0.5\,,2)$.

(c). The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 1 - y/2 & -x/2 \\ y/2 & -1/4 + x/2 \end{pmatrix}.$$

At the critical point $(0\,,0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,0) = \begin{pmatrix} 1 & 0 \\ 0 & -\,1/4 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = 1\,,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix};\ \ r_2 = -\,1/4\,,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The eigenvalues are of opposite sign, hence the origin is a *saddle*, which is *unstable*.

At the critical point $(0.5\,,2)$, the coefficient matrix of the linearized system is

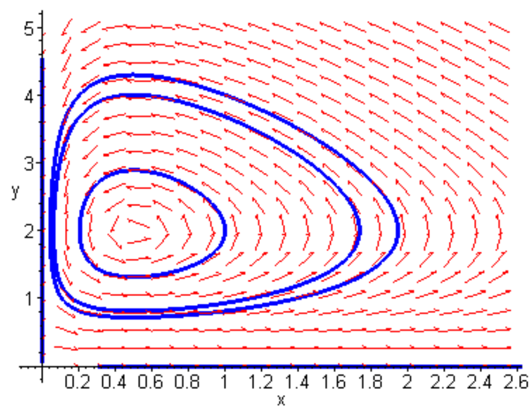
$$\mathbf{J}(0.5\,,2) = \begin{pmatrix} 0 & -\,1/4 \\ 1 & 0 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = i/2\,,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ -\,2\,i \end{pmatrix};\ \ r_2 = -\,i/2\,,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 2\,i \end{pmatrix}.$$

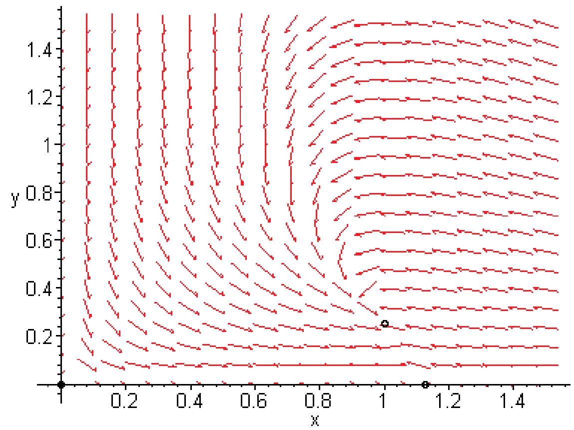
The eigenvalues are purely imaginary. Hence the critical point is a *center*, which is *stable*.

(d, e).



(f). Except for solutions along the coordinate axes, almost all trajectories are closed curves about the critical point $(0.5\,,2)$.

$4(a)$.



(b). The critical points are the solution set of the system of equations

$$x(9/8 - x - y/2) = 0$$
$$y(-1 + x) = 0\,.$$

The three critical points are $(0\,,0)$, $(9/8\,,0)$ and $(1\,,1/4)$.

(c). The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 9/8 - 2x - y/2 & -x/2 \\ y & -1 + x \end{pmatrix}.$$

At the critical point $(0\,,0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,0) = \begin{pmatrix} 9/8 & 0 \\ 0 & -1 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = 9/8\,,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix};\ \ r_2 = -1\,,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The eigenvalues are of opposite sign, hence the origin is a *saddle*, which is *unstable*.

At the critical point $(9/8\,,0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(9/8\,,0) = \begin{pmatrix} -9/8 & -9/16 \\ 0 & 1/8 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = -\frac{9}{8}\,,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix};\ \ r_2 = \frac{1}{8}\,,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 9 \\ -20 \end{pmatrix}.$$

The eigenvalues are of opposite sign, hence the critical point $(9/8\,,0)$ is a *saddle*, which is *unstable*.

At the critical point $(1\,,1/4)$, the coefficient matrix of the linearized system is

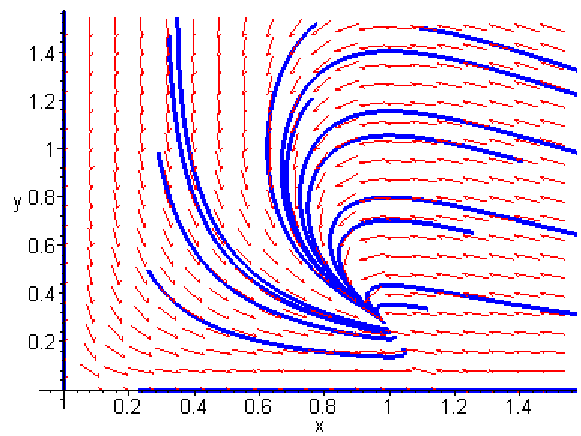
$$\mathbf{J}(1\,,1/4) = \begin{pmatrix} -1 & -1/2 \\ 1/4 & 0 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = \frac{-2+\sqrt{2}}{4}\,,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} -2+\sqrt{2} \\ 1 \end{pmatrix}\,;\ r_2 = \frac{-2-\sqrt{2}}{4}\,,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} -2-\sqrt{2} \\ 1 \end{pmatrix}.$$

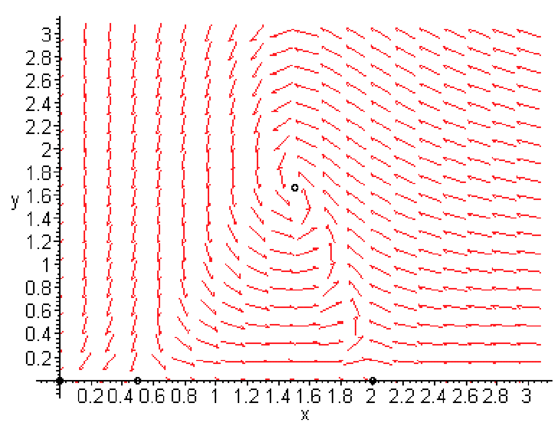
The eigenvalues are both negative. Hence the critical point is a *stable node*, which is *asymptotically stable*.

(d, e).



(f). Except for solutions along the coordinate axes, all solutions converge to the critical point $(1\,,1/4)$.

$5(a)$.



(b). The critical points are solutions of the system of equations

$$x\big(-1+2.5\,x-0.3\,y-x^2\big)=0$$
$$y(-1.5+x)=0\,.$$

The four critical points are $(0\,,0)$, $(1/2\,,0)$, $(2\,,0)$ and $(3/2\,,5/3)$.

(c). The Jacobian matrix of the vector field is

$$\mathbf{J}=\begin{pmatrix}-1+5x-3x^2-3y/10 & -3x/10\\ y & -3/2+x\end{pmatrix}.$$

At the critical point $(0\,,0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(0\,,0)=\begin{pmatrix}-1 & 0\\ 0 & -3/2\end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1=-1\,,\ \boldsymbol{\xi}^{(1)}=\begin{pmatrix}1\\0\end{pmatrix};\ \ r_2=-3/2\,,\ \boldsymbol{\xi}^{(2)}=\begin{pmatrix}0\\1\end{pmatrix}.$$

The eigenvalues are both negative, hence the critical point $(0\,,0)$ is a *stable node*, which is *asymptotically stable*.

At the critical point $(1/2\,,0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(1/2\,,0)=\begin{pmatrix}3/4 & -3/20\\ 0 & -1\end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1=\frac{3}{4}\,,\ \boldsymbol{\xi}^{(1)}=\begin{pmatrix}1\\0\end{pmatrix};\ \ r_2=-1\,,\ \boldsymbol{\xi}^{(2)}=\begin{pmatrix}3\\35\end{pmatrix}.$$

The eigenvalues are of opposite sign, hence the critical point $(1/2\,,0)$ is a *saddle*, which is *unstable*.

At the critical point $(2\,,0)$, the coefficient matrix of the linearized system is

$$\mathbf{J}(2\,,0)=\begin{pmatrix}-3 & -3/5\\ 0 & 1/2\end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1=-3\,,\ \boldsymbol{\xi}^{(1)}=\begin{pmatrix}1\\0\end{pmatrix};\ r_2=1/2\,,\ \boldsymbol{\xi}^{(2)}=\begin{pmatrix}6\\-35\end{pmatrix}.$$

The eigenvalues are of opposite sign, hence the critical point $(2\,,0)$ is a *saddle*, which is *unstable*.

At the critical point $(3/2\,,5/3)$, the coefficient matrix of the linearized system is

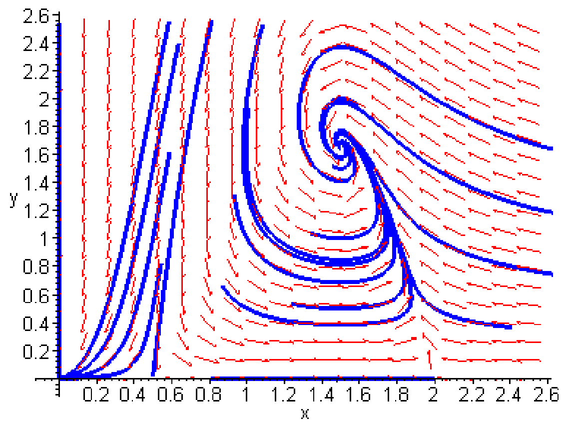
$$\mathbf{J}(3/2\,,5/3) = \begin{pmatrix} -3/4 & -9/20 \\ 5/3 & 0 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$r_1 = \frac{-3 + i\sqrt{39}}{8}\,,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} \frac{-9+i\,3\sqrt{39}}{40} \\ 1 \end{pmatrix}\,;\ r_2 = \frac{-3 - i\sqrt{39}}{8}\,,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} \frac{-9-i\,3\sqrt{39}}{40} \\ 1 \end{pmatrix}.$$

The eigenvalues are complex conjugates. Hence the critical point $(3/2\,,5/3)$ is a *stable spiral*, which is *asymptotically stable*.

(d, e).



(f). The single solution curve that converges to the node at $(1/2\,,0)$ is a *separatrix*. Except for initial conditions on the coordinate axes, trajectories on either side of the separatrix converge to the node at $(0\,,0)$ or the stable spiral at $(3/2\,,5/3)$.

6. Given that t is measured from the time that x is a *maximum*, we have

$$x = \frac{c}{\gamma} + \frac{cK}{\gamma}\,cos\big(\sqrt{ac}\,t\big)$$
$$y = \frac{a}{\alpha} + K\frac{a}{\alpha}\sqrt{\frac{c}{\alpha}}\,sin\big(\sqrt{ac}\,t\big)\,.$$

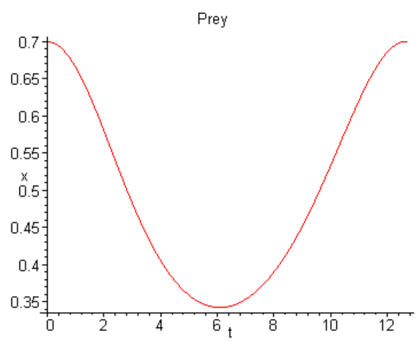
The *period* of oscillation is evidently $T = 2\pi/\sqrt{ac}$. Both populations oscillate about a mean value. The following is based on the properties of the cos and sin functions

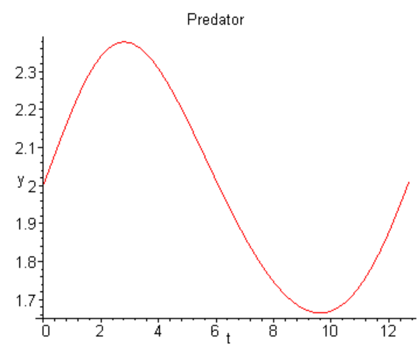
The prey population (x) is *maximum* at $t = 0$ and $t = T$. It is a *minimum* at $t = T/2$. Its rate of increase is greatest at $t = 3\,T/4$. The rate of *decrease* of the prey population is greatest at $t = T/4$.

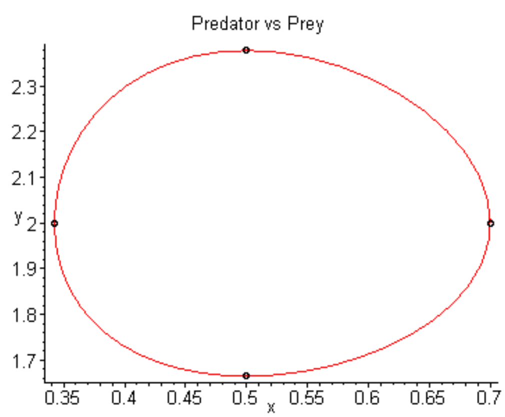
The predator population (y) is *maximum* at $t = T/4$. It is a *minimum* at $t = 3\,T/4$.

The rate of increase of the predator population is greatest at $t = 0$ and $t = T$. The rate of *decrease* of the predator population is greatest at $t = T/2$.

In the following example, the system in Problem 2 is solved numerically with the initial conditions $x(0) = 0.7$ and $y(0) = 2$. The critical point of interest is at $(0.5\,, 2)$. Since $a = 1$ and $c = 1/4$, it follows that the period of oscillation is $T = 4\,\pi$.







$8(a)$. The *period* of oscillation for the linear system is $T = 2\pi/\sqrt{ac}$. In system (2), $a = 1$ and $c = 0.75$. Hence the period is estimated as $T = 2\pi/\sqrt{0.75} \approx 7.2552$.

(b). The estimated period appears to agree with the graphic in Figure $9.5.3$.

(c). The critical point of interest is at $(3\,, 2)$. The system is solved numerically, with $y(0) = 2$ and $x(0) = 3.5\,, 4.0\,, 4.5\,, 5.0$. The resulting periods are shown in the table:

	$x(0) = 3.5$	$x(0) = 4.0$	$x(0) = 4.5$	$x(0) = 5.0$
T	7.26	7.29	7.34	7.42

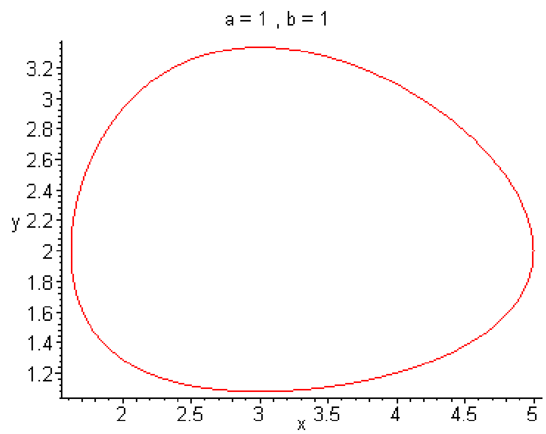
The actual amplitude steadily *increases* as the amplitude increases.

9. The system

$$\frac{dx}{dt} = a\,x\Big(1 - \frac{y}{2}\Big)$$
$$\frac{dy}{dt} = b\,y\Big(-1 + \frac{x}{3}\Big)$$

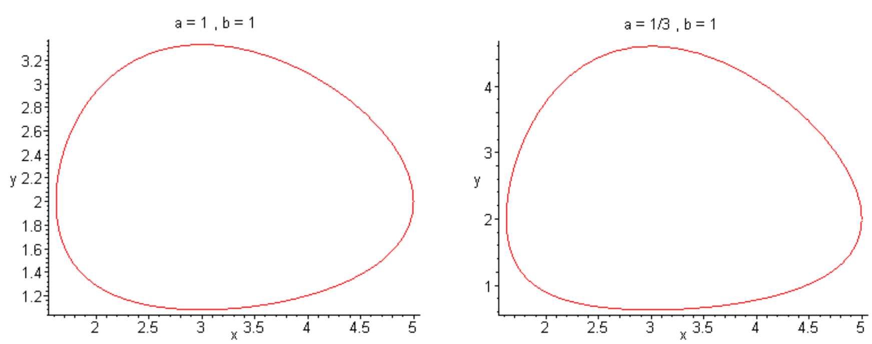
is solved numerically for various values of the parameters. The initial conditions are $x(0) = 5$, $y(0) = 2$.

(a). $a = 1$ and $b = 1$:



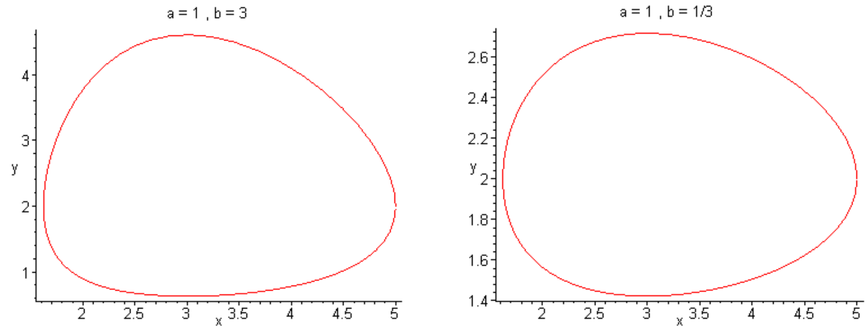
The period is estimated by observing when the trajectory becomes a closed curve. In this case, $T \approx 6.45$.

(b). $a = 3$ and $a = 1/3$, with $b = 1$:



For $a = 3$, $T \approx 3.69$. For $a = 1/3$, $T \approx 11.44$.

(c). $b = 3$ and $b = 1/3$, with $a = 1$:



For $b = 3$, $T \approx 3.82$. For $b = 1/3$, $T \approx 11.06$.

(d). It appears that if one of the parameters is fixed, the period varies *inversely* with the other parameter. Hence one might postulate the relation

$$T = \frac{k}{f(a\,,b)}\,.$$

$10(a)$. Since $T = 2\pi/\sqrt{ac}$, we first note that

$$\int_A^{A+T} cos\big(\sqrt{ac}\;t+\phi\big)dt = \int_A^{A+T} sin\big(\sqrt{ac}\;t+\phi\big)dt = 0\,.$$

Hence

$$\overline{x} = \frac{1}{T}\int_A^{A+T}\frac{c}{\gamma}\,dt = \frac{c}{\gamma} \text{ and } \overline{y} = \frac{1}{T}\int_A^{A+T}\frac{a}{\alpha}\,dt = \frac{a}{\alpha}\,.$$

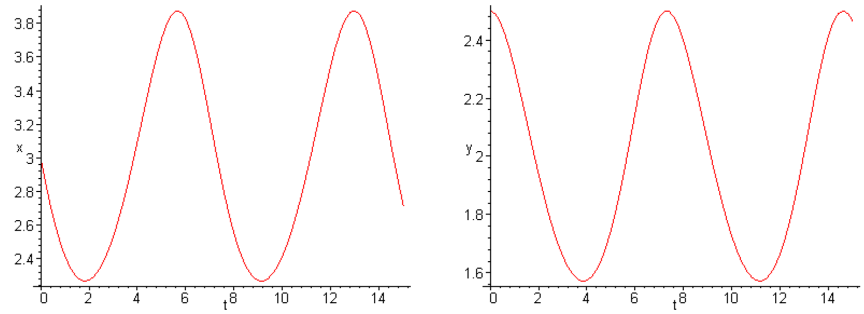
(b). One way to estimate the mean values is to find a horizontal line such that the area above the line is approximately equal to the area under the line. From Figure $9.5.3$, it appears that $\overline{x} \approx 3.25$ and $\overline{y} \approx 2.0$. In Example 1, $a = 1$, $c = 0.75$, $\alpha = 0.5$ and $\gamma = 0.25$. Using the result in Part (a), $\overline{x} = 3$ and $\overline{y} = 2$.

(c). The system

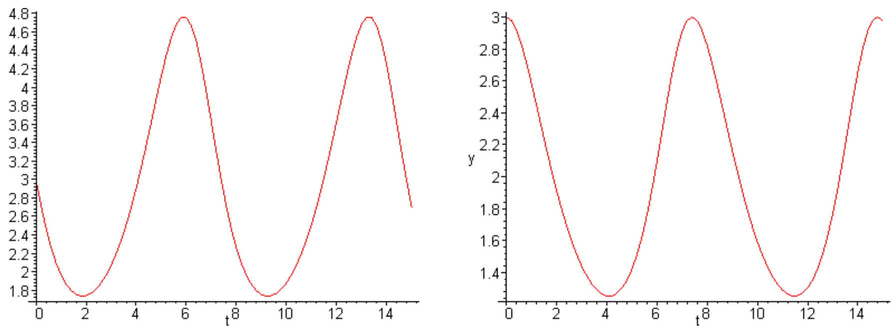
$$\frac{dx}{dt} = x\Big(1 - \frac{y}{2}\Big)$$
$$\frac{dy}{dt} = y\Big(-\frac{3}{4} + \frac{x}{4}\Big)$$

is solved numerically for various initial conditions.

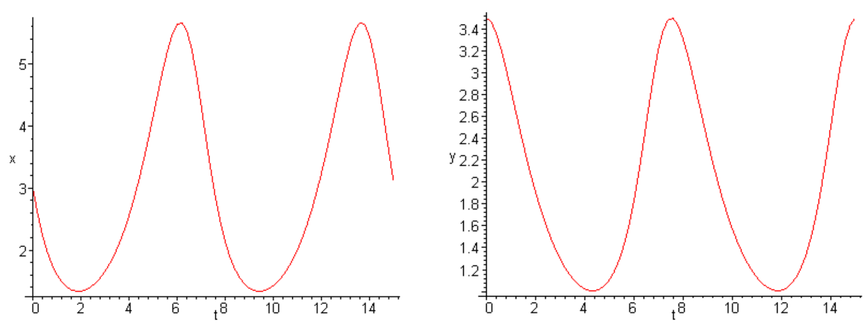
$x(0) = 3$ and $y(0) = 2.5$:



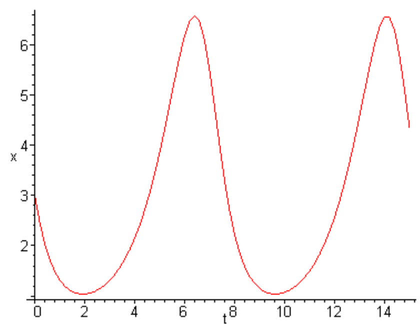
$x(0) = 3$ and $y(0) = 3.0$:

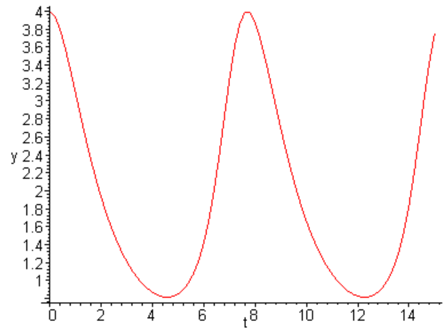


$x(0) = 3$ and $y(0) = 3.5$:



$x(0) = 3$ and $y(0) = 4.0$:





It is evident that the mean values *increase* as the amplitude increases. That is, the mean values increase as the initial conditions move farther from the critical point.

12. The system of equations in model (1) is given by

$$\frac{dx}{dt} = x(a - \alpha\, y)$$
$$\frac{dy}{dt} = y(\,-c + \gamma\, x)\,.$$

Based on the hypothesis, let the *death* rate of the insect population and the predators be $p\,x$ and $q\,y$, respectively. The modified system of equations becomes

$$\frac{dx}{dt} = x(a - \alpha\, y) - p\,x$$
$$\frac{dy}{dt} = y(\,-c + \gamma\, x) - q\,y\,,$$

in which $p > 0\,,\, q > 0$. The critical points are solutions of the system of equations

$$x(a - p - \alpha\, y) = 0$$
$$y(\,-c - q + \gamma\, x) = 0\,.$$

It is easy to see that the critical points are now at $(0\,,0)$ and $\left(\frac{c+q}{\gamma}\,,\,\frac{a-p}{\alpha}\right)$. Furthermore, since $(c+q)/\gamma > c/\gamma$, the equilibrium level of the insect population has *increased*. On the other hand, since $(a-p)/\alpha < a/\alpha$, equilibrium level of the predators has *decreased*. Indeed, the introduction of insecticide creates a potential to significantly affect the predator population $(a \approx p)$.

Section 9.6

2. We consider the function $V(x,y) = a\,x^2 + c\,y^2$. The rate of change of V *along any trajectory* is

$$\begin{aligned}\dot{V} &= V_x \frac{dx}{dt} + V_y \frac{dy}{dt} \\ &= 2ax\Big(-\frac{1}{2}x^3 + 2xy^2\Big) + 2cy\big(-y^3\big) \\ &= -ax^4 + 4ax^2y^2 - 2cy^4.\end{aligned}$$

Let $u = x^2$, $v = y^2$, $\alpha = -a$, $\beta = 4a$, and $\gamma = -2c$. We then have

$$-ax^4 + 4ax^2y^2 - 2cy^4 = \alpha\,u^2 + \beta\,uv + \gamma\,v^2.$$

If $a > 0$ and $c > 0$, then $V(x,y)$ is *positive definite*. Furthermore, $\alpha < 0$. Recall that Theorem 9.6.4 asserts that if $4\alpha\gamma - \beta^2 = 8ac - 16\,a^2 > 0$, then the function

$$\alpha\,u^2 + \beta\,uv + \gamma\,v^2$$

is *negative definite*. Hence if $c > 2a$, then $\dot{V}(x,y)$ is *negative definite*. One such example is $V(x,y) = x^2 + 3\,y^2$. It follows from Theorem 9.6.1 that the origin is an asymptotically stable critical point.

4. Given $V(x,y) = a\,x^2 + c\,y^2$, the rate of change of V *along any trajectory* is

$$\begin{aligned}\dot{V} &= V_x \frac{dx}{dt} + V_y \frac{dy}{dt} \\ &= 2ax\big(x^3 - y^3\big) + 2cy\big(2xy^2 + 4x^2y + 2y^3\big) \\ &= 2a\,x^4 + (4c - 2a)xy^3 + 8c\,x^2y^2 + 4c\,y^4.\end{aligned}$$

Setting $a = 2c$,

$$\begin{aligned}\dot{V} &= 4c\,x^4 + 8c\,x^2y^2 + 4c\,y^4 \\ &\geq 4c\,x^4 + 4c\,y^4.\end{aligned}$$

As long as $a = 2c > 0$, the function $V(x,y)$ is *positive definite* and $\dot{V}(x,y)$ is also *positive definite.* It follows from Theorem 9.6.2 that $(0\,,0)$ is an unstable critical point.

5. Given $V(x,y) = c(x^2 + y^2)$, the rate of change of V *along any trajectory* is

$$\begin{aligned}\dot{V} &= V_x \frac{dx}{dt} + V_y \frac{dy}{dt} \\ &= 2c\,x[y - xf(x,y)] + 2cy[-x - yf(x,y)] \\ &= -2c\big(x^2 + y^2\big)f(x,y)\,.\end{aligned}$$

If $c > 0$, then $V(x,y)$ is *positive definite*. Furthermore, if $f(x,y)$ is *positive* in some neighborhood of the origin, then $\dot{V}(x,y)$ is *negative definite*. Theorem 9.6.1 asserts that the origin is an asymptotically stable critical point.

On the other hand, if $f(x,y)$ is *negative* in some neighborhood of the origin, then $V(x,y)$
and $\dot{V}(x,y)$ are both *positive definite*. It follows from Theorem 9.6.2 that the origin is an unstable critical point.

9(a). Letting $x = u$ and $y = u'$, we obtain the system of equations

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -g(x) - y\,.$$

Since $g(0) = 0$, it is evident that $(0\,,0)$ is a critical point of the system. Consider the function

$$V(x,y) = \frac{1}{2}y^2 + \int_0^x g(s)ds\,.$$

It is clear that $V(0,0) = 0$. Since $g(u)$ is an *odd* function in a neighborhood of $u = 0$,

$$\int_0^x g(s)ds > 0 \text{ for } x > 0\,,$$

and

$$\int_0^x g(s)ds = -\int_x^0 g(s)ds > 0 \text{ for } x < 0\,.$$

Therefore $V(x,y)$ is *positive definite*.

The rate of change of V along any trajectory is

$$\begin{aligned}\dot{V} &= V_x\frac{dx}{dt} + V_y\frac{dy}{dt}\\ &= g(x)\cdot(y) + y[-g(x) - y]\\ &= -y^2\,.\end{aligned}$$

It follows that $\dot{V}(x,y)$ is only *negative semidefinite*. Hence the origin is a *stable* critical point.

(b). Given

$$V(x,y) = \frac{1}{2}y^2 + \frac{1}{2}y\,sin(x) + \int_0^x sin(s)ds\,,$$

It is easy to see that $V(0\,,0) = 0$. The rate of change of V along any trajectory is

$$
\begin{aligned}
\dot{V} &= V_x \frac{dx}{dt} + V_y \frac{dy}{dt} \\
&= \left[sin\, x + \frac{y}{2} cos\, x \right](y) + \left[y + \frac{1}{2} sin\, x \right][-\, sin\, x - y] \\
&= \frac{1}{2} y^2\, cos\, x - \frac{1}{2} sin^2 x - \frac{y}{2} sin\, x - y^2 \,.
\end{aligned}
$$

For $-\pi/2 < x < \pi/2$, we can write $sin\, x = x - \alpha\, x^3/6$ and $cos\, x = 1 - \beta\, x^2/2$, in which $\alpha = \alpha(x)\,, \beta = \beta(x)$. Note that $0 < \alpha\,, \beta < 1$. Then

$$
\dot{V}(x\,,y) = \frac{y^2}{2}\left(1 - \frac{\beta\, x^2}{2}\right) - \frac{1}{2}\left(x - \frac{\alpha\, x^3}{6}\right)^2 - \frac{y}{2}\left(x - \frac{\alpha\, x^3}{6}\right) - y^2 \;.
$$

Using polar coordinates,

$$
\begin{aligned}
\dot{V}(r\,,\theta) \;&=\; -\,\frac{r^2}{2}\,[1 + sin\,\theta\, cos\,\theta + h(r\,,\theta)] \\
&=\; -\,\frac{r^2}{2}\left[1 + \frac{1}{2} sin\, 2\theta + h(r\,,\theta)\right].
\end{aligned}
$$

It is easy to show that

$$
|h(r\,,\theta)| \le \frac{1}{2} r^2 + \frac{1}{72} r^4.
$$

So if r is *sufficiently small*, then $|h(r\,,\theta)| < 1/2$ and $\left|\frac{1}{2} sin\, 2\theta + h(r\,,\theta)\right| < 1$. Hence $\dot{V}(x\,,y)$ is *negative definite.*

Now we show that $V(x, y)$ is *positive definite*. Since $g(u) = sin\, u$,

$$
V(x, y) = \frac{1}{2} y^2 + \frac{1}{2} y\, sin(x) + 1 - cos\, x\,.
$$

This time we set

$$
cos\, x = 1 - \frac{x^2}{2} + \gamma\, \frac{x^4}{24}\,.
$$

Note that $0 < \gamma < 1$ for $-\pi/2 < x < \pi/2$. Converting to polar coordinates,

$$
\begin{aligned}
V(r, \theta) \;&=\; \frac{r^2}{2}\left[1 + sin\,\theta\, cos\,\theta - \frac{r^2}{12} sin\,\theta\, cos^3\theta - \gamma\, \frac{r^2}{24} cos^4\theta\right] \\
&=\; \frac{r^2}{2}\left[1 + \frac{1}{2} sin\, 2\theta - \frac{r^2}{12} sin\,\theta\, cos^3\theta - \gamma\, \frac{r^2}{24} cos^4\theta\right].
\end{aligned}
$$

Now

$$-\frac{r^2}{12}sin\,\theta\, cos^3\theta - \gamma\,\frac{r^2}{24}cos^4\theta > \,-\frac{1}{8}\text{ for } r < 1\,.$$

It follows that when $r > 0$,

$$V(r,\theta) > \frac{r^2}{2}\left[\frac{7}{8} + \frac{1}{2}sin\,2\theta\right] \geq \frac{3\,r^2}{16} > 0\,.$$

Therefore $V(x,y)$ is indeed *positive definite*, and by Theorem 9.6.1, the origin is an asymptotically stable critical point.

$12(a)$. We consider the linear system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Let $V(x,y) = Ax^2 + Bxy + Cy^2$, in which

$$A = \,-\,\frac{a_{21}^2 + a_{22}^2 + (a_{11}a_{22} - a_{12}a_{21})}{2\Delta}$$
$$B = \frac{a_{12}a_{22} + a_{11}a_{21}}{\Delta}$$
$$C = \,-\,\frac{a_{11}^2 + a_{12}^2 + (a_{11}a_{22} - a_{12}a_{21})}{2\Delta}\,,$$

and $\Delta = (a_{11} + a_{22})(a_{11}a_{22} - a_{12}a_{21})$. Based on the hypothesis, the coefficients A and B are negative. Therefore, except for the origin, $V(x,y)$ is *negative* on each of the coordinate axes. Along each trajectory,

$$\begin{aligned}\dot{V} &= (2Ax + By)(a_{11}\,x + a_{12}\,y) + (2Cy + Bx)(a_{21}\,x + a_{22}\,y) \\ &= \,-\,x^2 - y^2.\end{aligned}$$

Hence $\dot{V}(x,y)$ is *negative definite*. Theorem 9.6.2 asserts that the origin is an *unstable* critical point.

(b). We now consider the system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} F_1(x\,,y) \\ G_1(x\,,y) \end{pmatrix},$$

in which $F_1(x\,,y)/r \to 0$ and $G_1(x\,,y)/r \to 0$ as $r \to 0$. Let

$$V(x,y) = Ax^2 + Bxy + Cy^2,$$

in which

$$\begin{aligned} A &= \frac{a_{21}^2 + a_{22}^2 + (a_{11}a_{22} - a_{12}a_{21})}{2\Delta} \\ B &= -\frac{a_{12}a_{22} + a_{11}a_{21}}{\Delta} \\ C &= \frac{a_{11}^2 + a_{12}^2 + (a_{11}a_{22} - a_{12}a_{21})}{2\Delta}, \end{aligned}$$

and $\Delta = (a_{11} + a_{22})(a_{11}a_{22} - a_{12}a_{21})$. Based on the hypothesis, $A, B > 0$. Except for the origin, $V(x, y)$ is *positive* on each of the coordinate axes. Along each trajectory,

$$\dot{V} = x^2 + y^2 + (2Ax + By)F_1(x, y) + (2Cy + Bx)G_1(x, y).$$

Converting to polar coordinates, for $r \neq 0$,

$$\begin{aligned} \dot{V} &= r^2 + r(2A\cos\theta + B\sin\theta)\,F_1 + r(2C\sin\theta + B\cos\theta)\,G_1 \\ &= r^2 + r^2\left[(2A\cos\theta + B\sin\theta)\,\frac{F_1}{r} + (2C\sin\theta + B\cos\theta)\,\frac{G_1}{r}\right]. \end{aligned}$$

Since the system is *almost linear*, there is an R such that

$$\left|(2A\cos\theta + B\sin\theta)\,\frac{F_1}{r} + (2C\sin\theta + B\cos\theta)\,\frac{G_1}{r}\right| < \frac{1}{2},$$

and hence

$$(2A\cos\theta + B\sin\theta)\,\frac{F_1}{r} + (2C\sin\theta + B\cos\theta)\,\frac{G_1}{r} > -\frac{1}{2}$$

for $r < R$. It follows that

$$\dot{V} > \frac{1}{2}r^2$$

as long as $0 < r < R$. Hence $\dot{V}$ is *positive definite* on the domain

$$D = \{(x, y) \mid x^2 + y^2 < R^2\}.$$

By Theorem 9.6.2, the origin is an *unstable* critical point.

Section 9.7

3. The equilibrium solutions of the ODE

$$\frac{dr}{dt} = r(r-1)(r-3)$$

are given by $r_1 = 0\,, r_2 = 1$ and $r_3 = 3\,.$ Note that

$$\frac{dr}{dt} > 0 \text{ for } 0 < r < 1 \text{ and } r > 3 \;; \quad \frac{dr}{dt} < 0 \text{ for } 1 < r < 3\,.$$

$r = 0$ corresponds to an *unstable* critical point. The equilibrium solution $r_2 = 1$ is *asymptotically stable*, whereas the equilibrium solution $r_3 = 3$ is *unstable*. Since the critical values are *isolated*, a limit cycle is given by

$$r = 1\,, \theta = t + t_0$$

which is *asymptotically stable.* Another periodic solution is found to be

$$r = 3\,, \theta = t + t_0$$

which is *unstable.*

5. The equilibrium solutions of the ODE

$$\frac{dr}{dt} = sin\,\pi r$$

are given by $r = n\,,\ n = 0\,,1\,,2\,,\cdots.$ Based on the *sign* of r' in the neighborhood of each critical value, the equilibrium solutions $r = 2k\,, k = 1\,,2\,,\cdots$ correspond to *unstable* periodic solutions, with $\theta = t + t_0\,.$ The equilibrium solutions $r = 2k + 1\,,$ $k = 0\,,1\,,2\,,\cdots$ correspond to *stable* limit cycles, with $\theta = t + t_0\,.$ The solution $r = 0$ represents an *unstable* critical point.

10. Given $F(x\,,y) = a_{11}\,x + a_{12}\,y$ and $G(x\,,y) = a_{21}\,x + a_{22}\,y$, it follows that

$$F_x + G_y = a_{11} + a_{22}\,.$$

Based on the hypothesis, $F_x + G_y$ is either *positive* or *negative* on the entire plane. By Theorem 9.7.2 , the system cannot have a nontrivial periodic solution.

12. Given that $F(x\,,y) = \,-\,2x - 3y - xy^2$ and $G(x\,,y) = y + x^3 - x^2 y\,,$

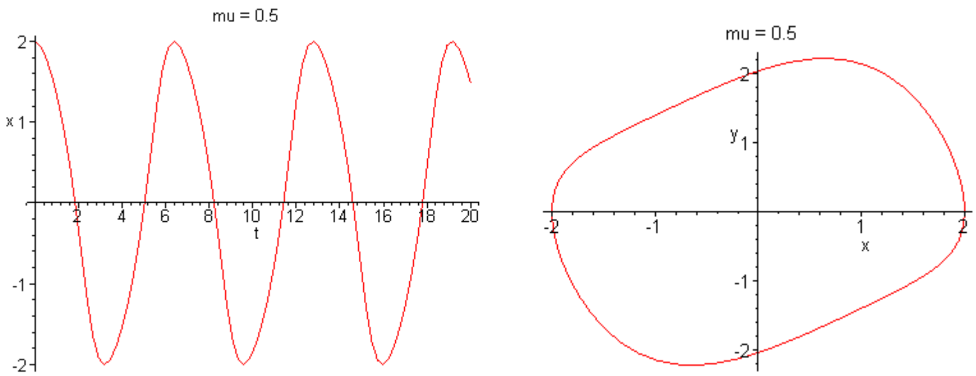
$$F_x + G_y = \,-\,1 - x^2 - y^2.$$

Since $F_x + G_y < 0$ on the entire plane, Theorem 9.7.2 asserts that the system cannot have a nontrivial periodic solution.

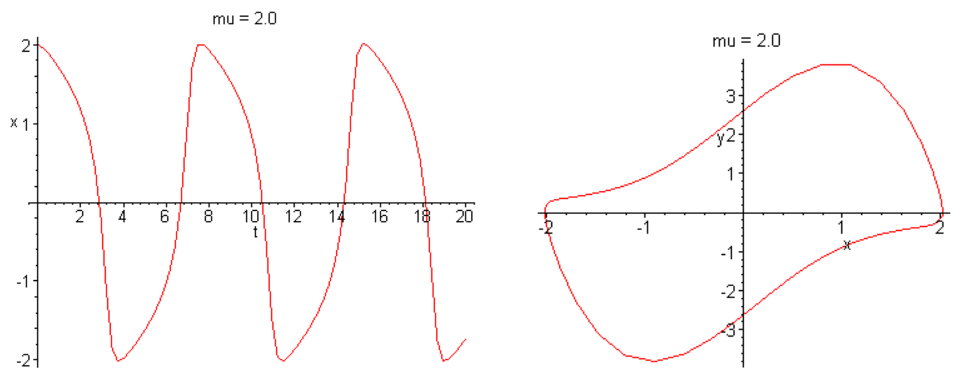
$14(a)$. Based on the given graphs, the following table shows the estimated values:

$\mu = 0.2$	$T \approx 6.29$
$\mu = 1.0$	$T \approx 6.66$
$\mu = 5.0$	$T \approx 11.60$

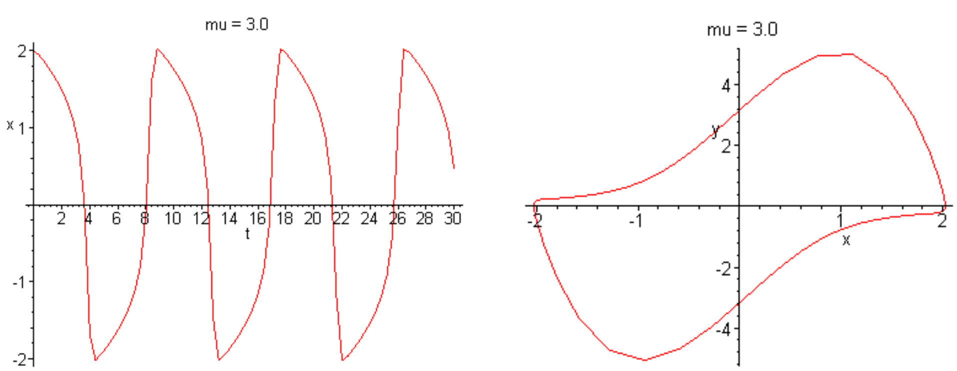
(b). The initial conditions were chosen as $x(0) = 2$, $y(0) = 0$.



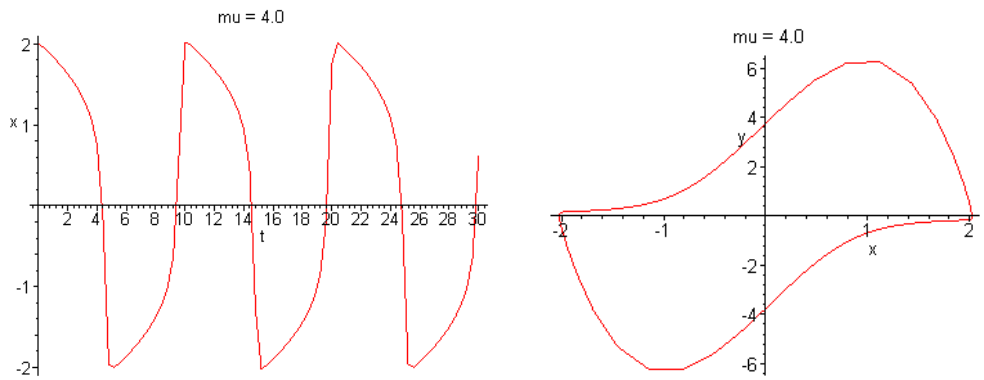
$T \approx 6.38$.



$T \approx 7.65$.

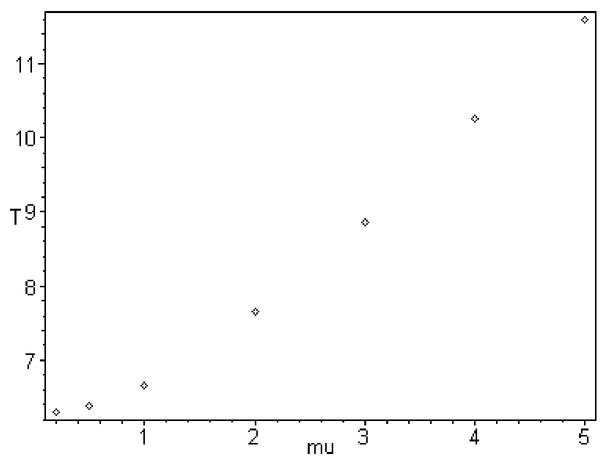


$T \approx 8.86$.



$T \approx 10.25$.

(c). The period, T , appears to be a *quadratic* function of μ .



$15(a)$. Setting $x = u$ and $y = u'$, we obtain the system of equations

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -x + \mu\left(1 - \frac{1}{3}y^2\right)y .$$

(b). Evidently, $y = 0$. It follows that $x = 0$. Hence the only critical point of the system is at $(0\,,0)$. The components of the vector field are infinitely differentiable everywhere. Therefore the system is *almost linear*.

The Jacobian matrix of the vector field is

$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ -1 & \mu - \mu y^2 \end{pmatrix}.$$

At the critical point $(0\,,0)$, the coefficient matrix of the linearized system is

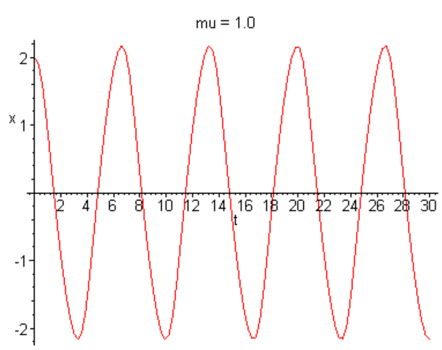
$$\mathbf{J}(0\,,0) = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix},$$

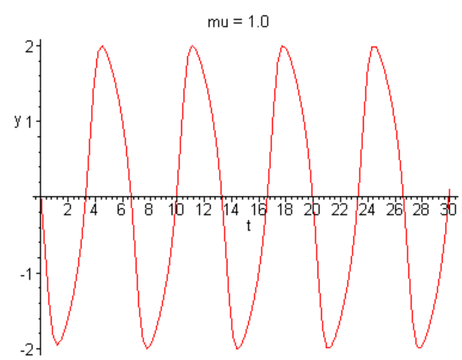
with eigenvalues

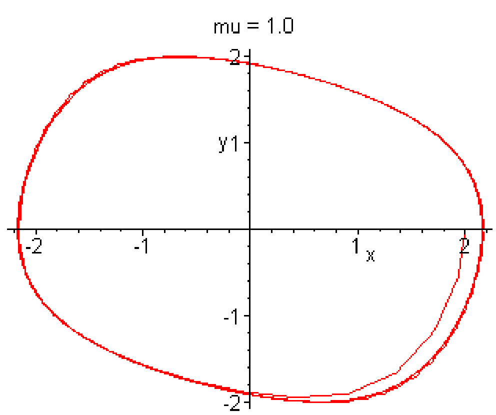
$$r_{1,2} = \frac{\mu}{2} \pm \frac{1}{2}\sqrt{\mu^2 - 4}\;.$$

If $\mu = 0$, the equation reduces to the ODE for a simple harmonic oscillator. For the case $0 < \mu < 2$, the eigenvalues are *complex*, and the critical point is an *unstable spiral*. For $\mu \geq 2$, the eigenvalues are *real*, and the origin is an *unstable node*.

(c). The initial conditions were chosen as $x(0) = 2\,,\, y(0) = 0$.

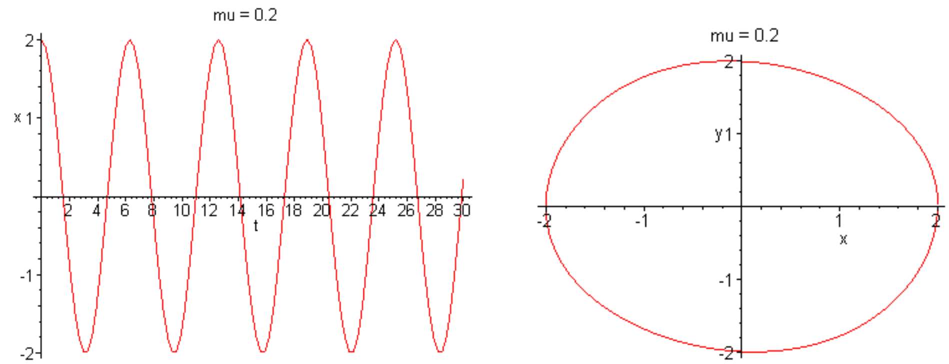




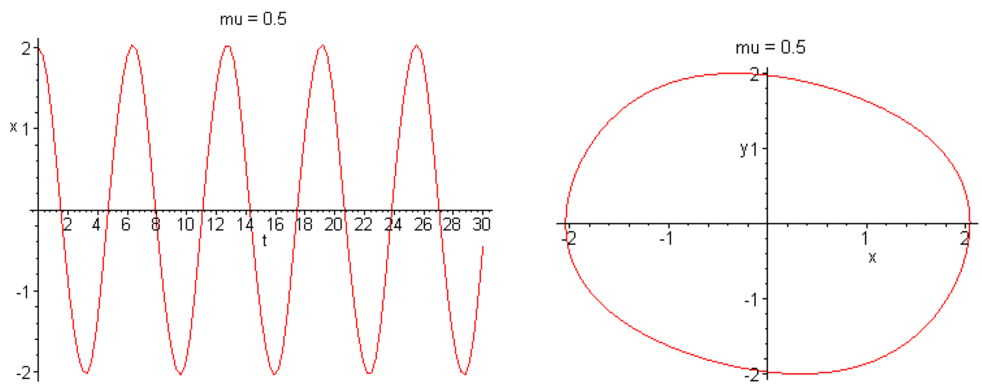


$A \approx 2.16$ and $T \approx 6.65$.

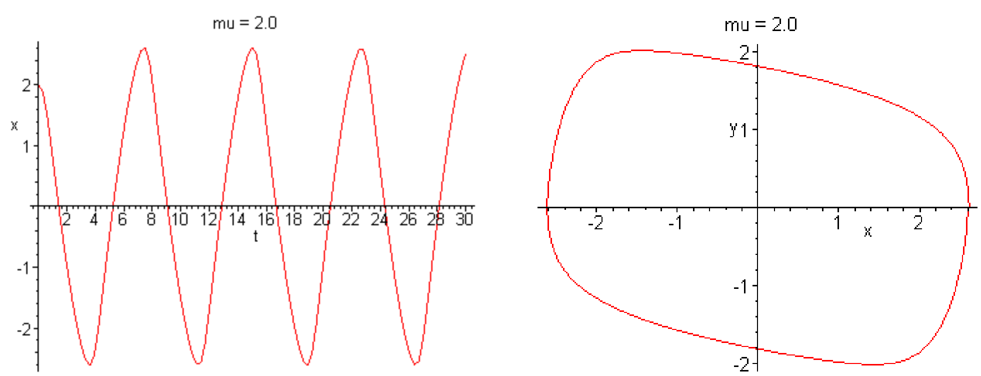
(d).



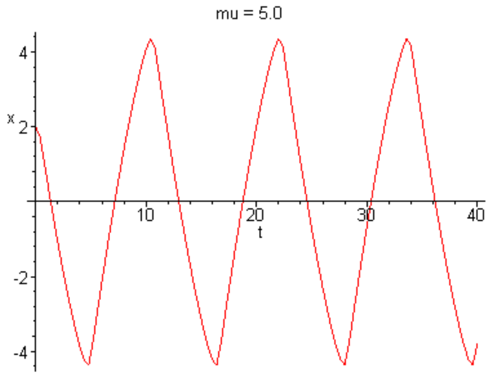
$A \approx 2.00$ and $T \approx 6.30$.

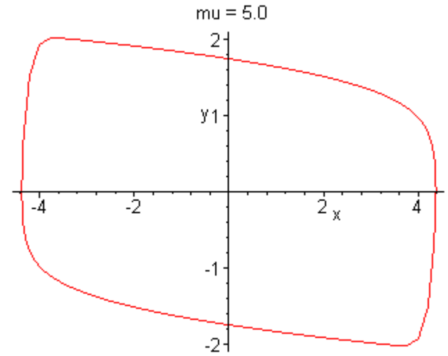


$A \approx 2.04$ and $T \approx 6.38$.



$A \approx 2.6$ and $T \approx 7.62$.

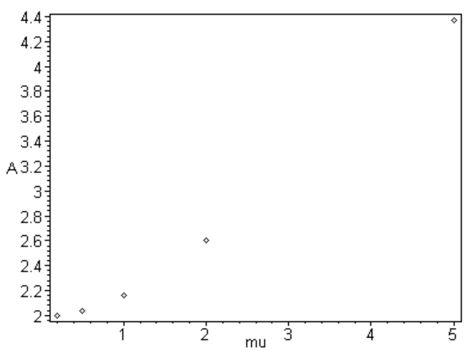


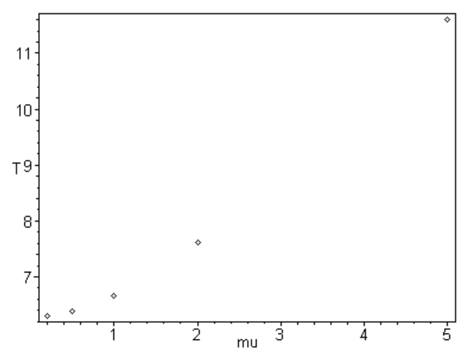


$A \approx 4.37$ and $T \approx 11.61$.

(e).

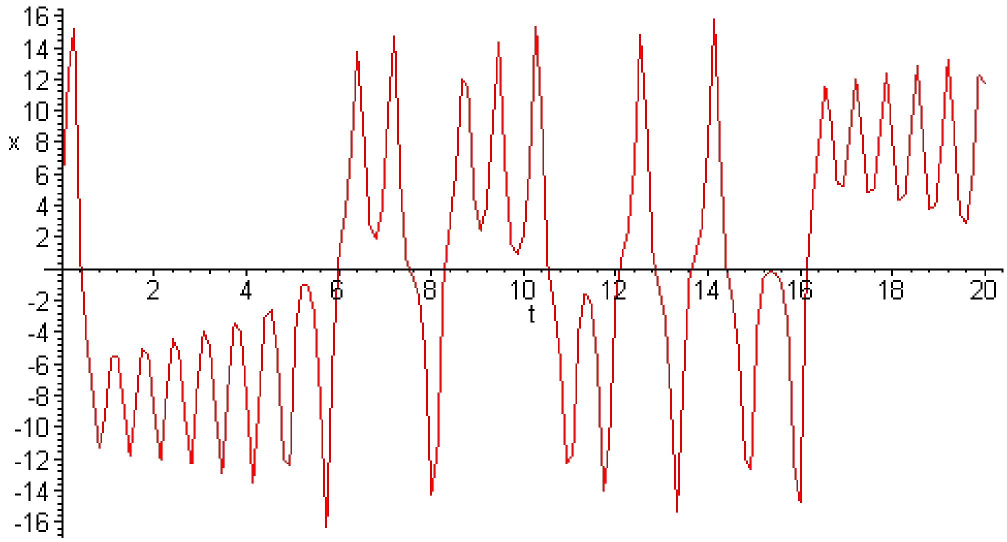
	A	T
$\mu = 0.2$	2.00	6.30
$\mu = 0.5$	2.04	6.38
$\mu = 1.0$	2.16	6.65
$\mu = 2.0$	2.6	7.62
$\mu = 5.0$	4.37	11.61



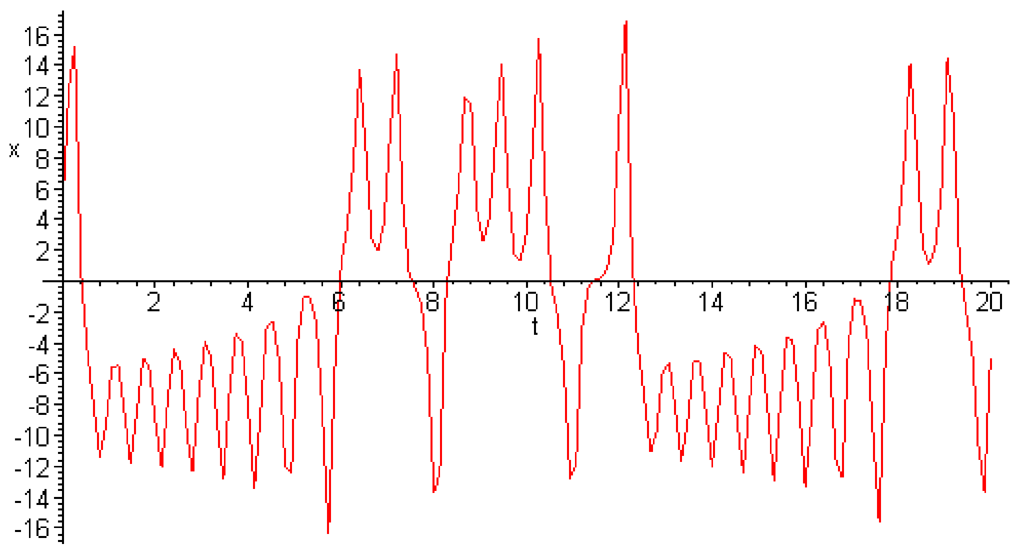


Section 9.8

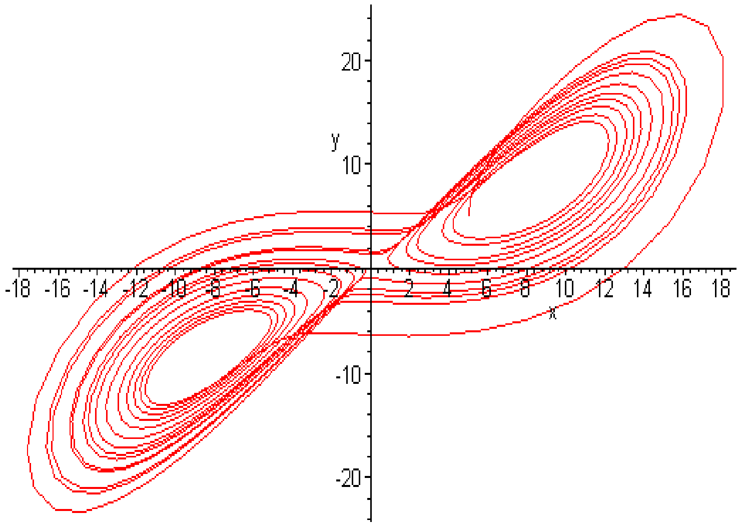
6. $r = 28$, with initial point $(5, 5, 5)$:

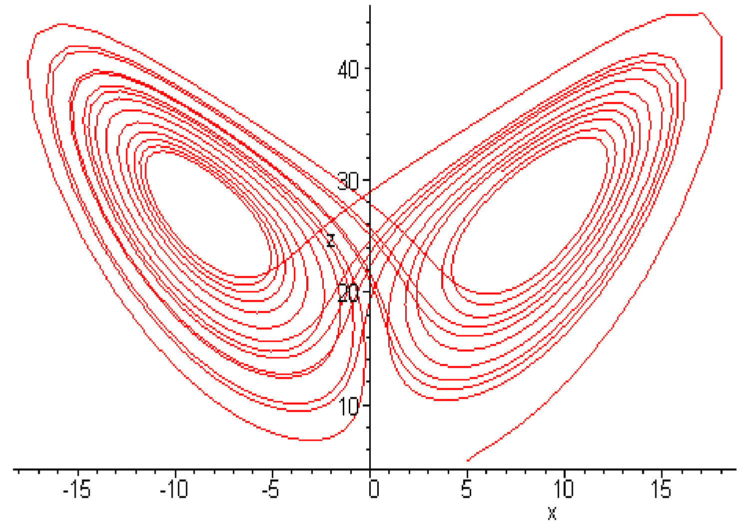


$r = 28$, with initial point $(5.01, 5, 5)$:

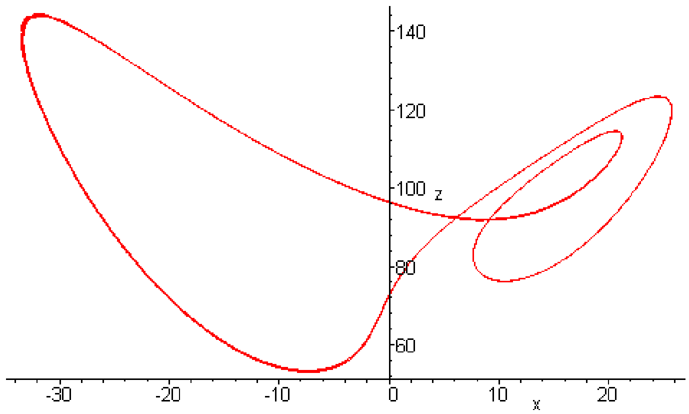


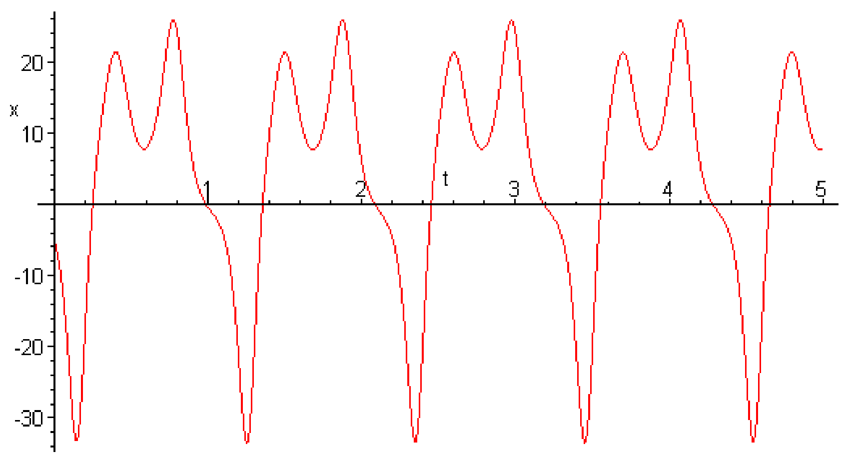
7. $r = 28$:


20
y
10
-18 -16 -14 -12 -10 -8 -6 -4 -2 2 4 6 8 10 12 14 16 18
x
-10
-20


40
30
z
20
10
-15 -10 -5 0 5 10 15
x

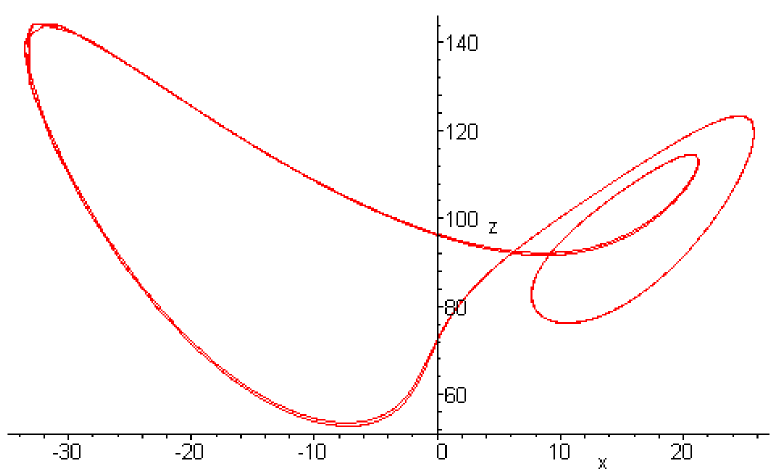
9(a). $r = 100$, initial point $(-5, -13, 55)$:



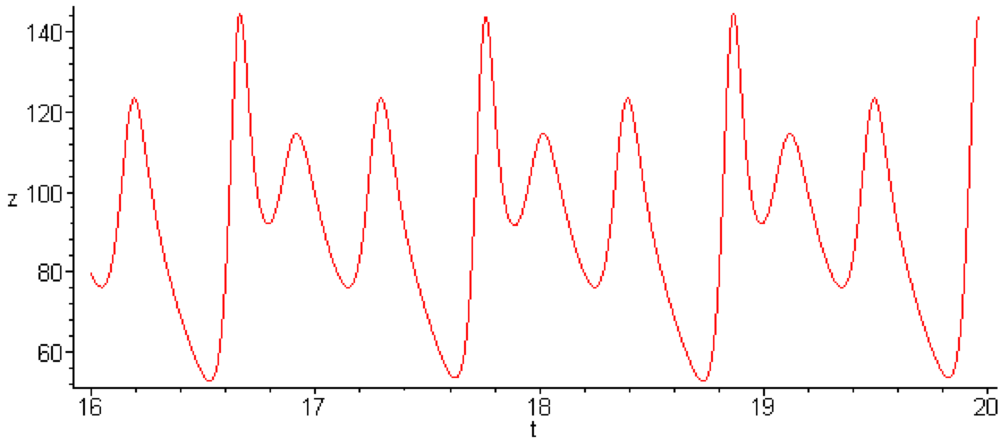


The period appears to be $T \approx 1.12$.

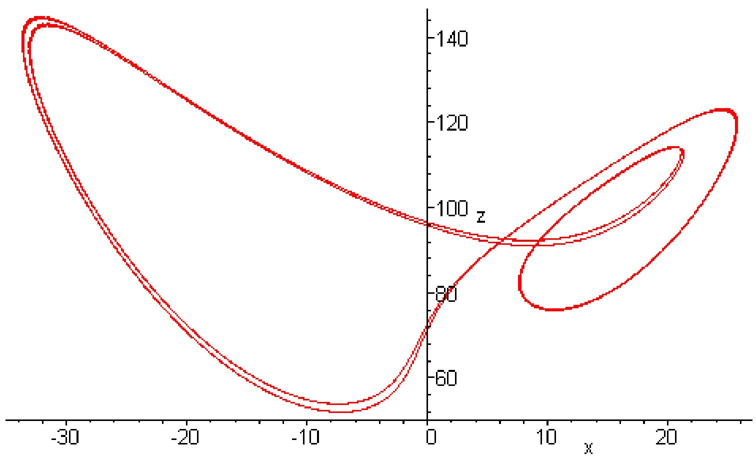
(b). $r = 99.94$, initial point $(-5, -13, 55)$:

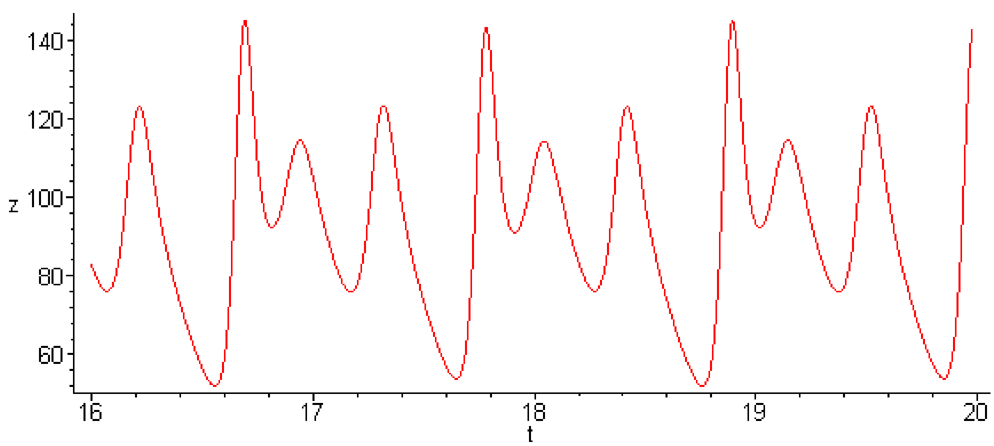


The periodic trajectory appears to have split into two strands, indicative of a period-doubling. Closer examination reveals that the peak values of $z(t)$ are slightly different:

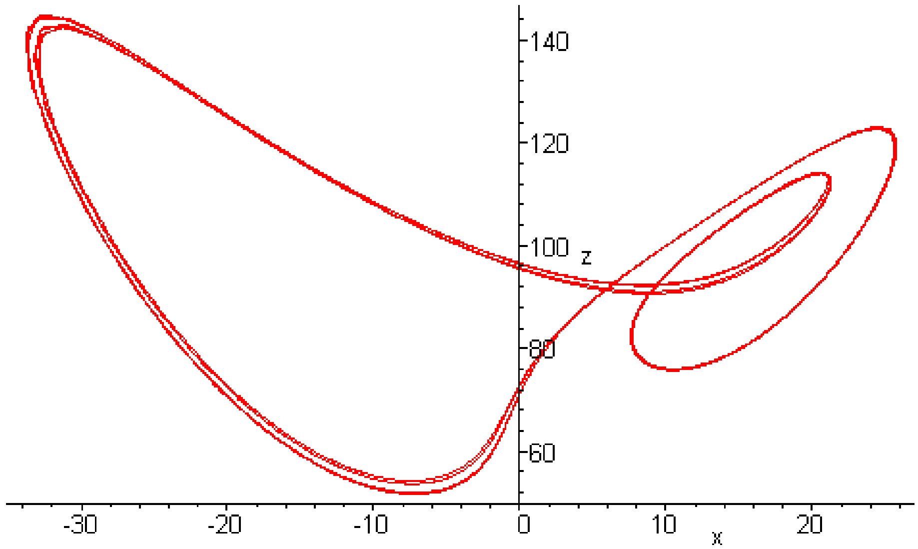


$r = 99.7$, initial point $(-5, -13, 55)$:

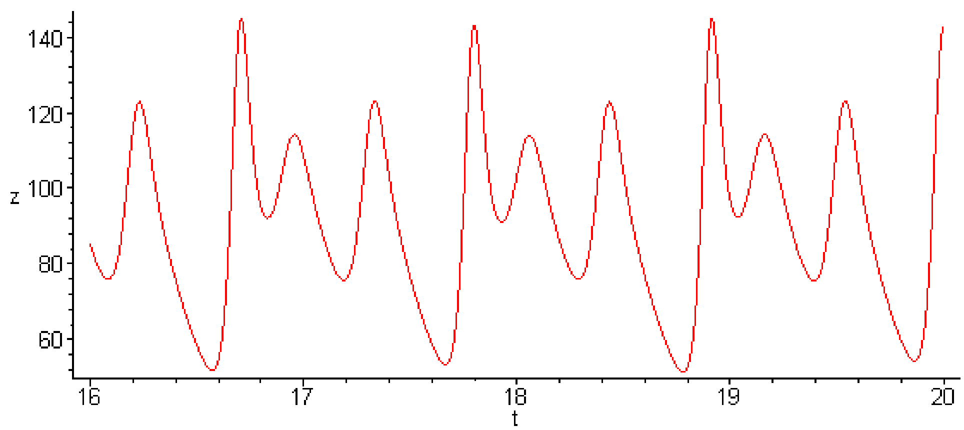




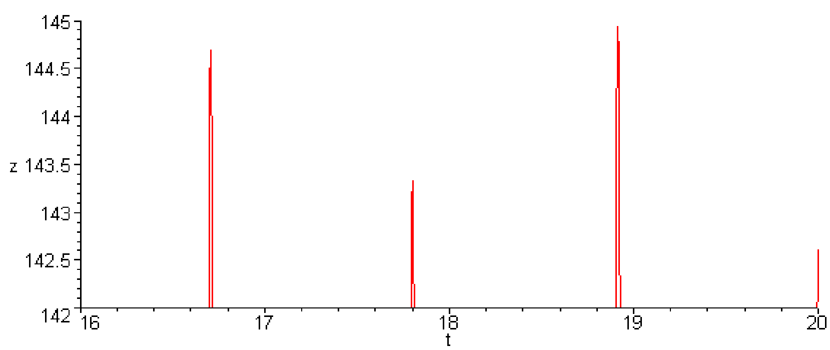
(c). $r = 99.6$, initial point $(-5, -13, 55)$:



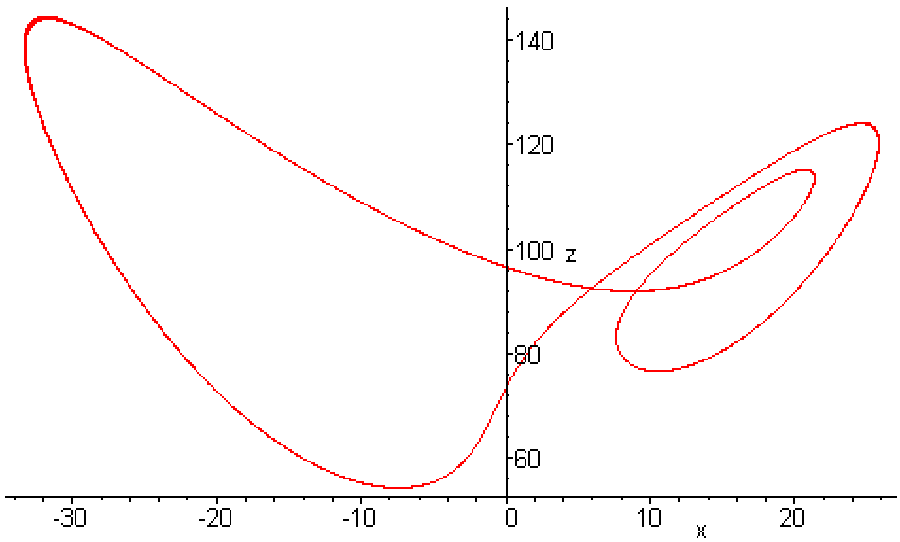
The strands again appear to have split.

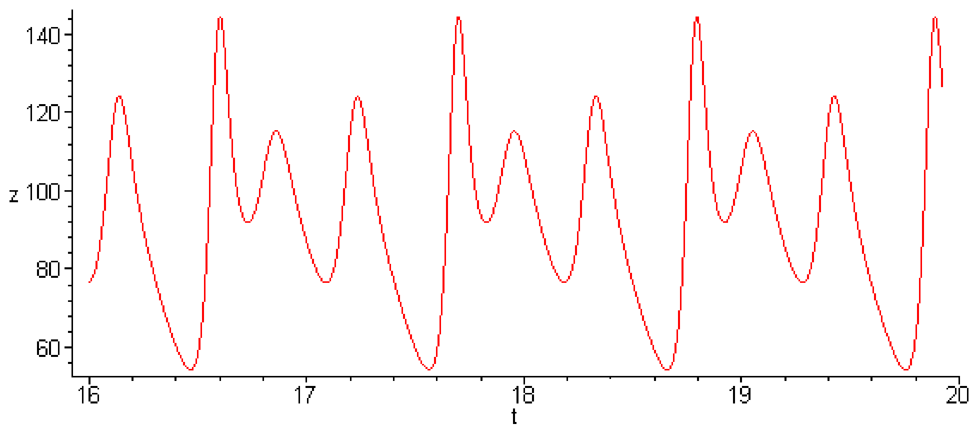


Closer examination reveals that the peak values of $z(t)$ are different:

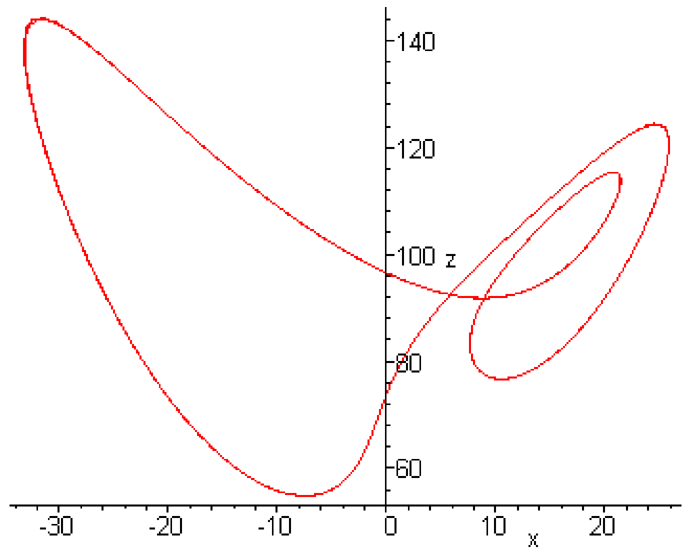


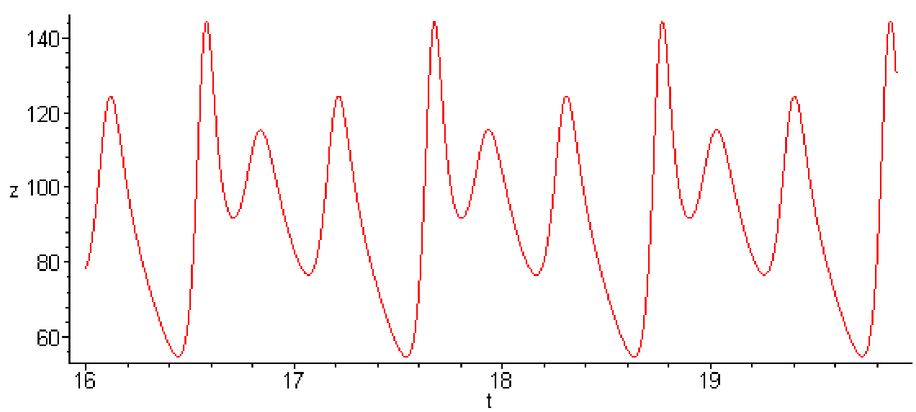
$10(a)$. $r = 100.5$, initial point $(-5, -13, 55)$:



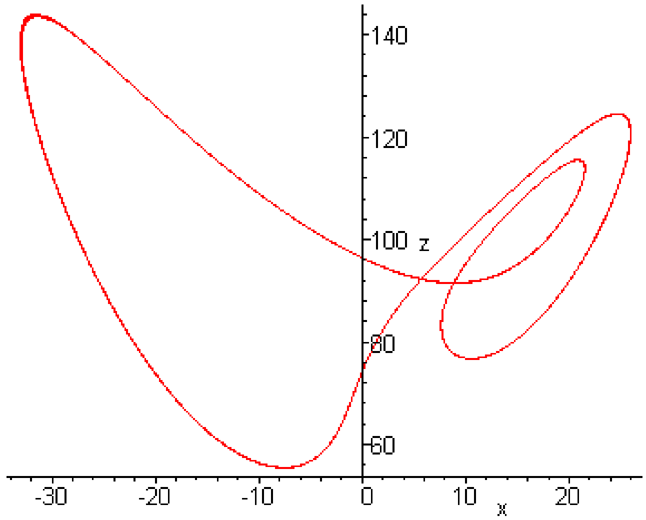


$r = 100.7$, initial point $(-5, -13, 55)$:

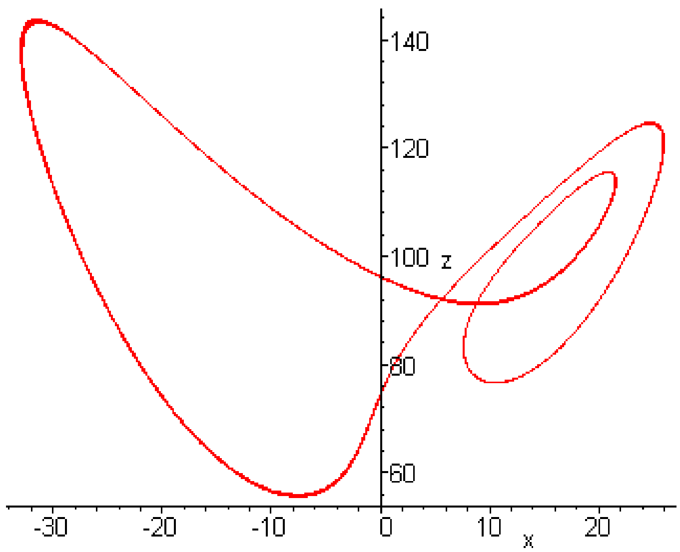




(b). $r = 100.8$, initial point $(-5, -13, 55)$:

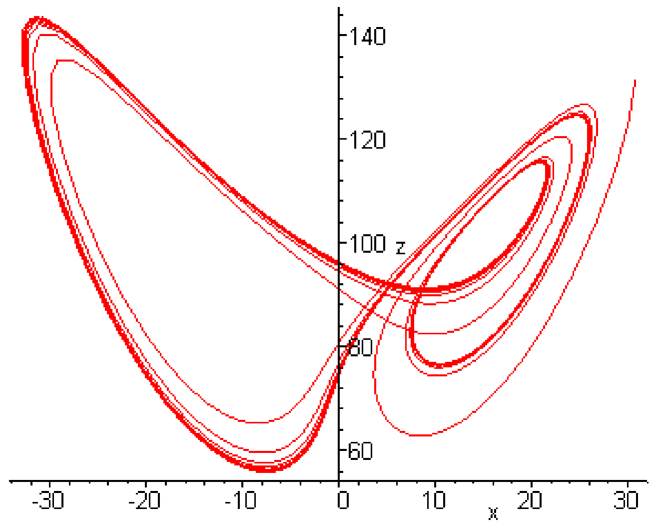


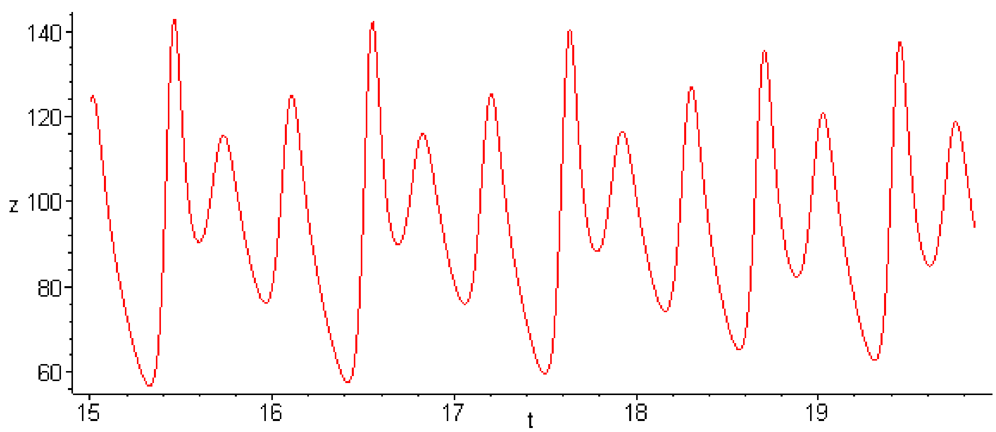
$r = 100.81$, initial point $(-5, -13, 55)$:



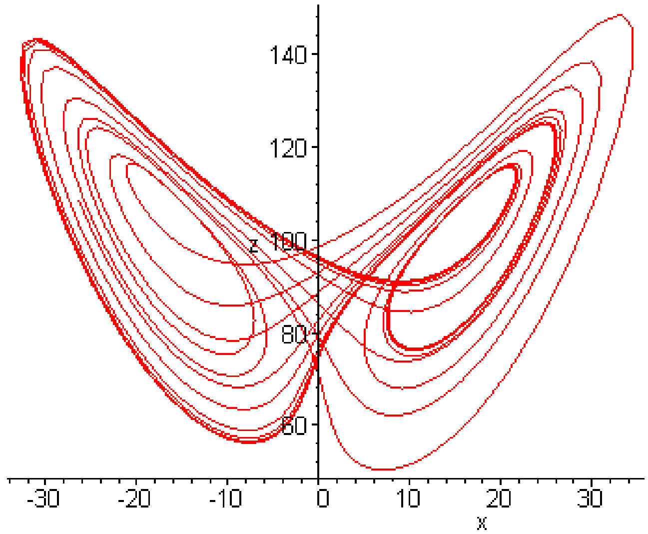
The strands of the periodic trajectory are beginning to split apart.

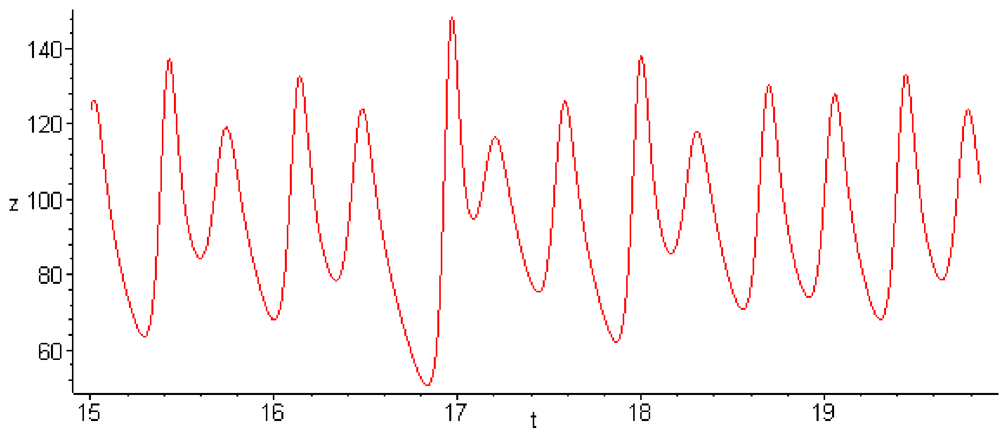
$r = 100.82$, initial point $(-5, -13, 55)$:





$r = 100.83$, initial point $(-5, -13, 55)$:





$r = 100.84$, initial point $(-5, -13, 55)$: