Chapter Seven

Section 7.1

1. Introduce the variables $x_1 = u$ and $x_2 = u'$. It follows that $x_1' = x_2$ and

$$\begin{aligned} x_2' &= u'' \\ &= -2u - 0.5\,u'. \end{aligned}$$

In terms of the new variables, we obtain the system of two first order ODEs

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -2x_1 - 0.5\,x_2\,. \end{aligned}$$

3. First divide both sides of the equation by t^2, and write

$$u'' = -\frac{1}{t}\,u' - \left(1 - \frac{1}{4t^2}\right)u\,.$$

Set $x_1 = u$ and $x_2 = u'$. It follows that $x_1' = x_2$ and

$$\begin{aligned} x_2' &= u'' \\ &= -\frac{1}{t}\,u' - \left(1 - \frac{1}{4t^2}\right)u\,. \end{aligned}$$

We obtain the system of equations

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -\left(1 - \frac{1}{4t^2}\right)x_1 - \frac{1}{t}\,x_2\,. \end{aligned}$$

6. One of the ways to transform the system is to assign the variables

$$y_1 = x_1\ ,\ \ y_2 = x_1'\ ,\ y_3 = x_2\ ,\ \ y_4 = x_2'\,.$$

Before proceeding, note that

$$\begin{aligned} x_1'' &= \frac{1}{m_1}[-(k_1 + k_2)x_1 + k_2x_2 + F_1(t)] \\ x_2'' &= \frac{1}{m_2}[k_2x_1 - (k_2 + k_3)x_2 + F_2(t)]\,. \end{aligned}$$

Differentiating the new variables, we obtain the system of four first order equations

$$\begin{aligned}
y_1' &= y_2 \\
y_2' &= \frac{1}{m_1}[-(k_1+k_2)y_1 + k_2 y_3 + F_1(t)] \\
y_3' &= y_4 \\
y_4' &= \frac{1}{m_2}[k_2 y_1 - (k_2+k_3)y_3 + F_2(t)] \,.
\end{aligned}$$

$7(a)$. Solving the *first* equation for x_2, we have $x_2 = x_1' + 2x_1$. Substitution into the second equation results in

$$(x_1' + 2x_1)' = x_1 - 2(x_1' + 2x_1)\,.$$

That is, $x_1'' + 4\,x_1' + 3\,x_1 = 0$. The resulting equation is a second order differential equation with *constant coefficients*. The general solution is

$$x_1(t) = c_1 e^{-t} + c_2 e^{-3t}\,.$$

With x_2 given in terms of x_1, it follows that

$$x_2(t) = c_1 e^{-t} - c_2 e^{-3t}\,.$$

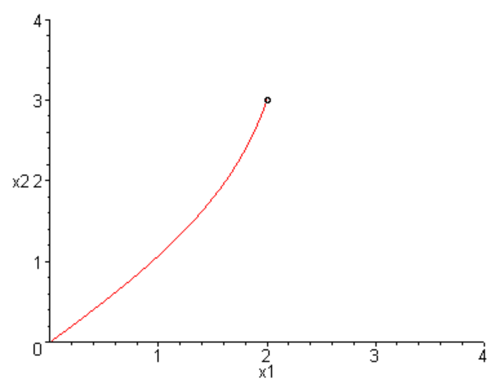
(b). Imposing the specified initial conditions, we obtain

$$\begin{aligned}
c_1 + c_2 &= 2 \\
c_1 - c_2 &= 3\,,
\end{aligned}$$

with solution $c_1 = 5/2$ and $c_2 = -1/2$. Hence

$$x_1(t) = \frac{5}{2}e^{-t} - \frac{1}{2}e^{-3t} \text{ and } x_2(t) = \frac{5}{2}e^{-t} + \frac{1}{2}e^{-3t}\,.$$

(c).



10. Solving the *first* equation for x_2, we obtain $x_2 = (x_1 - x_1')/2$. Substitution into

the second equation results in

$$(x_1 - x_1')'/2 = 3\,x_1 - 2(x_1 - x_1')\,.$$

Rearranging the terms, the single differential equation for x_1 is

$$x_1'' + 3\,x_1' + 2\,x_1 = 0\,.$$

The general solution is

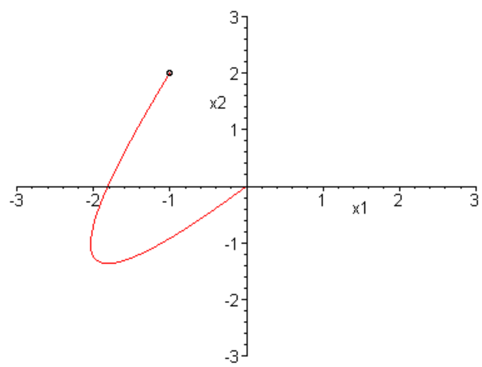
$$x_1(t) = c_1 e^{-t} + c_2 e^{-2t}\,.$$

With x_2 given in terms of x_1, it follows that

$$x_2(t) = c_1 e^{-t} + \frac{3}{2} c_2 e^{-3t}\,.$$

Invoking the specified *initial conditions*, $c_1 = -7$ and $c_2 = 6$. Hence

$$x_1(t) = -7e^{-t} + 6\,e^{-2t} \text{ and } x_2(t) = -7e^{-t} + 9\,e^{-3t}\,.$$



11. Solving the *first* equation for x_2, we have $x_2 = x_1'/2$. Substitution into the second equation results in

$$x_1''/2 = -2\,x_1\,.$$

The resulting equation is $x_1'' + 4\,x_1 = 0$, with general solution

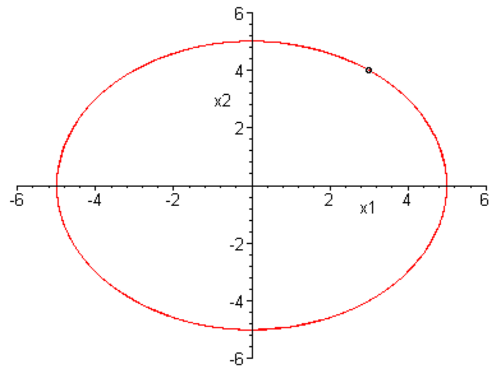
$$x_1(t) = c_1 cos\,2t + c_2 sin\,2t\,.$$

With x_2 given in terms of x_1, it follows that

$$x_2(t) = -c_1 sin\,2t + c_2\,cos\,2t\,.$$

Imposing the specified initial conditions, we obtain $c_1 = 3$ and $c_2 = 4$. Hence

$$x_1(t) = 3\,cos\,2t + 4sin\,2t \text{ and } x_2(t) = -3sin\,2t + 4cos\,2t\,.$$



12. Solving the *first* equation for x_2, we obtain $x_2 = x_1'/2 + x_1/4$. Substitution into the second equation results in

$$x_1''/2 + x_1'/4 = -2x_1 - (x_1'/2 + x_1/4)/2\,.$$

Rearranging the terms, the single differential equation for x_1 is

$$x_1'' + x_1' + \frac{17}{4}x_1 = 0\,.$$

The general solution is

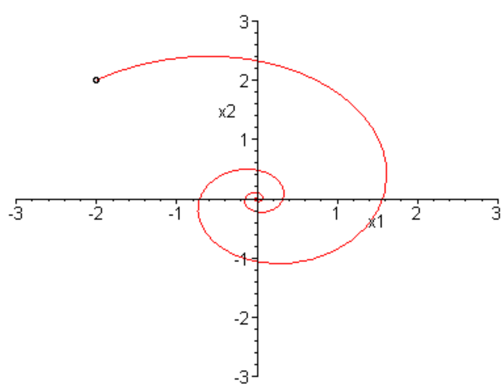
$$x_1(t) = e^{-t/2}[c_1 cos\,2t + c_2 sin\,2t].$$

With x_2 given in terms of x_1, it follows that

$$x_2(t) = e^{-t/2}[-c_1 cos\,2t + c_2 sin\,2t]\,.$$

Imposing the specified initial conditions, we obtain $c_1 = -2$ and $c_2 = 2$. Hence

$$x_1(t) = e^{-t/2}[-2cos\,2t + 2sin\,2t] \text{ and } x_2(t) = e^{-t/2}[2cos\,2t + 2sin\,2t]\,.$$



13. Solving the *first* equation for V, we obtain $V = L \cdot I'$. Substitution into the second equation results in

$$L \cdot I'' = -\frac{I}{C} - \frac{L}{RC} I'.$$

Rearranging the terms, the single differential equation for I is

$$LRC \cdot I'' + L \cdot I' + R \cdot I = 0.$$

15. Direct substitution results in

$$\begin{aligned}(c_1 x_1(t) + c_2 x_2(t))' &= p_{11}(t)[c_1 x_1(t) + c_2 x_2(t)] + p_{12}(t)[c_1 y_1(t) + c_2 y_2(t)] \\ (c_1 y_1(t) + c_2 y_2(t))' &= p_{21}(t)[c_1 x_1(t) + c_2 x_2(t)] + p_{22}(t)[c_1 y_1(t) + c_2 y_2(t)].\end{aligned}$$

Expanding the left-hand-side of the *first* equation,

$$\begin{aligned}c_1 x_1'(t) + c_2 x_2'(t) &= c_1[p_{11}(t)x_1(t) + p_{12}(t)y_1(t)] + \\ &\quad + c_2[p_{11}(t)x_2(t) + p_{12}(t)y_2(t)].\end{aligned}$$

Repeat with the second equation to show that the system of ODEs is identically satisfied.

16. Based on the hypothesis,

$$\begin{aligned}x_1'(t) &= p_{11}(t)x_1(t) + p_{12}(t)y_1(t) + g_1(t) \\ x_2'(t) &= p_{11}(t)x_2(t) + p_{12}(t)y_2(t) + g_1(t).\end{aligned}$$

Subtracting the two equations,

$$x_1'(t) - x_2'(t) = p_{11}(t)[x_1'(t) - x_2'(t)] + p_{12}(t)[y_1'(t) - y_2'(t)].$$

Similarly,

$$y_1'(t) - y_2'(t) = p_{21}(t)[x_1'(t) - x_2'(t)] + p_{22}(t)[y_1'(t) - y_2'(t)].$$

Hence the *difference* of the two solutions satisfies the *homogeneous* ODE.

17. For *rectilinear motion* in one dimension, Newton's second law can be stated as

$$\sum F = m\, x''.$$

The $resisting$ force exerted by a linear spring is given by $F_s = k\,\delta$, in which δ is the *displacement* of the end of a spring from its equilibrium configuration. Hence, with $0 < x_1 < x_2$, the first two springs are in *tension*, and the last spring is in *compression*. The *sum* of the spring forces on m_1 is

$$F_s^1 = -k_1 x_1 - k_2(x_2 - x_1).$$

The *total* force on m_1 is

$$\sum F^1 = -k_1 x_1 + k_2(x_2 - x_1) + F_1(t)\,.$$

Similarly, the *total* force on m_2 is

$$\sum F^2 = -k_2(x_2 - x_1) - k_3 x_2 + F_2(t)\,.$$

18(a). Taking a *clockwise* loop around each of the paths, it is easy to see that voltage drops are given by $V_1 - V_2 = 0$, and $V_2 - V_3 = 0$.

(b). Consider the *right node*. The *current in* is given by $I_1 + I_2$. The current *leaving* the node is $-I_3$. Hence the current passing through the node is $(I_1 + I_2) - (-I_3)$. Based on Kirchhoff's first law, $I_1 + I_2 + I_3 = 0$.

(c). In the capacitor,

$$C\,V_1{}' = I_1.$$

In the resistor,

$$V_2 = R\,I_2\,.$$

In the inductor,

$$L\,I_3' = V_3\,.$$

(d). Based on part (a), $V_3 = V_2 = V_1$. Based on part (b),

$$C\,V_1{}' + \frac{1}{R}V_2 + I_3 = 0\,.$$

It follows that

$$C\,V_1{}' = -\frac{1}{R}V_1 - I_3 \;\text{ and }\; L\,I_3' = V_1.$$

20. Let $I_1, I_2, I_3,$ and I_4 be the current through the resistors, inductor, and capacitor, respectively. Assign $V_1, V_2, V_3,$ and V_4 as the respective voltage drops. Based on Kirchhoff's second law, the net voltage drops, around each loop, satisfy

$$V_1 + V_3 + V_4 = 0\,,\; V_1 + V_3 + V_2 = 0 \;\text{ and }\; V_4 - V_2 = 0\,.$$

Applying Kirchhoff's first law to the upper-right node,

$$I_3 - (I_2 + I_4) = 0\,.$$

Likewise, in the remaining nodes,

$$I_1 - I_3 = 0 \text{ and } I_2 + I_4 - I_1 = 0\,.$$

That is,

$$V_4 - V_2 = 0\,,\; V_1 + V_3 + V_4 = 0 \text{ and } I_2 + I_4 - I_3 = 0\,.$$

Using the current-voltage relations,

$$V_1 = R_1 I_1,\; V_2 = R_2 I_2,\; L\,I_3' = V_3\,,\; C\,V_4' = I_4.$$

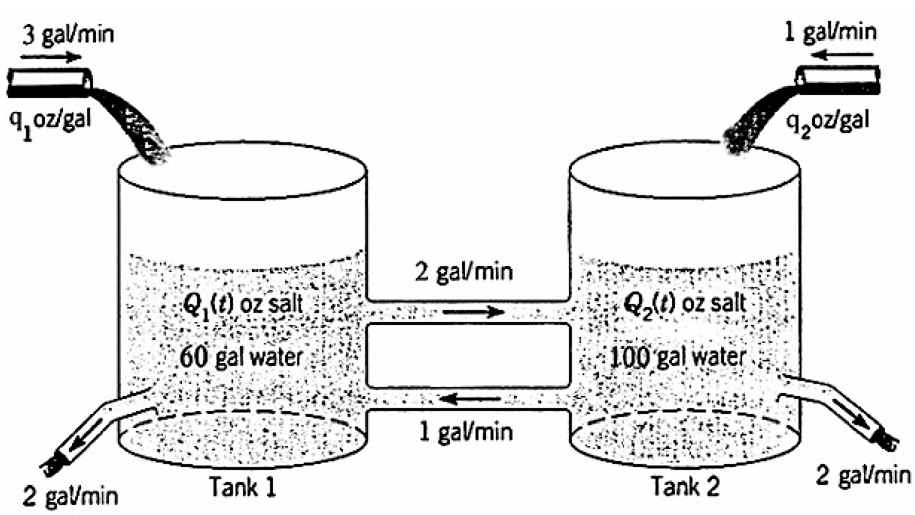
Combining these equations,

$$R_1 I_3 + L\,I_3' + V_4 = 0 \text{ and } C\,V_4' = I_3 - \frac{V_4}{R_2}.$$

Now set $I_3 = I$ and $V_4 = V$, to obtain the system of equations

$$L\,I' = -R_1 I - V \quad \text{and} \quad C\,V' = I - \frac{V}{R_2}.$$

22(a).



Let $Q_1(t)$ and $Q_2(t)$ be the *amount* of salt in the respective tanks at time t . Note that the *volume* of each tank remains constant. Based on conservation of mass, the *rate of increase* of salt, in any given tank, is given by

$$rate\,of\;increase = rate\,in\; - \;rate\,out\,.$$

For Tank 1, the rate of salt flowing *into* Tank 1 is

$$\begin{aligned} r_{in} &= \left[q_1 \frac{oz}{gal}\right]\left[3 \frac{gal}{min}\right] + \left[\frac{Q_2}{100} \frac{oz}{gal}\right]\left[1 \frac{gal}{min}\right] \\ &= 3\, q_1 + \frac{Q_2}{100}\ \frac{oz}{min}\,. \end{aligned}$$

The rate at which salt flow *out* of Tank 1 is

$$r_{out} = \left[\frac{Q_1}{60} \frac{oz}{gal}\right]\left[4 \frac{gal}{min}\right] = \frac{Q_1}{15} \frac{oz}{min}\,.$$

Hence

$$\frac{dQ_1}{dt} = 3\, q_1 + \frac{Q_2}{100} - \frac{Q_1}{15}\,.$$

Similarly, for Tank 2,

$$\frac{dQ_2}{dt} = q_2 + \frac{Q_1}{30} - \frac{3Q_2}{100}\,.$$

The process is modeled by the system of equations

$$\begin{aligned} Q_1' &= -\frac{Q_1}{15} + \frac{Q_2}{100} + 3\, q_1 \\ Q_2' &= \frac{Q_1}{30} - \frac{3Q_2}{100} + q_2\,. \end{aligned}$$

The initial conditions are $Q_1(0) = Q_1^0$ and $Q_2(0) = Q_2^0$.

(b). The *equilibrium values* are obtain by solving the system

$$\begin{aligned} -\frac{Q_1}{15} + \frac{Q_2}{100} + 3\, q_1 &= 0 \\ \frac{Q_1}{30} - \frac{3Q_2}{100} + q_2 &= 0\,. \end{aligned}$$

Its solution leads to $Q_1^E = 54\, q_1 + 6\, q_2$ and $Q_2^E = 60\, q_1 + 40\, q_2$.

(c). The question refers to possible solution of the system

$$\begin{aligned} 54\, q_1 + 6\, q_2 &= 60 \\ 60\, q_1 + 40\, q_2 &= 50\,. \end{aligned}$$

It is possible for formally solve the system of equations, but the unique solution gives

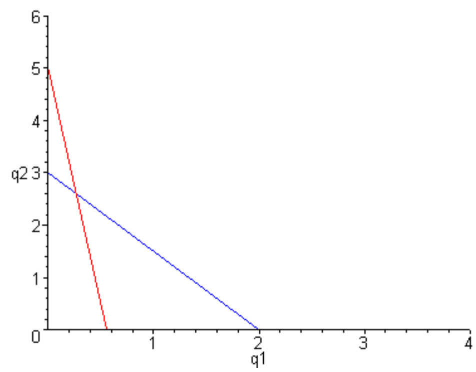
$$q_1 = \frac{7}{6} \frac{oz}{gal} \quad \text{and} \quad q_2 = -\frac{1}{2} \frac{oz}{gal}\,,$$

which is *not* physically possible.

(d). We can write

$$q_2 = -9\,q_1 + \frac{Q_1^E}{6}$$
$$q_2 = -\frac{3}{2}\,q_1 + \frac{Q_2^E}{40}\,,$$

which are the equations of two lines in the $q_1 q_2$-plane:



The intercepts of the *first* line are $Q_1^E/54$ and $Q_1^E/6$. The intercepts of the *second* line are $Q_2^E/60$ and $Q_2^E/40$. Therefore the system will have a unique solution, in the *first quadrant*, as long as $Q_1^E/54 \le Q_2^E/60$ or $Q_2^E/40 \le Q_1^E/6$. That is,

$$\frac{10}{9} \le \frac{Q_2^E}{Q_1^E} \le \frac{20}{3}\,.$$

Section 7.2

$2(a)$.

$$\mathbf{A}-2\,\mathbf{B}=\begin{pmatrix}1+i-2i & -1+2i-6\\ 3+2i-4 & 2-i+4i\end{pmatrix}=\begin{pmatrix}1-i & -7+2i\\ -1+2i & 2+3i\end{pmatrix}.$$

(b).

$$3\,\mathbf{A}+\mathbf{B}=\begin{pmatrix}3+3i+i & -3+6i+3\\ 9+6i+2 & 6-3i-2i\end{pmatrix}=\begin{pmatrix}3+4i & 6i\\ 11+6i & 6-5i\end{pmatrix}.$$

(c).

$$\begin{aligned}\mathbf{AB}&=\begin{pmatrix}(1+i)i+2(-1+2i) & 3(1+i)+(-1+2i)(-2i)\\ (3+2i)i+2(2-i) & 3(3+2i)+(2-i)(-2i)\end{pmatrix}\\ &=\begin{pmatrix}-3+5i & 7+5i\\ 2+i & 7+2i\end{pmatrix}.\end{aligned}$$

(d).

$$\begin{aligned}\mathbf{BA}&=\begin{pmatrix}(1+i)i+3(3+2i) & (-1+2i)i+3(2-i)\\ 2(1+i)+(-2i)(3+2i) & 2(-1+2i)+(-2i)(2-i)\end{pmatrix}\\ &=\begin{pmatrix}8+7i & 4-4i\\ 6-4i & -4\end{pmatrix}.\end{aligned}$$

3.

$$\begin{aligned}\mathbf{A}^T+\mathbf{B}^T&=\begin{pmatrix}-2 & 1 & 2\\ 1 & 0 & -1\\ 2 & -3 & 1\end{pmatrix}+\begin{pmatrix}1 & 3 & -2\\ 2 & -1 & 1\\ 3 & -1 & 0\end{pmatrix}\\ &=\begin{pmatrix}-1 & 4 & 0\\ 3 & -1 & 0\\ 5 & -4 & 1\end{pmatrix}\\ &=(\mathbf{A}+\mathbf{B})^T.\end{aligned}$$

$4(b)$.

$$\overline{\mathbf{A}}=\begin{pmatrix}3+2i & 1-i\\ 2+i & -2-3i\end{pmatrix}.$$

(c). By definition, $\mathbf{A}^*=(\overline{\mathbf{A}^T})=(\overline{\mathbf{A}})^T$.

5.

$$2(\mathbf{A}+\mathbf{B}) = 2\begin{pmatrix} 5 & 3 & -2 \\ 0 & 2 & 5 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 10 & 6 & -4 \\ 0 & 4 & 10 \\ 2 & 4 & 6 \end{pmatrix}.$$

7. Let $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$. The given operations in $(a) - (d)$ are performed elementwise. That is,

(a). $a_{ij} + b_{ij} = b_{ij} + a_{ij}$.
(b). $a_{ij} + (b_{ij} + c_{ij}) = (a_{ij} + b_{ij}) + c_{ij}$.
(c). $\alpha(a_{ij} + b_{ij}) = \alpha\, a_{ij} + \alpha\, b_{ij}$.
(d). $(\alpha + \beta)\, a_{ij} = \alpha\, a_{ij} + \alpha\, a_{ij}$.

In the following, let $\mathbf{A} = (a_{ij})$, $\mathbf{B} = (b_{ij})$and $\mathbf{C} = (c_{ij})$.

(e). Calculating the generic element,

$$(\mathbf{BC})_{ij} = \sum_{k=1}^{n} b_{ik}\, c_{kj}\,.$$

Therefore

$$\begin{aligned}[\mathbf{A}(\mathbf{BC})]_{ij} &= \sum_{r=1}^{n} a_{ir}\left(\sum_{k=1}^{n} b_{rk}\, c_{kj}\right) \\ &= \sum_{r=1}^{n}\sum_{k=1}^{n} a_{ir}\, b_{rk}\, c_{kj} \\ &= \sum_{k=1}^{n}\left[\left(\sum_{r=1}^{n} a_{ir}\, b_{rk}\right) c_{kj}\right].\end{aligned}$$

The last summation is recognized as

$$\sum_{r=1}^{n} a_{ir}\, b_{rk} = (\mathbf{AB})_{ik}\,,$$

which is the ik-*th* element of the matrix $\mathbf{AB}$.

(f). Likewise,

$$
\begin{aligned}
[\mathbf{A}(\mathbf{B}+\mathbf{C})]_{ij} &= \sum_{k=1}^{n} a_{ik}(\,b_{kj} + \,c_{kj}) \\
&= \sum_{k=1}^{n} a_{ik}\, b_{kj} + \sum_{k=1}^{n} a_{ik}\, c_{kj} \\
&= (\mathbf{AB})_{ij} + (\mathbf{AC})_{ij}\,.
\end{aligned}
$$

$8(a)$. $\mathbf{x}^T\mathbf{y} = 2(\,-1+i) + 2(3i) + (1-i)(3-i) = 4i\,.$
(b). $\mathbf{y}^T\mathbf{y} = (\,-1+i)^2 + 2^2 + (3-i)^2 = 12 - 8i\,.$
(c). $(\mathbf{x}\,,\mathbf{y}) = 2(\,-1-i) + 2(3i) + (1-i)(3+i) = 2+2i\,.$
(d). $(\mathbf{y}\,,\mathbf{y}) = (\,-1+i)(\,-1-i) + 2^2 + (3-i)(3+i) = 16\,.$

9. Indeed,

$$
\mathbf{x}^T\mathbf{y} = \sum_{j=1}^{n} x_j\, y_j = \mathbf{y}^T\mathbf{x}\,,
$$

and

$$
(\mathbf{x}\,,\mathbf{y}) = \sum_{j=1}^{n} x_j\, \overline{y_j} = \sum_{j=1}^{n} \overline{y_j}\, x_j = \overline{\sum_{j=1}^{n} y_j\, \overline{x_j}} = \overline{(\mathbf{y}\,,\mathbf{x})}\,.
$$

11. First *augment* the given matrix by the identity matrix:

$$
[\mathbf{A}\,|\,\mathbf{I}\,] = \begin{pmatrix} 3 & -1 & 1 & 0 \\ 6 & 2 & 0 & 1 \end{pmatrix}.
$$

Divide the *first row* by 3, to obtain

$$
\begin{pmatrix} 1 & -\frac{1}{3} & \frac{1}{3} & 0 \\ 6 & 2 & 0 & 1 \end{pmatrix}.
$$

Adding -6 times the *first row* to the *second row* results in

$$
\begin{pmatrix} 1 & -\frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 4 & -2 & 1 \end{pmatrix}.
$$

Divide the *second row* by 4, to obtain

$$
\begin{pmatrix} 1 & -\frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{4} \end{pmatrix}.
$$

Finally, adding $1/3$ times the *second row* to the *first row* results in

$$\begin{pmatrix} 1 & 0 & \frac{1}{6} & \frac{1}{12} \\ 0 & 1 & -\frac{1}{2} & \frac{1}{4} \end{pmatrix}.$$

Hence

$$\begin{pmatrix} 3 & -1 \\ 6 & 2 \end{pmatrix}^{-1} = \frac{1}{12}\begin{pmatrix} 2 & 1 \\ -6 & 3 \end{pmatrix}.$$

13. The augmented matrix is

$$\begin{pmatrix} 1 & 1 & -1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{pmatrix}.$$

Combining the elements of the *first row* with the elements of the *second* and *third* rows results in

$$\begin{pmatrix} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -3 & 3 & -2 & 1 & 0 \\ 0 & 0 & 3 & -1 & 0 & 1 \end{pmatrix}.$$

Divide the elements of the *second row* by -3, and the elements of the *third row* by 3. Now subtracting the new *second row* from the *first row* yields

$$\begin{pmatrix} 1 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & -1 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & 0 & \frac{1}{3} \end{pmatrix}.$$

Finally, combine the *third row* with the *second row* to obtain

$$\begin{pmatrix} 1 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & 0 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & 0 & \frac{1}{3} \end{pmatrix}.$$

Hence

$$\begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \\ 1 & 1 & 2 \end{pmatrix}^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix}.$$

15. Elementary row operations yield

$$\begin{pmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{pmatrix} \to$$

$$\begin{pmatrix} 1 & 0 & -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{4} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{4} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

Finally, combining the $first$ and $third$ rows results in

$$\begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{4} & \frac{1}{8} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{4} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

16. Elementary row operations yield

$$\begin{pmatrix} 1 & -1 & -1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 3 & -2 & 1 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 3 & 2 & -2 & 1 & 0 \\ 0 & 1 & 4 & -3 & 0 & 1 \end{pmatrix} \to$$

$$\begin{pmatrix} 1 & 0 & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{10}{3} & -\frac{7}{3} & -\frac{1}{3} & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & \frac{1}{10} & \frac{3}{10} & \frac{1}{10} \\ 0 & 1 & 0 & -\frac{3}{15} & \frac{2}{5} & -\frac{1}{5} \\ 0 & 0 & \frac{10}{3} & -\frac{7}{3} & -\frac{1}{3} & 1 \end{pmatrix}.$$

Finally, normalizing the *last row* results in

$$\begin{pmatrix} 1 & 0 & 0 & \frac{1}{10} & \frac{3}{10} & \frac{1}{10} \\ 0 & 1 & 0 & -\frac{3}{15} & \frac{2}{5} & -\frac{1}{5} \\ 0 & 0 & 1 & -\frac{7}{10} & -\frac{1}{10} & \frac{3}{10} \end{pmatrix}.$$

17. Elementary row operations on the augmented matrix yield the row-reduced form of the augmented matrix

$$\begin{pmatrix} 1 & 0 & -\frac{1}{7} & 0 & \frac{1}{7} & \frac{2}{7} \\ 0 & 1 & \frac{3}{7} & 0 & \frac{4}{7} & \frac{1}{7} \\ 0 & 0 & 0 & 1 & -2 & -1 \end{pmatrix}.$$

The *left submatrix* cannot be converted to the identity matrix. Hence the given matrix is singular.

18. Elementary row operations on the augmented matrix yield

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow$$
$$\begin{pmatrix} 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

19. Elementary row operations on the augmented matrix yield

$$\begin{pmatrix} 1 & -1 & 2 & 0 & 1 & 0 & 0 & 0 \\ -1 & 2 & -4 & 2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 3 & 0 & 0 & 1 & 0 \\ -2 & 2 & 0 & -1 & 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 3 & -1 & 0 & 1 & 0 \\ 0 & 0 & 4 & -1 & 2 & 0 & 0 & 1 \end{pmatrix} \rightarrow$$
$$\begin{pmatrix} 1 & 0 & 0 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -2 & -1 & 1 & 0 \\ 0 & 0 & 4 & -1 & 2 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 4 & -3 & -1 & 2 & 0 \\ 0 & 0 & 1 & 1 & -2 & -1 & 1 & 0 \\ 0 & 0 & 0 & -5 & 10 & 4 & -4 & 1 \end{pmatrix}.$$

Normalizing the *last row* and combining it with the others results in

$$\begin{pmatrix} 1 & 0 & 0 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 4 & -3 & -1 & 2 & 0 \\ 0 & 0 & 1 & 1 & -2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & -\frac{4}{5} & \frac{4}{5} & -\frac{1}{5} \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 6 & \frac{13}{5} & -\frac{8}{5} & \frac{2}{5} \\ 0 & 1 & 0 & 0 & 5 & \frac{11}{5} & -\frac{6}{5} & \frac{4}{5} \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & 0 & 1 & -2 & -\frac{4}{5} & \frac{4}{5} & -\frac{1}{5} \end{pmatrix}.$$

20. Suppose that **A** is *nonsingular*, and that there exist matrices **B** and **C**, such that $\mathbf{AB} = \mathbf{I}$ and $\mathbf{AC} = \mathbf{I}$. Based on the properties of matrices, it follows that

$$\mathbf{A}(\mathbf{B} - \mathbf{C}) = \mathbf{AY} = \mathbf{0}_{n\times n}\,.$$

Write the *difference* of the two matrices, **Y** , in terms of its *columns* as

$$\mathbf{Y} = \left[\mathbf{y}^{\mathbf{1}} \,|\, \mathbf{y}^{\mathbf{2}} |\cdots|\, \mathbf{y}^{\boldsymbol{n}}\right].$$

The *j-th* column of the product matrix, $\mathbf{AY}$, can be expressed as $\mathbf{A}\,\mathbf{y}^{\mathbf{j}}$. Now since *all* columns of the product matrix consist only of *zeros*, we end up with n homogeneous systems of linear equations

$$\mathbf{A}\,\mathbf{y}^{\mathbf{j}} = \mathbf{0}_{n\times 1}\,,\ j = 1, 2, \cdots, n\,.$$

Since **A** is *nonsingular*, each system must have a *trivial solution*. That is, $\mathbf{y}^{\mathbf{j}} = \mathbf{0}_{n\times 1}$, for $j = 1, 2, \cdots, n$. Hence $\mathbf{Y} = \mathbf{0}_{n\times n}$ and $\mathbf{B} = \mathbf{C}$.

$21(a)$.

$$\mathbf{A}+3\mathbf{B} = \begin{pmatrix} e^t & 2e^{-t} & e^{2t} \\ 2e^t & e^{-t} & -e^{2t} \\ -e^t & 3e^{-t} & 2e^{2t} \end{pmatrix} + \begin{pmatrix} 6e^t & 3e^{-t} & 9e^{2t} \\ -3e^t & 6e^{-t} & 3e^{2t} \\ 9e^t & -3e^{-t} & -3e^{2t} \end{pmatrix}$$
$$= \begin{pmatrix} 7e^t & 5e^{-t} & 10e^{2t} \\ -e^t & 7e^{-t} & 2e^{2t} \\ 8e^t & 0 & -e^{2t} \end{pmatrix}.$$

(b). Based on the standard definition of *matrix multiplication*,

$$\mathbf{AB} = \begin{pmatrix} 2e^{2t}-2+3e^{3t} & 1+4e^{-2t}-e^t & 3e^{3t}+2e^t-e^{4t} \\ 4e^{2t}-1-3e^{3t} & 2+2e^{-2t}+e^t & 6e^{3t}+e^t+e^{4t} \\ -2e^{2t}-3+6e^{3t} & -1+6e^{-2t}-2e^t & -3e^{3t}+3e^t-2e^{4t} \end{pmatrix}.$$

(c).

$$\frac{d\mathbf{A}}{dt} = \begin{pmatrix} e^t & -2e^{-t} & 2e^{2t} \\ 2e^t & -e^{-t} & -2e^{2t} \\ -e^t & -3e^{-t} & 4e^{2t} \end{pmatrix}.$$

(d). Note that

$$\int \mathbf{A}(t)dt = \begin{pmatrix} e^t & -2e^{-t} & e^{2t}/2 \\ 2e^t & -e^{-t} & -e^{2t}/2 \\ -e^t & -3e^{-t} & e^{2t} \end{pmatrix} + \mathbf{C}\,.$$

Therefore

$$\int_0^1 \mathbf{A}(t)dt = \begin{pmatrix} e & -2e^{-1} & e^2/2 \\ 2e & -e^{-1} & -e^2/2 \\ -e & -3e^{-1} & e^2 \end{pmatrix} - \begin{pmatrix} 1 & -2 & 1/2 \\ 2 & -1 & -1/2 \\ -1 & -3 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} e-1 & 2-2e^{-1} & e^2/2-1/2 \\ 2e-2 & 1-e^{-1} & 1/2-e^2/2 \\ 1-e & 3-3e^{-1} & e^2-1 \end{pmatrix}.$$

The result can also be written as

$$(e-1)\begin{pmatrix} 1 & \frac{2}{e} & \frac{1}{2}(e+1) \\ 2 & \frac{1}{e} & -\frac{1}{2}(e+1) \\ -1 & \frac{3}{e} & e+1 \end{pmatrix}.$$

23. First note that

$$\mathbf{x}' = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + 2\begin{pmatrix} 1 \\ 1 \end{pmatrix}\left(e^t + t\,e^t\right) = \begin{pmatrix} 3e^t + 2t\,e^t \\ 2e^t + 2t\,e^t \end{pmatrix}.$$

We also have

$$\begin{aligned}\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\mathbf{x} &= \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix}\left(t\,e^t\right) \\ &= \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^t + \begin{pmatrix} 2 \\ 2 \end{pmatrix}\left(t\,e^t\right) \\ &= \begin{pmatrix} 2e^t + 2t\,e^t \\ 3e^t + 2t\,e^t \end{pmatrix}.\end{aligned}$$

It follows that

$$\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t = \begin{pmatrix} 3e^t + 2t\,e^t \\ 2e^t + 2t\,e^t \end{pmatrix}.$$

24. It is easy to see that

$$\mathbf{x}' = \begin{pmatrix} -6 \\ 8 \\ 4 \end{pmatrix} e^{-t} + \begin{pmatrix} 0 \\ 4 \\ -4 \end{pmatrix} e^{2t} = \begin{pmatrix} -6e^{-t} \\ 8e^{-t} + 4e^{2t} \\ 4e^{-t} - 4e^{-2t} \end{pmatrix}.$$

On the other hand,

$$\begin{aligned}\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}\mathbf{x} &= \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 6 \\ -8 \\ -4 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix} e^{2t} \\ &= \begin{pmatrix} -6 \\ 8 \\ 4 \end{pmatrix} e^{-t} + \begin{pmatrix} 0 \\ 4 \\ -4 \end{pmatrix} e^{2t}.\end{aligned}$$

26. Differentiation, elementwise, results in

$$\mathbf{\Psi}' = \begin{pmatrix} e^t & -2e^{-2t} & 3e^{3t} \\ -4e^t & 2e^{-2t} & 6e^{3t} \\ -e^t & 2e^{-2t} & 3e^{3t} \end{pmatrix}.$$

On the other hand,

$$
\begin{aligned}
\begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} \mathbf{\Psi} &= \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} e^{t} & e^{-2t} & e^{3t} \\ -4e^{t} & -e^{-2t} & 2e^{3t} \\ -e^{t} & -e^{-2t} & e^{3t} \end{pmatrix} \\
&= \begin{pmatrix} e^{t} & -2e^{-2t} & 3e^{3t} \\ -4e^{t} & 2e^{-2t} & 6e^{3t} \\ -e^{t} & 2e^{-2t} & 3e^{3t} \end{pmatrix}.
\end{aligned}
$$

Section 7.3

4. The augmented matrix is

$$\begin{pmatrix} 1 & 2 & -1 & | & 0 \\ 2 & 1 & 1 & | & 0 \\ 1 & -1 & 2 & | & 0 \end{pmatrix}.$$

Adding -2 times the *first row* to the *second row* and subtracting the *first row* from the *third row* results in

$$\begin{pmatrix} 1 & 2 & -1 & | & 0 \\ 0 & -3 & 3 & | & 0 \\ 0 & -3 & 3 & | & 0 \end{pmatrix}.$$

Adding the *negative* of the *second row* to the *third row* results in

$$\begin{pmatrix} 1 & 2 & -1 & | & 0 \\ 0 & -3 & 3 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}.$$

We evidently end up with an equivalent system of equations

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 0 \\ -x_2 + x_3 &= 0\,. \end{aligned}$$

Since there is no unique solution, let $x_3 = \alpha$, where α is arbitrary. It follows that $x_2 = \alpha$, and $x_1 = -\alpha$. Hence all solutions have the form

$$\mathbf{x} = \alpha \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

5. The augmented matrix is

$$\begin{pmatrix} 1 & 0 & -1 & | & 0 \\ 3 & 1 & 1 & | & 0 \\ -1 & 1 & 2 & | & 0 \end{pmatrix}.$$

Adding -3 times the *first row* to the *second row* and adding the *first row* to the *last row* yields

$$\begin{pmatrix} 1 & 0 & -1 & | & 0 \\ 0 & 1 & 3 & | & 0 \\ 0 & 1 & 1 & | & 0 \end{pmatrix}.$$

Now add the negative of the *second row* to the *third row* to obtain

$$\begin{pmatrix} 1 & 0 & -1 & | & 0 \\ 0 & 1 & 3 & | & 0 \\ 0 & 0 & -2 & | & 0 \end{pmatrix}.$$

We end up with an equivalent linear system

$$\begin{aligned} x_1 - x_3 &= 0 \\ x_2 + 3\,x_3 &= 0 \\ x_3 &= 0\,. \end{aligned}$$

Hence the unique solution of the given system of equations is $x_1 = x_2 = x_3 = 0$.

7. Write the given vectors as *columns* of the matrix

$$\mathbf{X} = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is evident that $det(\mathbf{X}) = 0$. Hence the vectors are *linearly dependent*. In order to find a linear relationship between them, write $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + c_3\mathbf{x}^{(3)} = \mathbf{0}$. The latter equation is equivalent to

$$\begin{pmatrix} 2 & 0 & -1 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Performing elementary row operations,

$$\begin{pmatrix} 2 & 0 & -1 & | & 0 \\ 1 & 1 & 2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1/2 & | & 0 \\ 0 & 1 & 5/2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}.$$

We obtain the system of equations

$$\begin{aligned} c_1 - c_3/2 &= 0 \\ c_2 + 5c_3/2 &= 0\,. \end{aligned}$$

Setting $c_3 = 2$, it follows that $c_1 = 1$ and $c_3 = -5$. Hence

$$\mathbf{x}^{(1)} - 5\mathbf{x}^{(2)} + 2\mathbf{x}^{(3)} = \mathbf{0}\,.$$

9. The matrix containing the given vectors as *columns* is

$$\mathbf{X} = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 3 & 0 & -1 \\ -1 & 1 & 2 & 1 \\ 0 & -1 & 2 & 3 \end{pmatrix}.$$

We find that $det(\mathbf{X}) = -70$. Hence the given vectors are *linearly independent*.

10. Write the given vectors as *columns* of the matrix

$$\mathbf{X} = \begin{pmatrix} 1 & 3 & 2 & 4 \\ 2 & 1 & -1 & 3 \\ -2 & 0 & 1 & -2 \end{pmatrix}.$$

The *four* vectors are necessarily *linearly dependent*. Hence there are nonzero scalars such that $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + c_3\mathbf{x}^{(3)} + c_4\mathbf{x}^{(4)} = \mathbf{0}$. The latter equation is equivalent to

$$\begin{pmatrix} 1 & 3 & 2 & 4 \\ 2 & 1 & -1 & 3 \\ -2 & 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Performing elementary row operations,

$$\left(\begin{array}{cccc|c} 1 & 3 & 2 & 4 & 0 \\ 2 & 1 & -1 & 3 & 0 \\ -2 & 0 & 1 & -2 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right).$$

We end up with an equivalent linear system

$$\begin{aligned} c_1 + c_4 &= 0 \\ c_2 + c_4 &= 0 \\ c_3 &= 0\,. \end{aligned}$$

Let $c_4 = -1$. Then $c_1 = 1$ and $c_2 = 1$. Therefore we find that

$$\mathbf{x}^{(1)} + \mathbf{x}^{(2)} - \mathbf{x}^{(4)} = \mathbf{0}\,.$$

11. The matrix containing the given vectors as *columns*, $\mathbf{X}$, is of size $n \times m$. Since $n < m$, we can augment the matrix with $m - n$ rows of *zeros*. The resulting matrix, $\widetilde{\mathbf{X}}$, is of size $m \times m$. Since $\widetilde{\mathbf{X}}$ is square matrix, with *at least* one row of *zeros*, it follows that $det(\widetilde{\mathbf{X}}) = 0$. Hence the column vectors of $\widetilde{\mathbf{X}}$ are linearly dependent. That is, there is a *nonzero* vector, $\mathbf{c}$, such that $\widetilde{\mathbf{X}}\,\mathbf{c} = \mathbf{0}_{m\times 1}$. If we write only the first n rows of the latter equation, we have $\mathbf{X}\,\mathbf{c} = \mathbf{0}_{n\times 1}$. Therefore the column vectors of $\mathbf{X}$ are *linearly dependent*.

12. By inspection, we find that

$$\mathbf{x}^{(1)}(t) - 2\,\mathbf{x}^{(2)}(t) = \begin{pmatrix} -e^{-t} \\ 0 \end{pmatrix}.$$

Hence $3\,\mathbf{x}^{(1)}(t) - 6\,\mathbf{x}^{(2)}(t) + \mathbf{x}^{(3)}(t) = \mathbf{0}$, and the vectors are *linearly dependent*.

13. Two vectors are *linearly dependent* if and only if one is a *nonzero* scalar multiple

of the other. However, there is no *nonzero* scalar, c, such that $2\,sin\,t = c\,sin\,t$ and $sin\,t = 2c\,sin\,t$ for all $t \in (-\infty, \infty)$. Therefore the vectors are *linearly independent*.

16. The eigenvalues λ and eigenvectors **x** satisfy the equation

$$\begin{pmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $(3-\lambda)(-1-\lambda)+8=0$, that is,

$$\lambda^2 - 2\lambda + 5 = 0\,.$$

The eigenvalues are $\lambda_1 = 1-2\,i$ and $\lambda_2 = 1+2\,i$. The components of the eigenvector $\mathbf{x}^{(1)}$ are solutions of the system

$$\begin{pmatrix} 2+2\,i & -2 \\ 4 & -2+2\,i \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The two equations reduce to $(1+i)x_1 = x_2$. Hence $\mathbf{x}^{(1)} = (1\,, 1+i)^T$. Now setting $\lambda = \lambda_2 = 1+2\,i$, we have

$$\begin{pmatrix} 2-2\,i & -2 \\ 4 & -2-2\,i \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

with solution given by $\mathbf{x}^{(2)} = (1\,, 1-i)^T$.

17. The eigenvalues λ and eigenvectors **x** satisfy the equation

$$\begin{pmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $(-2-\lambda)(-2-\lambda)-1=0$, that is,

$$\lambda^2 + 4\lambda + 3 = 0\,.$$

The eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = -1$. For $\lambda_1 = -3$, the system of equations becomes

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which reduces to $x_1 + x_2 = 0$. A solution vector is given by $\mathbf{x}^{(1)} = (1\,, -1)^T$. Substituting $\lambda = \lambda_2 = -1$, we have

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The equations reduce to $x_1 = x_2$. Hence a solution vector is given by $\mathbf{x}^{(2)} = (1\,, 1)^T$.

19. The eigensystem is obtained from analysis of the equation

$$\begin{pmatrix} 1-\lambda & \sqrt{3} \\ \sqrt{3} & -1-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, the determinant of the coefficient matrix must be zero. That is,

$$\lambda^2 - 4 = 0\,.$$

Hence the eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = 2$. Substituting the first eigenvalue, $\lambda = -2$, yields

$$\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The system is equivalent to the equation $\sqrt{3}\,x_1 + x_2 = 0$. A solution vector is given by $\mathbf{x}^{(1)} = \left(1\,,\,-\sqrt{3}\right)^T$. Substitution of $\lambda = 2$ results in

$$\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which reduces to $x_1 = \sqrt{3}\,x_2$. A corresponding solution vector is $\mathbf{x}^{(2)} = \left(\sqrt{3}\,,1\right)^T$.

20. The eigenvalues λ and eigenvectors $\mathbf{x}$ satisfy the equation

$$\begin{pmatrix} -3-\lambda & 3/4 \\ -5 & 1-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $(-3-\lambda)(1-\lambda) + 15/4 = 0$, that is,

$$\lambda^2 + 2\lambda + 3/4 = 0\,.$$

Hence the eigenvalues are $\lambda_1 = -3/2$ and $\lambda_2 = -1/2$. In order to determine the eigenvector corresponding to λ_1, set $\lambda = -3/2$. The system of equations becomes

$$\begin{pmatrix} -3/2 & 3/4 \\ -5 & 5/2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which reduces to $-2\,x_1 + x_2 = 0$. A solution vector is given by $\mathbf{x}^{(1)} = (1\,,2)^T$. Substitution of $\lambda = \lambda_2 = -1/2$ results in

$$\begin{pmatrix} -5/2 & 3/4 \\ -5 & 3/2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which reduces to $10\,x_1 = 3\,x_2$. A corresponding solution vector is $\mathbf{x}^{(2)} = (3\,,10)^T$.

22. The eigensystem is obtained from analysis of the equation

$$\begin{pmatrix} 3-\lambda & 2 & 2 \\ 1 & 4-\lambda & 1 \\ -2 & -4 & -1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The characteristic equation of the coefficient matrix is $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$, with roots $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 3$. Setting $\lambda = \lambda_1 = 1$, we have

$$\begin{pmatrix} 2 & 2 & 2 \\ 1 & 3 & 1 \\ -2 & -4 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This system is reduces to the equations

$$\begin{aligned} x_1 + x_3 &= 0 \\ x_2 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\mathbf{x}^{(1)} = (1\,, 0\,, -1)^T$. Setting $\lambda = \lambda_2 = 2$, the *reduced* system of equations is

$$\begin{aligned} x_1 + 2\,x_2 &= 0 \\ x_3 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\mathbf{x}^{(2)} = (-2\,, 1\,, 0)^T$. Finally, setting $\lambda = \lambda_3 = 3$, the *reduced* system of equations is

$$\begin{aligned} x_1 &= 0 \\ x_2 + x_3 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\mathbf{x}^{(3)} = (0\,, 1\,, -1)^T$.

23. For computational purposes, note that if λ is an eigenvalue of $\mathbf{B}$, then $c\,\lambda$ is an eigenvalue of the matrix $\mathbf{A} = c\,\mathbf{B}$. Eigenvectors are unaffected, since they are only determined up to a scalar multiple. So with

$$\mathbf{B} = \begin{pmatrix} 11 & -2 & 8 \\ -2 & 2 & 10 \\ 8 & 10 & 5 \end{pmatrix},$$

the associated characteristic equation is $\mu^3 - 18\mu^2 - 81\mu + 1458 = 0$, with roots $\mu_1 = -9$, $\mu_2 = 9$ and $\mu_3 = 18$. Hence the eigenvalues of the given matrix, $\mathbf{A}$, are $\lambda_1 = -1$, $\lambda_2 = 1$ and $\lambda_3 = 2$. Setting $\lambda = \lambda_1 = -1$, (which corresponds to using $\mu_1 = -9$ in the *modified* problem) the *reduced* system of equations is

$$\begin{aligned} 2\,x_1 + x_3 &= 0 \\ x_2 + x_3 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\mathbf{x}^{(1)} = (1\,, 2\,, -2)^T$. Setting $\lambda = \lambda_2 = 1$, the *reduced* system of equations is

$$x_1 + 2\,x_3 = 0$$
$$x_2 - 2\,x_3 = 0\,.$$

A corresponding solution vector is given by $\mathbf{x}^{(2)} = (2\,,\,-2\,,\,-1)^T$. Finally, setting $\lambda = \lambda_2 = 1$, the *reduced* system of equations is

$$x_1 - x_3 = 0$$
$$2\,x_2 - x_3 = 0\,.$$

A corresponding solution vector is given by $\mathbf{x}^{(3)} = (2\,,1\,,2)^T$.

25. Suppose that $\mathbf{Ax} = \mathbf{0}$, but that $\mathbf{x} \neq \mathbf{0}$. Let $\mathbf{A} = (a_{ij})$. Using elementary row operations, it is possible to transform the matrix into one that is *not* upper triangular. If it were upper triangular, backsubstitution would imply that $\mathbf{x} = \mathbf{0}$. Hence a linear combination of all the rows results in a row containing only *zeros*. That is, there are n scalars, β_i, one for each row and not all zero, such that for each for column j,

$$\sum_{i=1}^{n} \beta_i\, a_{ij} = 0\,.$$

Now consider $\mathbf{A}^* = (b_{ij})$. By definition, $b_{ij} = \overline{a_{ji}}$, or $a_{ij} = \overline{b_{ji}}$. It follows that for each j,

$$\sum_{i=1}^{n} \beta_i\,\overline{b_{ji}} = \sum_{k=1}^{n} \overline{b_{jk}}\,\beta_k = \sum_{k=1}^{n} b_{jk}\,\overline{\beta_k} = 0\,.$$

Let $\mathbf{y} = \left(\overline{\beta_1}, \overline{\beta_2}, \cdots, \overline{\beta_n}\right)^T$. We therefore have *nonzero* vector, $\mathbf{y}$, such that $\mathbf{A}^*\mathbf{y} = \mathbf{0}$.

26. By definition,

$$\begin{aligned}(\mathbf{Ax}\,,\mathbf{y}) &= \sum_{i=0}^{n} (\mathbf{Ax})_i\,\overline{y_i} \\ &= \sum_{i=0}^{n}\sum_{j=0}^{n} a_{ij}\,x_j\,\overline{y_i}\,.\end{aligned}$$

Let $b_{ij} = \overline{a_{ji}}$, so that $a_{ij} = \overline{b_{ji}}$. Now interchanging the order or summation,

$$\begin{aligned}(\mathbf{Ax}\,,\mathbf{y}) &= \sum_{j=0}^{n} x_j \sum_{i=0}^{n} a_{ij}\,\overline{y_i} \\ &= \sum_{j=0}^{n} x_j \sum_{i=0}^{n} \overline{b_{ji}}\,\overline{y_i}\,.\end{aligned}$$

Now note that

$$\sum_{i=0}^{n} \overline{b_{ji}}\,\overline{y_i} = \overline{\sum_{i=0}^{n} b_{ji}\,y_i} = \overline{(\mathbf{A}^*\mathbf{y})}_j\,.$$

Therefore

$$(\mathbf{Ax}\,,\mathbf{y}) = \sum_{j=0}^{n} x_j\,\overline{(\mathbf{A}^*\mathbf{y})}_j = (\mathbf{x}\,,\mathbf{A}^*\mathbf{y})\,.$$

28. By linearity,

$$\begin{aligned}\mathbf{A}\big(\mathbf{x}^{(0)} + \alpha\,\boldsymbol{\xi}\big) &= \mathbf{Ax}^{(0)} + \alpha\,\mathbf{A}\boldsymbol{\xi} \\ &= \mathbf{b} + \mathbf{0} \\ &= \mathbf{b}\,.\end{aligned}$$

29. Let $c_{ij} = \overline{a_{ji}}$. By the hypothesis, there is a nonzero vector, $\mathbf{y}$, such that

$$\sum_{j=1}^{n} c_{ij}\,y_j = \sum_{j=1}^{n} \overline{a_{ji}}\;y_j = 0\,,\; i = 1, 2, \cdots, n\,.$$

Taking the *conjugate* of both sides, and interchanging the indices, we have

$$\sum_{i=1}^{n} a_{ij}\;\overline{y_i} = 0\,.$$

This implies that a linear combination of *each row* of $\mathbf{A}$ is equal to *zero*. Now consider the augmented matrix $[\mathbf{A}\,|\mathbf{b}]$. Replace the *last* row by

$$\sum_{i=1}^{n}\;\overline{y_i}\,[a_{i1}\,, a_{i2}\,, \cdots, a_{in}\,, b_i] = \left[0\,, 0\,, \cdots, 0\,, \sum_{i=1}^{n}\;\overline{y_i}\,b_i\right].$$

We find that if $(\mathbf{b}\,,\mathbf{y}) = 0$, then the last row of the augmented matrix contains only zeros. Hence there are $n-1$ remaining equations. We can now set $x_n = \alpha$, some parameter, and solve for the other variables in terms of α . Therefore the system of equations $\mathbf{Ax} = \mathbf{b}$ has a solution.

30. If $\lambda = 0$ is an eigenvalue of $\mathbf{A}$, then there is a nonzero vector, $\mathbf{x}$, such that

$$\mathbf{Ax} = \lambda\,\mathbf{x} = \mathbf{0}\,.$$

That is, $\mathbf{Ax} = \mathbf{0}$ has a nonzero solution. This implies that the mapping defined by $\mathbf{A}$ is *not 1-to-1*, and hence not invertible. On the other hand, if $\mathbf{A}$ is singular, then $det(\mathbf{A}) = 0$.
Thus, $\mathbf{Ax} = \mathbf{0}$ has a nonzero solution. The latter equation can be written as $\mathbf{Ax} = 0\,\mathbf{x}$.

31. As shown in Prob. 26 , $(\mathbf{Ax}\,,\mathbf{y}) = (\mathbf{x}\,,\mathbf{A}^*\mathbf{y})$. By definition of a *Hermitian* matrix,

$\mathbf{A} = \mathbf{A}^*$.

32(a). Based on Prob. 31, $(\mathbf{Ax}, \mathbf{x}) = (\mathbf{x}, \mathbf{Ax})$.

(b). Let $\mathbf{x}$ be an eigenvector corresponding to an eigenvalue λ. It then follows that $(\mathbf{Ax}, \mathbf{x}) = (\lambda\mathbf{x}, \mathbf{x})$ and $(\mathbf{x}, \mathbf{Ax}) = (\mathbf{x}, \lambda\mathbf{x})$. Based on the properties of the inner product, $(\lambda\mathbf{x}, \mathbf{x}) = \lambda(\mathbf{x}, \mathbf{x})$ and $(\mathbf{x}, \lambda\mathbf{x}) = \overline{\lambda}(\mathbf{x}, \mathbf{x})$. Then from Part (a),

$$\lambda(\mathbf{x}, \mathbf{x}) = \overline{\lambda}(\mathbf{x}, \mathbf{x}).$$

(c). From Part (b),

$$\left(\lambda - \overline{\lambda}\right)(\mathbf{x}, \mathbf{x}) = 0.$$

Based on the definition of an eigenvector, $(\mathbf{x}, \mathbf{x}) = \|\mathbf{x}\|^2 > 0$. Hence we must have $\lambda - \overline{\lambda} = 0$, which implies that λ is *real*.

33. From Prob. 31,

$$\left(\mathbf{Ax}^{(1)}, \mathbf{x}^{(2)}\right) = \left(\mathbf{x}^{(1)}, \mathbf{Ax}^{(2)}\right).$$

Hence

$$\lambda_1\left(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right) = \overline{\lambda_2}\left(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right) = \lambda_2\left(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right),$$

since the eigenvalues are real. Therefore

$$(\lambda_1 - \lambda_2)\left(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right) = 0.$$

Given that $\lambda_1 \neq \lambda_2$, we must have $\left(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right) = 0$.

Section 7.4

3. Eq. (14) states that the Wronskian satisfies the first order linear ODE

$$\frac{dW}{dt} = (p_{11} + p_{22} + \cdots + p_{nn})W.$$

The general solution is

$$W(t) = C\, exp\left[\int (p_{11} + p_{22} + \cdots + p_{nn})dt\right],$$

in which C is an arbitrary constant. Let $\mathbf{X}_1$ and $\mathbf{X}_2$ be matrices representing two sets of fundamental solutions. It follows that

$$det(\mathbf{X}_1) = W_1(t) = C_1 exp\left[\int (p_{11} + p_{22} + \cdots + p_{nn})dt\right]$$
$$det(\mathbf{X}_2) = W_2(t) = C_2 exp\left[\int (p_{11} + p_{22} + \cdots + p_{nn})dt\right].$$

Hence $det(\mathbf{X}_1)/det(\mathbf{X}_2) = C_1/C_2$. Note that $C_2 \neq 0$.

4. First note that $p_{11} + p_{22} = -p(t)$. As shown in Prob. (3),

$$W\left[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right] = c\, e^{-\int p(t)dt}.$$

For second order linear ODE, the Wronskian (as defined in Chap. 3) satisfies the first order differential equation $W' + p(t)W = 0$. It follows that

$$W\left[y^{(1)}, y^{(2)}\right] = c_1\, e^{-\int p(t)dt}.$$

Alternatively, based on the hypothesis,

$$y^{(1)} = \alpha_{11}\, x_{11} + \alpha_{12}\, x_{12}$$
$$y^{(2)} = \alpha_{21}\, x_{11} + \alpha_{22}\, x_{12}\,.$$

Direct calculation shows that

$$\begin{aligned} W\left[y^{(1)}, y^{(2)}\right] &= \begin{vmatrix} \alpha_{11}\, x_{11} + \alpha_{12}\, x_{12} & \alpha_{21}\, x_{11} + \alpha_{22}\, x_{12} \\ \alpha_{11}\, x_{11}' + \alpha_{12}\, x_{12}' & \alpha_{21}\, x_{11}' + \alpha_{22}\, x_{12}' \end{vmatrix} \\ &= (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})x_{11}x_{12}' - (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})x_{12}x_{11}' \\ &= (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})x_{11}x_{22} - (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})x_{12}x_{21}\,. \end{aligned}$$

Here we used the fact that $x_1' = x_2$. Hence

$$W\left[y^{(1)}, y^{(2)}\right] = (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})W\left[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right].$$

5. The *particular solution* satisfies the ODE $\left[\mathbf{x}^{(p)}\right]' = \mathbf{P}(t)\mathbf{x}^{(p)} + \mathbf{g}(t)$. Now let

$\mathbf{x} = \boldsymbol{\phi}(t)$ be any solution of the homogeneous equation. That is, $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$. We know that $\mathbf{x} = \mathbf{x}^c$, in which $\mathbf{x}^c$ is a linear combination of some fundamental solution. By linearity of the differential equation, it follows that $\mathbf{x} = \mathbf{x}^{(p)} + \mathbf{x}^c$ is a solution of the ODE. Based on the *uniqueness theorem*, all solutions must have this form.

$7(a)$. By definition,

$$W\left[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right] = \begin{vmatrix} t^2 & e^t \\ 2t & e^t \end{vmatrix} = \left(t^2 - 2t\right)e^t.$$

(b). The Wronskian vanishes at $t_0 = 0$ and $t_0 = 2$. Hence the vectors are linearly independent on $\mathcal{D} = (-\infty, 0) \cup (0, 2) \cup (2, \infty)$.

(c). It follows from Theorem 7.4.3 that one or more of the coefficients of the ODE must be discontinuous at $t_0 = 0$ and $t_0 = 2$. If not, the Wronskian would not vanish.

(d). Let

$$\mathbf{x} = c_1\begin{pmatrix} t^2 \\ 2t \end{pmatrix} + c_2\begin{pmatrix} e^t \\ e^t \end{pmatrix}.$$

Then

$$\mathbf{x}' = c_1\begin{pmatrix} 2t \\ 2 \end{pmatrix} + c_2\begin{pmatrix} e^t \\ e^t \end{pmatrix}.$$

On the other hand,

$$\begin{aligned} \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}\mathbf{x} &= c_1\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}\begin{pmatrix} t^2 \\ 2t \end{pmatrix} + c_2\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}\begin{pmatrix} e^t \\ e^t \end{pmatrix} \\ &= \begin{pmatrix} c_1[p_{11}t^2 + 2p_{12}t] + c_2[p_{11} + p_{12}]e^t \\ c_1[p_{21}t^2 + 2p_{22}t] + c_2[p_{21} + p_{22}]e^t \end{pmatrix}. \end{aligned}$$

Comparing coefficients, we find that

$$\begin{aligned} p_{11}t^2 + 2p_{12}t &= 2t \\ p_{11} + p_{12} &= 1 \\ p_{21}t^2 + 2p_{22}t &= 2 \\ p_{21} + p_{22} &= 1\,. \end{aligned}$$

Solution of this system of equations results in

$$p_{11}(t) = 0\ , p_{12}(t) = 1\ , p_{21}(t) = \frac{2 - 2t}{t^2 - 2t}\ , p_{22}(t) = \frac{t^2 - 2}{t^2 - 2t}\,.$$

Hence the vectors are solutions of the ODE

$$\mathbf{x}' = \frac{1}{t^2 - 2t}\begin{pmatrix} 0 & t^2 - 2t \\ 2 - 2t & t^2 - 2 \end{pmatrix}\mathbf{x}\,.$$

8. Suppose that the solutions $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}, \cdots, \mathbf{x}^{(m)}$ are linearly *dependent* at $t = t_0$. Then there are constants $c_1, c_2, \cdots, c_m$ (not all zero) such that

$$c_1\mathbf{x}^{(1)}(t_0) + c_2\,\mathbf{x}^{(2)}(t_0) + \cdots + c_m\,\mathbf{x}^{(m)}(t_0) = \mathbf{0}\,.$$

Now let $\mathbf{z}(t) = c_1\mathbf{x}^{(1)}(t) + c_2\,\mathbf{x}^{(2)}(t) + \cdots + c_m\,\mathbf{x}^{(m)}(t)$. Then clearly, $\mathbf{z}(t)$ is a solution of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$, with $\mathbf{z}(t_0) = 0$. Furthermore, $\mathbf{y}(t) \equiv \mathbf{0}$ is also a solution, with $\mathbf{y}(t_0) = 0$. By the *uniqueness theorem*, $\mathbf{z}(t) = \mathbf{y}(t) = \mathbf{0}$. Hence

$$c_1\mathbf{x}^{(1)}(t) + c_2\,\mathbf{x}^{(2)}(t) + \cdots + c_m\,\mathbf{x}^{(m)}(t) = \mathbf{0}$$

on the entire interval $\alpha < t < \beta$. Going in the other direction is trivial.

9(a). Let $\mathbf{y}(t)$ be *any* solution of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$. It follows that

$$\mathbf{z}(t) + \mathbf{y}(t) = c_1\mathbf{x}^{(1)}(t) + c_2\,\mathbf{x}^{(2)}(t) + \cdots + c_n\,\mathbf{x}^{(n)}(t) + \mathbf{y}(t)$$

is also a solution. Now let $t_0 \in (\alpha\,, \beta)$. Then the collection of vectors

$$\mathbf{x}^{(1)}(t_0), \mathbf{x}^{(2)}(t_0), \cdots,\ \mathbf{x}^{(n)}(t_0),\ \mathbf{y}(t_0)$$

constitutes $n+1$ vectors, each with n components. Based on the assertion in Prob. 11, Section 7.3, these vectors are necessarily linearly *dependent*. That is, there are $n+1$ constants $b_1, b_2, \cdots, b_n, b_{n+1}$ (not all zero) such that

$$b_1\mathbf{x}^{(1)}(t_0) + b_2\,\mathbf{x}^{(2)}(t_0) + \cdots + b_n\,\mathbf{x}^{(n)}(t_0) + b_{n+1}\,\mathbf{y}(t_0) = \mathbf{0}\,.$$

From Prob. 8, we have

$$b_1\mathbf{x}^{(1)}(t) + b_2\,\mathbf{x}^{(2)}(t) + \cdots + b_n\,\mathbf{x}^{(n)}(t) + b_{n+1}\,\mathbf{y}(t) = \mathbf{0}$$

for all $t \in (\alpha\,, \beta)$. Now $b_{n+1} \neq 0$, otherwise that would contradict the fact that the first n vectors are linearly independent. Hence

$$\mathbf{y}(t) = -\frac{1}{b_{n+1}}\left(b_1\mathbf{x}^{(1)}(t) + b_2\,\mathbf{x}^{(2)}(t) + \cdots + b_n\,\mathbf{x}^{(n)}(t)\right),$$

and the assertion is true.

(b). Consider $\mathbf{z}(t) = c_1\mathbf{x}^{(1)}(t) + c_2\,\mathbf{x}^{(2)}(t) + \cdots + c_n\,\mathbf{x}^{(n)}(t)$, and suppose that we also have

$$\mathbf{z}(t) = k_1\mathbf{x}^{(1)}(t) + k_2\,\mathbf{x}^{(2)}(t) + \cdots + k_n\,\mathbf{x}^{(n)}(t)\,.$$

Based on the assumption,

$$(k_1 - c_1)\mathbf{x}^{(1)}(t) + (k_2 - c_2)\mathbf{x}^{(2)}(t) + \cdots + (k_n - c_n)\mathbf{x}^{(n)}(t) = \mathbf{0}\,.$$

The collection of vectors

$$\mathbf{x}^{(1)}(t), \mathbf{x}^{(2)}(t), \cdots, \ \mathbf{x}^{(n)}(t)$$

is linearly *independent* on $\alpha < t < \beta$. It follows that $k_i - c_i = 0$, for $i = 1, 2, \cdots, n$.

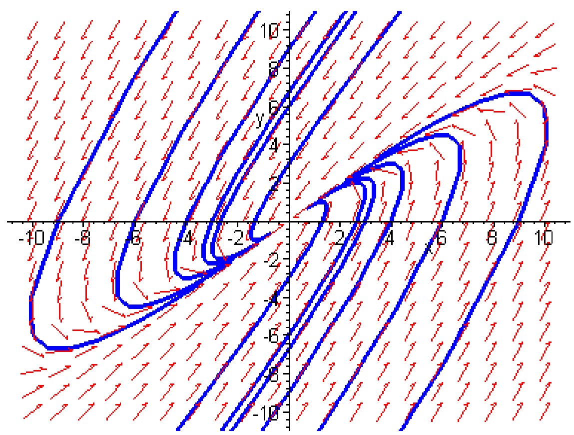
Section 7.5

2. Setting $\mathbf{x} = \boldsymbol{\xi}\, e^{rt}$, and substituting into the ODE, we obtain the algebraic equations

$$\begin{pmatrix} 1-r & -2 \\ 3 & -4-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $det(\mathbf{A} - r\,\mathbf{I}) = r^2 + 3r + 2 = 0$. The roots of the characteristic equation are $r_1 = -1$ and $r_2 = -2$. For $r = -1$, the two equations reduce to $\xi_1 = \xi_2$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1\,,1)^T$. Substitution of $r = -2$ results in the single equation $3\,\xi_1 = 2\,\xi_2$. A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (2\,,3)^T$. Since the eigenvalues are *distinct*, the general solution is

$$\mathbf{x} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2\begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{-2t}.$$

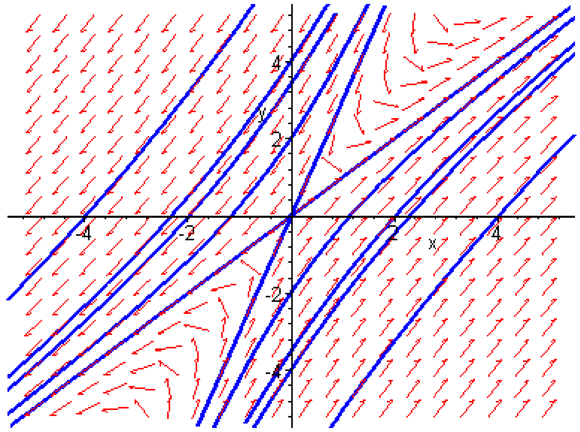


3. Setting $\mathbf{x} = \boldsymbol{\xi}\, e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} 2-r & -1 \\ 3 & -2-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $det(\mathbf{A} - r\,\mathbf{I}) = r^2 - 1 = 0$. The roots of the characteristic equation are $r_1 = 1$ and $r_2 = -1$. For $r = 1$, the system of equations reduces to $\xi_1 = \xi_2$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1\,,1)^T$. Substitution of $r = -1$ results in the single equation $3\,\xi_1 = \xi_2$. A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (1\,,3)^T$. Since the eigenvalues are *distinct*, the general solution is

$$\mathbf{x} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} + c_2\begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}.$$



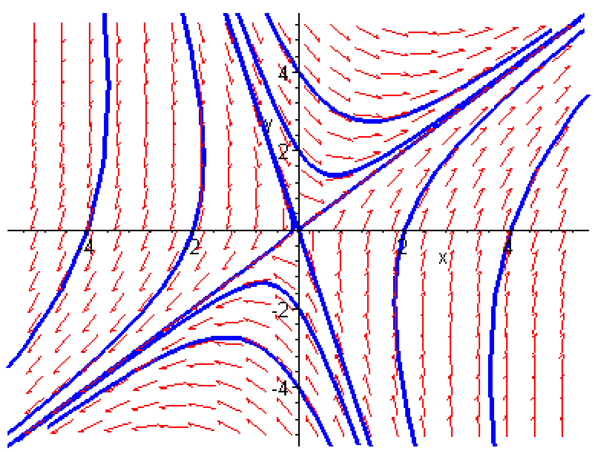
The system has an *unstable* eigendirection along $\boldsymbol{\xi}^{(1)} = (1\,,1)^T$. Unless $c_1 = 0$, all solutions will diverge.

4. Solution of the ODE requires analysis of the algebraic equations

$$\begin{pmatrix} 1-r & 1 \\ 4 & -2-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $det(\mathbf{A} - r\,\mathbf{I}) = r^2 + r - 6 = 0$. The roots of the characteristic equation are $r_1 = 2$ and $r_2 = -3$. For $r = 2$, the system of equations reduces to $\xi_1 = \xi_2$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1\,,1)^T$. Substitution of $r = -3$ results in the single equation $4\,\xi_1 + \xi_2 = 0$. A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (1\,,-4)^T$. Since the eigenvalues are *distinct*, the general solution is

$$\mathbf{x} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{2t} + c_2\begin{pmatrix} 1 \\ -4 \end{pmatrix}e^{-3t}.$$



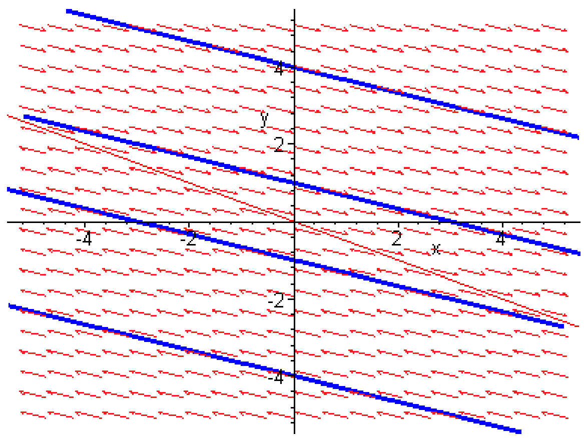
The system has an *unstable* eigendirection along $\boldsymbol{\xi}^{(1)} = (1\,,1)^T$. Unless $c_1 = 0$, all solutions will diverge.

8. Setting $\mathbf{x} = \boldsymbol{\xi}\,e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} 3-r & 6 \\ -1 & -2-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $det(\mathbf{A} - r\,\mathbf{I}) = r^2 - r = 0$. The roots of the characteristic equation are $r_1 = 1$ and $r_2 = 0$. With $r = 1$, the system of equations reduces to $\xi_1 + 3\xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (3\,,\,-1)^T$. For the case $r = 0$, the system is equivalent to the equation $\xi_1 + 2\,\xi_2 = 0$. An eigenvector is $\boldsymbol{\xi}^{(2)} = (2\,,\,-1)^T$. Since the eigenvalues are *distinct*, the general solution is

$$\mathbf{x} = c_1\begin{pmatrix} 3 \\ -1 \end{pmatrix}e^t + c_2\begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$



The *entire line* along the eigendirection $\boldsymbol{\xi}^{(2)} = (2\,,\,-1)^T$ consists of equilibrium points. All other solutions diverge. The direction field changes across the line $x_1 + 2x_2 = 0$. Eliminating the exponential terms in the solution, the trajectories are given by

$$x_1 + 3\,x_2 = -\,c_2\,.$$

10. The characteristic equation is given by

$$\begin{vmatrix} 2-r & 2+i \\ -1 & -1-i-r \end{vmatrix} = r^2 - (1-i)r - i = 0\,.$$

The equation has *complex* roots $r_1 = 1$ and $r_2 = -\,i$. For $r = 1$, the components of the solution vector must satisfy $\xi_1 + (2+i)\xi_2 = 0$. Thus the corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (2+i\,,\,-1)^T$. Substitution of $r = -\,i$ results in the single equation $\xi_1 + \xi_2 = 0$. A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (1\,,\,-1)^T$. Since the eigenvalues are *distinct*, the general solution is

$$\mathbf{x} = c_1\begin{pmatrix} 2+i \\ -1 \end{pmatrix}e^t + c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-it}.$$

11. Setting $\mathbf{x} = \boldsymbol{\xi}\,e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} 1-r & 1 & 2 \\ 1 & 2-r & 1 \\ 2 & 1 & 1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $det(\mathbf{A} - r\,\mathbf{I}) = r^3 - 4r^2 - r + 4 = 0$. The roots of the characteristic equation are $r_1 = 4$, $r_2 = 1$ and $r_3 = -1$. Setting $r = 4$, we have

$$\begin{pmatrix} -3 & 1 & 2 \\ 1 & -2 & 1 \\ 2 & 1 & -3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This system is reduces to the equations

$$\begin{aligned} \xi_1 - \xi_3 &= 0 \\ \xi_2 - \xi_3 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\boldsymbol{\xi}^{(1)} = (1\,,1\,,1)^T$. Setting $\lambda = 1$, the *reduced* system of equations is

$$\begin{aligned} \xi_1 - \xi_3 &= 0 \\ \xi_2 + 2\,\xi_3 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\boldsymbol{\xi}^{(2)} = (1\,,\,-2\,,1)^T$. Finally, setting $\lambda = -1$, the *reduced* system of equations is

$$\begin{aligned} \xi_1 + \xi_3 &= 0 \\ \xi_2 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\boldsymbol{\xi}^{(3)} = (1\,,0\,,\,-1)^T$. Since the eigenvalues are distinct, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} e^{t} + c_3 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t}.$$

12. The eigensystem is obtained from analysis of the equation

$$\begin{pmatrix} 3-r & 2 & 4 \\ 2 & -r & 2 \\ 4 & 2 & 3-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The characteristic equation of the coefficient matrix is $r^3 - 6r^2 - 15r - 8 = 0$, with roots $r_1 = 8$, $r_2 = -1$ and $r_3 = -1$. Setting $r = r_1 = 8$, we have

$$\begin{pmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This system is reduced to the equations

$$\begin{aligned} \xi_1 - \xi_3 &= 0 \\ 2\,\xi_2 - \xi_3 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\boldsymbol{\xi}^{(1)} = (2\,,1\,,2)^T$. Setting $r = -1$, the system of equations is reduced to the *single* equation

$$2\,\xi_1 + \xi_2 + 2\,\xi_3 = 0\,.$$

Two independent solutions are obtained as

$$\boldsymbol{\xi}^{(2)} = (1\,,\,-2\,,0)^T \text{ and } \boldsymbol{\xi}^{(3)} = (0\,,\,-2\,,1)^T.$$

Hence the general solution is

$$\mathbf{x} = c_1\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}e^{8t} + c_2\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}e^{-t} + c_3\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}e^{-t}.$$

13. Setting $\mathbf{x} = \boldsymbol{\xi}\,e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} 1-r & 1 & 1 \\ 2 & 1-r & -1 \\ -8 & -5 & -3-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $det(\mathbf{A} - r\,\mathbf{I}) = r^3 + r^2 - 4r - 4 = 0$. The roots of the characteristic equation are $r_1 = 2\,, r_2 = -2$ and $r_3 = -1$. Setting $r = 2$, we have

$$\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -8 & -5 & -5 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This system is reduces to the equations

$$\begin{aligned} \xi_1 &= 0 \\ \xi_2 + \xi_3 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\boldsymbol{\xi}^{(1)} = (0\,,1\,,\,-1)^T$. Setting $\lambda = -1$, the *reduced* system of equations is

$$\begin{aligned} 2\,\xi_1 + 3\,\xi_3 &= 0 \\ \xi_2 - 2\,\xi_3 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\boldsymbol{\xi}^{(2)} = (3\,,\,-4\,,\,-2)^T$. Finally, setting $\lambda = -2$, the *reduced* system of equations is

$$\begin{aligned} 7\,\xi_1 + 4\,\xi_3 &= 0 \\ 7\xi_2 - 5\,\xi_3 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\boldsymbol{\xi}^{(3)} = (4\,,\,-5\,,\,-7)^T$. Since the eigenvalues are distinct, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 3 \\ -4 \\ -2 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 4 \\ -5 \\ -7 \end{pmatrix} e^{-2t}.$$

15. Setting $\mathbf{x} = \boldsymbol{\xi}\, e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} 5-r & -1 \\ 3 & 1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $det(\mathbf{A} - r\,\mathbf{I}) = r^2 - 6r + 8 = 0$. The roots of the characteristic equation are $r_1 = 4$ and $r_2 = 2$. With $r = 4$, the system of equations reduces to $\xi_1 - \xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1\,,1)^T$. For the case $r = 2$, the system is equivalent to the equation $3\,\xi_1 - \xi_2 = 0$. An eigenvector is $\boldsymbol{\xi}^{(2)} = (1\,,3)^T$. Since the eigenvalues are *distinct*, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t}.$$

Invoking the initial conditions, we obtain the system of equations

$$\begin{aligned} c_1 + c_2 &= 2 \\ c_1 + 3\,c_2 &= -1\,. \end{aligned}$$

Hence $c_1 = 7/2$ and $c_2 = -3/2$, and the solution of the IVP is

$$\mathbf{x} = \frac{7}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} - \frac{3}{2} \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t}.$$

17. Setting $\mathbf{x} = \boldsymbol{\xi}\, e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} 1-r & 1 & 2 \\ 0 & 2-r & 2 \\ -1 & 1 & 3-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $det(\mathbf{A} - r\,\mathbf{I}) = r^3 - 6r^2 + 11r - 6 = 0$. The roots of the characteristic equation are $r_1 = 1$, $r_2 = 2$ and $r_3 = 3$. Setting $r = 1$, we have

$$\begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ -1 & 1 & 2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This system is reduces to the equations

$$\begin{aligned} \xi_1 &= 0 \\ \xi_2 + 2\,\xi_3 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\boldsymbol{\xi}^{(1)} = (0\,,\,-2\,,1)^T$. Setting $\lambda = 2$, the *reduced* system of equations is

$$\begin{aligned} \xi_1 - \xi_2 &= 0 \\ \xi_3 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\boldsymbol{\xi}^{(2)} = (1\,,1\,,0)^T$. Finally, upon setting $\lambda = 3$, the *reduced* system of equations is

$$\begin{aligned} \xi_1 - 2\,\xi_3 &= 0 \\ \xi_2 - 2\,\xi_3 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\boldsymbol{\xi}^{(3)} = (2\,,2\,,1)^T$. Since the eigenvalues are distinct, the general solution is

$$\mathbf{x} = c_1\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}e^t + c_2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}e^{2t} + c_3\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}e^{3t}.$$

Invoking the initial conditions, the coefficients must satisfy the equations

$$\begin{aligned} c_2 + 2\,c_3 &= 2 \\ -2\,c_1 + c_2 + 2\,c_3 &= 0 \\ c_1 + c_3 &= 1\,. \end{aligned}$$

It follows that $c_1 = 1$, $c_2 = 2$ and $c_3 = 0$. Hence the solution of the IVP is

$$\mathbf{x} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}e^t + 2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}e^{2t}.$$

18. The eigensystem is obtained from analysis of the equation

$$\begin{pmatrix} -r & 0 & -1 \\ 2 & -r & 0 \\ -1 & 2 & 4-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The characteristic equation of the coefficient matrix is $r^3 - 4r^2 - r + 4 = 0$, with roots $r_1 = -1$, $r_2 = 1$ and $r_3 = 4$. Setting $r = r_1 = -1$, we have

$$\begin{pmatrix} -1 & 0 & -1 \\ 2 & -1 & 0 \\ -1 & 2 & 3 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This system is reduced to the equations

$$\begin{aligned} \xi_1 - \xi_3 &= 0 \\ \xi_2 + 2\,\xi_3 &= 0\,. \end{aligned}$$

A corresponding solution vector is given by $\boldsymbol{\xi}^{(1)} = (1\,,\,-2\,,1)^T$. Setting $r = 1$, the system reduces to the equations

$$\begin{aligned} \xi_1 + \xi_3 &= 0 \\ \xi_2 + 2\,\xi_3 &= 0\,. \end{aligned}$$

The corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (1\,,2\,,\,-1)^T$. Finally, upon setting $r = 4$, the system is equivalent to the equations

$$\begin{aligned} 4\,\xi_1 + \xi_3 &= 0 \\ 8\,\xi_2 + \xi_3 &= 0\,. \end{aligned}$$

The corresponding eigenvector is $\boldsymbol{\xi}^{(3)} = (2\,,1\,,\,-8)^T$. Hence the general solution is

$$\mathbf{x} = c_1\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}e^{-t} + c_2\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}e^{t} + c_3\begin{pmatrix} 2 \\ 1 \\ -8 \end{pmatrix}e^{4t}.$$

Invoking the initial conditions,

$$\begin{aligned} c_1 + c_2 + 2\,c_3 &= 7 \\ -2\,c_1 + 2\,c_2 + c_3 &= 5 \\ c_1 - c_2 - 8\,c_3 &= 5\,. \end{aligned}$$

It follows that $c_1 = 3$, $c_2 = 6$ and $c_3 = -1$. Hence the solution of the IVP is

$$\mathbf{x} = 3\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}e^{-t} + 6\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}e^{t} - \begin{pmatrix} 2 \\ 1 \\ -8 \end{pmatrix}e^{4t}.$$

19. Set $\mathbf{x} = \boldsymbol{\xi}\,t^r$. Substitution into the system of differential equations results in

$$t \cdot rt^{r-1}\boldsymbol{\xi} \;= \mathbf{A}\,\boldsymbol{\xi}\,t^r,$$

which upon simplification yields is, $\mathbf{A}\,\boldsymbol{\xi} - r\boldsymbol{\xi} = \mathbf{0}$. Hence the vector $\boldsymbol{\xi}$ and constant r must satisfy $(\mathbf{A} - r\mathbf{I})\boldsymbol{\xi} = \mathbf{0}$.

21. Setting $\mathbf{x} = \boldsymbol{\xi}\,t^r$ results in the algebraic equations

$$\begin{pmatrix} 5-r & -1 \\ 3 & 1-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $det(\mathbf{A} - r\,\mathbf{I}) = r^2 - 6r + 8 = 0$. The roots of the characteristic equation are $r_1 = 4$ and $r_2 = 2$. With $r = 4$, the system of equations reduces to $\xi_1 - \xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1\,,1)^T$. For the case $r = 2$, the system is equivalent to the equation $3\,\xi_1 - \xi_2 = 0$. An eigenvector is $\boldsymbol{\xi}^{(2)} = (1\,,3)^T$. It follows that

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t^4 \text{ and } \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} t^2.$$

The Wronskian of this solution set is $W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}] = 2t^6$. Thus the solutions are linearly independent for $t > 0$. Hence the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} t^4 + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} t^2.$$

22. As shown in Prob. 19, solution of the ODE requires analysis of the equations

$$\begin{pmatrix} 4-r & -3 \\ 8 & -6-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $det(\mathbf{A} - r\,\mathbf{I}) = r^2 + 2r = 0$. The roots of the characteristic equation are $r_1 = 0$ and $r_2 = -2$. For $r = 0$, the system of equations reduces to $4\,\xi_1 = 3\,\xi_2$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (3\,,4)^T$. Setting $r = -2$ results in the single equation $2\,\xi_1 - \xi_2 = 0$. A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (1\,,2)^T$. It follows that

$$\mathbf{x}^{(1)} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \text{ and } \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} t^{-2}.$$

The Wronskian of this solution set is $W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}] = 2\,t^{-2}$. These solutions are linearly independent for $t > 0$. Hence the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 3 \\ 4 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} t^{-2}.$$

23. Setting $\mathbf{x} = \boldsymbol{\xi}\, t^r$ results in the algebraic equations

$$\begin{pmatrix} 3-r & -2 \\ 2 & -2-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $det(\mathbf{A} - r\,\mathbf{I}) = r^2 - r - 2 = 0$. The roots of the characteristic equation are $r_1 = 2$ and $r_2 = -1$. Setting $r = 2$, the system of equations reduces to $\xi_1 - 2\,\xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (2\,,1)^T$.

With $r = -1$, the system is equivalent to the equation $2\xi_1 - \xi_2 = 0$. An eigenvector is $\boldsymbol{\xi}^{(2)} = (1\,,2)^T$. It follows that

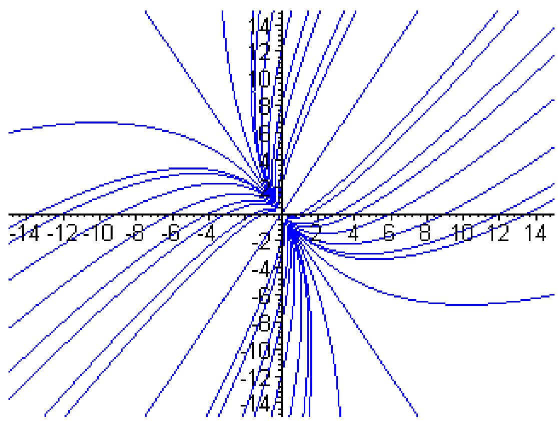
$$\mathbf{x}^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} t^2 \text{ and } \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} t^{-1}.$$

The Wronskian of this solution set is $W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}] = 3\,t$. Thus the solutions are linearly independent for $t > 0$. Hence the general solution is

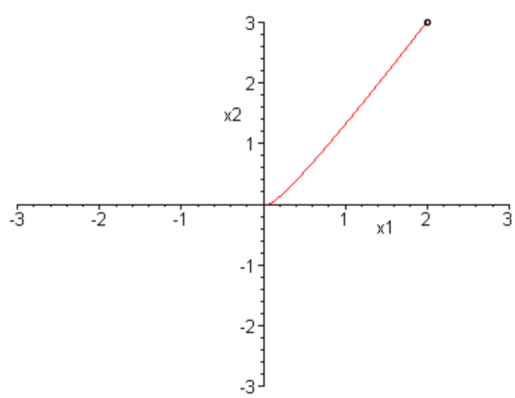
$$\mathbf{x} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} t^2 + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} t^{-1}.$$

24(a). The general solution is

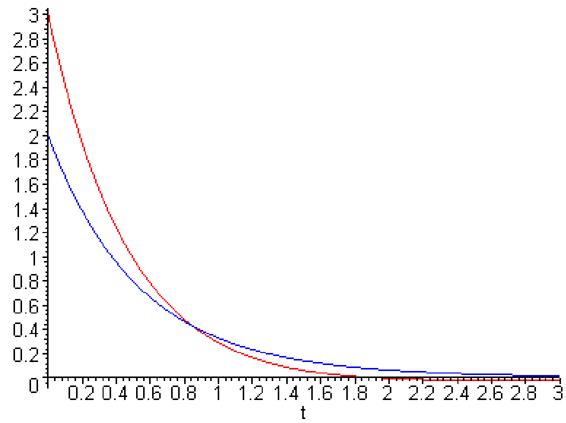
$$\mathbf{x} = c_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-2t}.$$



(b).



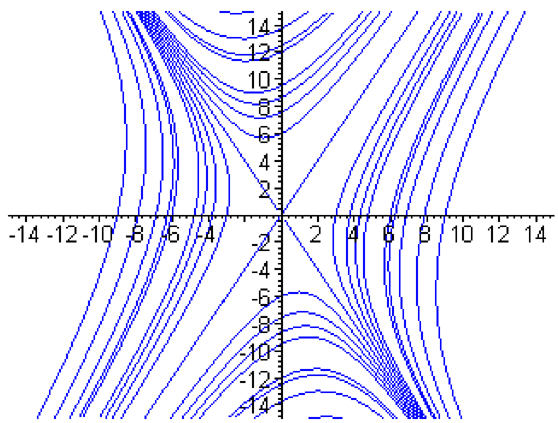
(c).



$26(a)$. The general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t}.$$

(b).



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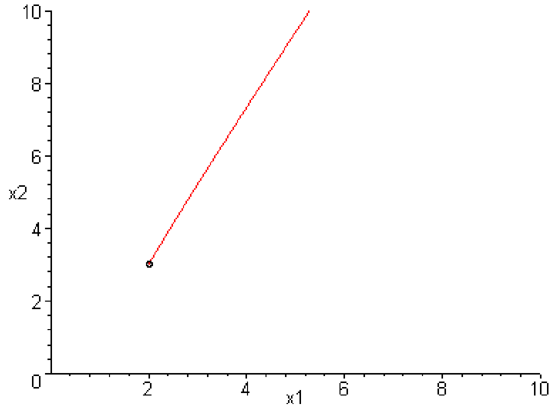

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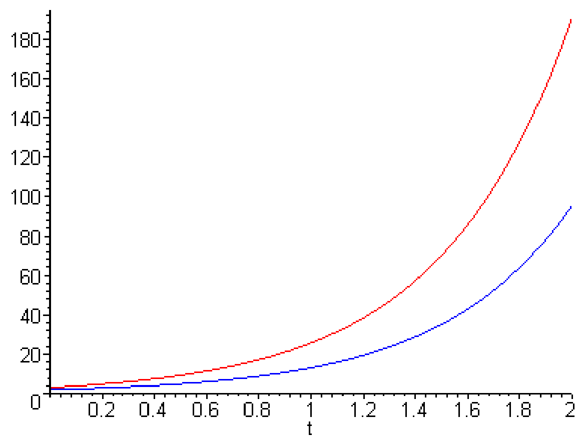


28(a). We note that $(\mathbf{A} - r_i\mathbf{I})\boldsymbol{\xi}^{(i)} = \mathbf{0}$, for $i = 1, 2$.

(b). It follows that $(\mathbf{A} - r_2\mathbf{I})\boldsymbol{\xi}^{(1)} = \mathbf{A}\,\boldsymbol{\xi}^{(1)} - r_2\boldsymbol{\xi}^{(1)} = r_1\boldsymbol{\xi}^{(1)} - r_2\boldsymbol{\xi}^{(1)}$.

(c). Suppose that $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ are linearly *dependent*. Then there exist constants c_1 and c_2, not both zero, such that $c_1\boldsymbol{\xi}^{(1)} + c_2\boldsymbol{\xi}^{(2)} = \mathbf{0}$. Assume that $c_1 \neq 0$. It is clear that $(\mathbf{A} - r_2\mathbf{I})\left(c_1\boldsymbol{\xi}^{(1)} + c_2\,\boldsymbol{\xi}^{(2)}\right) = \mathbf{0}$. On the other hand,

$$\begin{aligned}(\mathbf{A} - r_2\mathbf{I})\left(c_1\boldsymbol{\xi}^{(1)} + c_2\,\boldsymbol{\xi}^{(2)}\right) &= c_1(r_1 - r_2)\boldsymbol{\xi}^{(1)} + \mathbf{0} \\ &= c_1(r_1 - r_2)\boldsymbol{\xi}^{(1)}.\end{aligned}$$

Since $r_1 \neq r_2$, we must have $c_1 = 0$, which leads to a contradiction.

(d). Note that $(\mathbf{A} - r_1\mathbf{I})\boldsymbol{\xi}^{(2)} = (r_2 - r_1)\boldsymbol{\xi}^{(2)}$.

(e). Let $n = 3$, with $r_1 \neq r_2 \neq r_3$. Suppose that $\boldsymbol{\xi}^{(1)}$, $\boldsymbol{\xi}^{(2)}$and $\boldsymbol{\xi}^{(3)}$ are indeed linearly *dependent*. Then there exist constants c_1, c_2 and c_3, not all zero, such that

$$c_1\boldsymbol{\xi}^{(1)} + c_2\boldsymbol{\xi}^{(2)} + c_3\boldsymbol{\xi}^{(3)} = \mathbf{0}\,.$$

Assume that $c_1 \neq 0$. It is clear that $(\mathbf{A} - r_2\mathbf{I})\big(c_1\boldsymbol{\xi}^{(1)} + c_2\,\boldsymbol{\xi}^{(2)} + c_3\boldsymbol{\xi}^{(3)}\big) = \mathbf{0}$. On the other hand,

$$(\mathbf{A} - r_2\mathbf{I})\big(c_1\boldsymbol{\xi}^{(1)} + c_2\,\boldsymbol{\xi}^{(2)} + c_3\boldsymbol{\xi}^{(3)}\big) = c_1(r_1 - r_2)\boldsymbol{\xi}^{(1)} + c_3(r_3 - r_2)\boldsymbol{\xi}^{(3)}.$$

It follows that $c_1(r_1 - r_2)\boldsymbol{\xi}^{(1)} + c_3(r_3 - r_2)\boldsymbol{\xi}^{(3)} = \mathbf{0}$. Based on the result of Part (a), which is actually not dependent on the value of n, the vectors $\boldsymbol{\xi}^{(1)}$and $\boldsymbol{\xi}^{(3)}$ are linearly *independent*. Hence we must have $c_1(r_1 - r_2) = c_3(r_3 - r_2) = 0$, which leads to a contradiction.

29(a). Let $x_1 = y$ and $x_2 = y'$. It follows that $x_1' = x_2$ and

$$\begin{aligned} x_2' &= y'' \\ &= -\frac{1}{a}(c\,y + b\,y'). \end{aligned}$$

In terms of the new variables, we obtain the system of two first order ODEs

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -\frac{1}{a}(c\,x_1 + b\,x_2)\,. \end{aligned}$$

(b). The coefficient matrix is given by

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix}.$$

Setting $\mathbf{x} = \boldsymbol{\xi}\,e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} -r & 1 \\ -\frac{c}{a} & -\frac{b}{a} - r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have

$$det(\mathbf{A} - r\,\mathbf{I}) = r^2 + \frac{b}{a}r + \frac{c}{a} = 0\,.$$

Multiplying both sides of the equation by a, we obtain $a\,r^2 + b\,r + c = 0$.

30. Solution of the ODE requires analysis of the algebraic equations

$$\begin{pmatrix} 1 - r & 1 \\ 4 & -2 - r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $det(\mathbf{A} - r\,\mathbf{I}) = 0$. The characteristic equation is

$80\,r^2 + 24\,r + 1 = 0$, with roots $r_1 = -1/4$ and $r_2 = -1/20$. With $r = -1/4$, the system of equations reduces to $2\,\xi_1 + \xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1\,,\,-2)^T$. Substitution of $r = -1/20$ results in the equation $2\,\xi_1 - 3\,\xi_2 = 0$. A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (3\,,2)^T$. Since the eigenvalues are *distinct*, the general solution is

$$\mathbf{x} = c_1\begin{pmatrix}1\\-2\end{pmatrix}e^{-t/4} + c_2\begin{pmatrix}3\\2\end{pmatrix}e^{-t/20}.$$

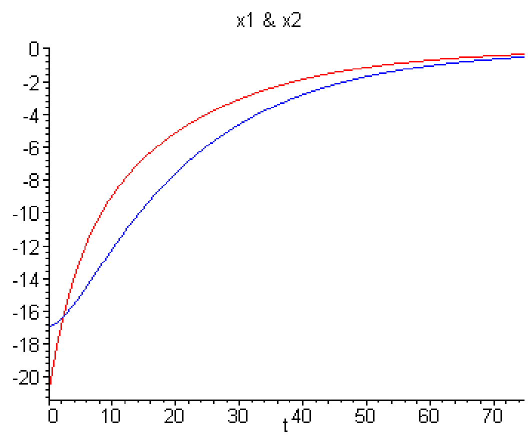
Invoking the initial conditions, we obtain the system of equations

$$\begin{aligned} c_1 + 3\,c_2 &= -17 \\ -2\,c_1 + 2\,c_2 &= -21\,. \end{aligned}$$

Hence $c_1 = 29/8$ and $c_2 = -55/8$, and the solution of the IVP is

$$\mathbf{x} = \frac{29}{8}\begin{pmatrix}1\\-2\end{pmatrix}e^{-t/4} - \frac{55}{8}\begin{pmatrix}3\\2\end{pmatrix}e^{-t/20}.$$

(b).



(c). Both functions are monotone increasing. It is easy to show that $-0.5 \le x_1(t) < 0$ and $-0.5 \le x_2(t) < 0$ provided that $t > T \approx 74.39$.

$31(a)$. For $\alpha = 1/2$, solution of the ODE requires that

$$\begin{pmatrix}-1-r & -1\\ -1/2 & -1-r\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}.$$

The characteristic equation is $2\,r^2 + 4\,r + 1 = 0$, with roots $r_1 = -1 + 1/\sqrt{2}$ and $r_2 = -1 - 1/\sqrt{2}$. With $r = -1 + 1/\sqrt{2}$, the system of equations reduces to $\sqrt{2}\,\xi_1 + 2\,\xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = \left(-\sqrt{2}\,,1\right)^T$. Substitution of $r = -1 - 1/\sqrt{2}$ results in the equation $\sqrt{2}\,\xi_1 - 2\,\xi_2 = 0$. An eigenvector is $\boldsymbol{\xi}^{(2)} = \left(\sqrt{2}\,,1\right)^T$. The general solution is

$$\mathbf{x} = c_1\begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{\left(-2+\sqrt{2}\right)t/2} + c_2\begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} e^{\left(-2-\sqrt{2}\right)t/2}.$$

The eigenvalues are distinct and both *negative*. The equilibrium point is a stable *node*.

(b). For $\alpha = 2$, the characteristic equation is given by $r^2 + 2\,r - 1 = 0$, with roots $r_1 = -1 + \sqrt{2}$ and $r_2 = -1 - \sqrt{2}$. With $r = -1 + \sqrt{2}$, the system of equations reduces to $\sqrt{2}\,\xi_1 + \xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = \left(1\,, -\sqrt{2}\right)^T$. Substitution of $r = -1 - \sqrt{2}$ results in the equation $\sqrt{2}\,\xi_1 - \xi_2 = 0$. An eigenvector is $\boldsymbol{\xi}^{(2)} = \left(1\,,\sqrt{2}\right)^T$. The general solution is

$$\mathbf{x} = c_1\begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} e^{\left(-1+\sqrt{2}\right)t} + c_2\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{\left(-1-\sqrt{2}\right)t}.$$

The eigenvalues are of opposite sign, hence the equilibrium point is a *saddle point*.

32. The system of differential equations is

$$\frac{d}{dt}\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & -\frac{5}{2} \end{pmatrix}\begin{pmatrix} I \\ V \end{pmatrix}.$$

Solution of the system requires analysis of the eigenvalue problem

$$\begin{pmatrix} -\frac{1}{2} - r & -\frac{1}{2} \\ \frac{3}{2} & -\frac{5}{2} - r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 + 3\,r + 2$, with roots $r_1 = -1$ and $r_2 = -2$. With $r = -1$, the equations reduce to $\xi_1 - \xi_2 = 0$. A corresponding eigenvector is given by $\boldsymbol{\xi}^{(1)} = (1\,,1)^T$. Setting $r = -2$, the system reduces to the equation $3\,\xi_1 - \xi_2 = 0$. An eigenvector is $\boldsymbol{\xi}^{(2)} = (1\,,3)^T$. Hence the general solution is

$$\begin{pmatrix} I \\ V \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2\begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-2t}.$$

(b). The eigenvalues are distinct and both *negative*. We find that the equilibrium point $(0\,,0)$ is a stable *node*. Hence all solutions converge to $(0\,,0)$.

$33(a)$. Solution of the ODE requires analysis of the algebraic equations

$$\begin{pmatrix} -\frac{R_1}{L} - r & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{CR_2} - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is

$$r^2 + \left(\frac{L + CR_1R_2}{LCR_2}\right) r + \frac{R_1 + R_2}{LCR_2} = 0\,.$$

The eigenvectors are *real* and *distinct*, provided that the *discriminant* is positive. That is,

$$\left(\frac{L + CR_1R_2}{LCR_2}\right)^2 - 4\left(\frac{R_1 + R_2}{LCR_2}\right) > 0\,,$$

which simplifies to the condition

$$\left(\frac{1}{CR_2} - \frac{R_1}{L}\right)^2 - \frac{4}{LC} > 0\,.$$

(b). The parameters in the ODE are all positive. Observe that the *sum* of the roots is

$$-\frac{L + CR_1R_2}{LCR_2} < 0\,.$$

Also, the *product* of the roots is

$$\frac{R_1 + R_2}{LCR_2} > 0\,.$$

It follows that *both* roots are negative. Hence the *equilibrium solution* $I = 0\,,\ V = 0$ represents a stable node, which attracts *all* solutions.

(c). If the condition in Part (a) is not satisfied, that is,

$$\left(\frac{1}{CR_2} - \frac{R_1}{L}\right)^2 - \frac{4}{LC} \leq 0\,,$$

then the *real part* of the eigenvalues is

$$Re(r_{1,2}) = -\frac{L + CR_1R_2}{2\,LCR_2}\,.$$

As long as the parameters are *all* positive, then the solutions will still converge to the equilibrium point $(0\,,0)$.

Section 7.6

2. Setting $\mathbf{x} = \boldsymbol{\xi}\, e^{rt}$ results in the algebraic equations

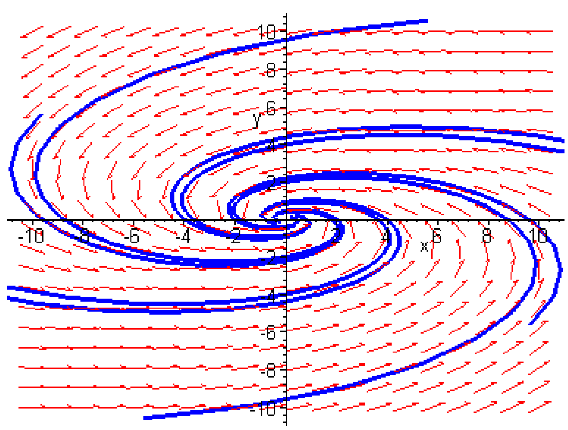
$$\begin{pmatrix} -1-r & -4 \\ 1 & -1-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we require that $det(\mathbf{A} - r\,\mathbf{I}) = r^2 + 2r + 5 = 0$. The roots of the characteristic equation are $r = -1 \pm 2i$. Substituting $r = -1 - 2i$, the two equations reduce to $\xi_1 + 2i\,\xi_2 = 0$. The two eigenvectors are $\boldsymbol{\xi}^{(1)} = (-2i\,,1)^T$ and $\boldsymbol{\xi}^{(2)} = (2i\,,1)^T$. Hence one of the *complex-valued* solutions is given by

$$\begin{aligned} \mathbf{x}^{(1)} &= \begin{pmatrix} -2i \\ 1 \end{pmatrix} e^{-(1+2i)t} \\ &= \begin{pmatrix} -2i \\ 1 \end{pmatrix} e^{-t}(cos\,2t - i\,sin\,2t) \\ &= e^{-t}\begin{pmatrix} -2\,sin\,2t \\ cos\,2t \end{pmatrix} + i\,e^{-t}\begin{pmatrix} -2\,cos\,2t \\ -\,sin\,2t \end{pmatrix}. \end{aligned}$$

Based on the real and imaginary parts of this solution, the general solution is

$$\mathbf{x} = c_1 e^{-t}\begin{pmatrix} -2\,sin\,2t \\ cos\,2t \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 2\,cos\,2t \\ sin\,2t \end{pmatrix}.$$



3. Solution of the ODEs is based on the analysis of the algebraic equations

$$\begin{pmatrix} 2-r & -5 \\ 1 & -2-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we require that $det(\mathbf{A} - r\,\mathbf{I}) = r^2 + 1 = 0$. The roots of the characteristic equation are $r = \pm i$. Setting $r = i$, the equations are equivalent to $\xi_1 - (2+i)\xi_2 = 0$. The eigenvectors are $\boldsymbol{\xi}^{(1)} = (2+i\,,1)^T$ and $\boldsymbol{\xi}^{(2)} = (2-i\,,1)^T$. Hence one of the *complex-valued* solutions is given by

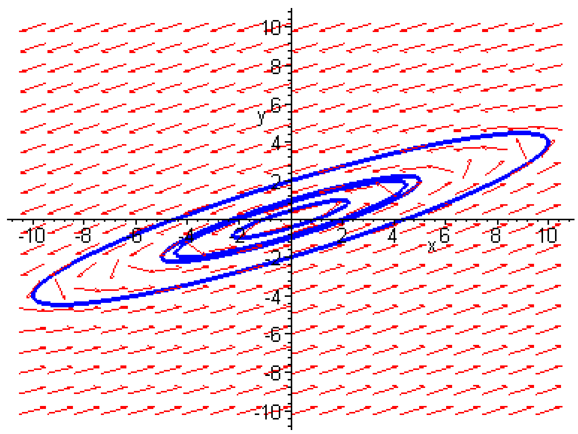
$$\begin{aligned}\mathbf{x}^{(1)} &= \begin{pmatrix} 2+i \\ 1 \end{pmatrix} e^{it} \\ &= \begin{pmatrix} 2+i \\ 1 \end{pmatrix} (cos\,t + i\,sin\,t) \\ &= \begin{pmatrix} 2\,cos\,t - sin\,t \\ cos\,t \end{pmatrix} + i \begin{pmatrix} cos\,t + 2\,sin\,t \\ sin\,t \end{pmatrix}.\end{aligned}$$

Therefore the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 2\,cos\,t - sin\,t \\ cos\,t \end{pmatrix} + c_2 \begin{pmatrix} cos\,t + 2\,sin\,t \\ sin\,t \end{pmatrix}.$$

The solution may also be written as

$$\mathbf{x} = c_1 \begin{pmatrix} 5\,cos\,t \\ 2\,cos\,t + sin\,t \end{pmatrix} + c_2 \begin{pmatrix} 5\,sin\,t \\ -\,cos\,t + 2\,sin\,t \end{pmatrix}.$$



4. Setting $\mathbf{x} = \boldsymbol{\xi}\, e^{rt}$ results in the algebraic equations

$$\begin{pmatrix} 2-r & -\,5/2 \\ 9/5 & -\,1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we require that $det(\mathbf{A} - r\,\mathbf{I}) = r^2 - r + \frac{5}{2} = 0$. The roots of the characteristic equation are $r = (1 \pm 3i)/2$. With $r = (1 + 3\,i)/2$, the equations reduce to the single equation $(3 - 3i)\xi_1 - 5\,\xi_2 = 0$. The corresponding eigenvector is given by $\boldsymbol{\xi}^{(1)} = (5\,, 3 - 3\,i)^T$. Hence one of the *complex-valued* solutions is

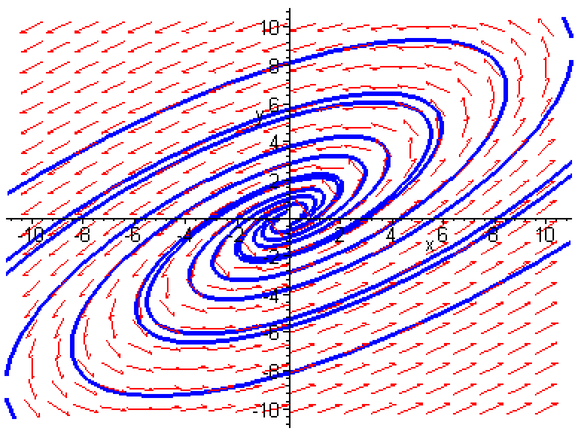
$$\begin{aligned}\mathbf{x}^{(1)} &= \begin{pmatrix} 5 \\ 3-3i \end{pmatrix} e^{(1+3i)t/2} \\ &= \begin{pmatrix} 2+i \\ 1 \end{pmatrix} e^{t/2}\left(\cos\frac{3}{2}t + i\sin\frac{3}{2}t\right) \\ &= e^{t/2}\begin{pmatrix} 2\cos\frac{3}{2}t - \sin\frac{3}{2}t \\ \cos\frac{3}{2}t \end{pmatrix} + ie^{t/2}\begin{pmatrix} \cos\frac{3}{2}t + 2\sin\frac{3}{2}t \\ \sin\frac{3}{2}t \end{pmatrix}.\end{aligned}$$

The general solution is

$$\mathbf{x} = c_1 e^{t/2}\begin{pmatrix} 2\cos\frac{3}{2}t - \sin\frac{3}{2}t \\ \cos\frac{3}{2}t \end{pmatrix} + c_2 e^{t/2}\begin{pmatrix} \cos\frac{3}{2}t + 2\sin\frac{3}{2}t \\ \sin\frac{3}{2}t \end{pmatrix}.$$

The solution may also be written as

$$\mathbf{x} = c_1 e^{t/2}\begin{pmatrix} 5\cos\frac{3}{2}t \\ 3\cos\frac{3}{2}t + 3\sin\frac{3}{2}t \end{pmatrix} + c_2 e^{t/2}\begin{pmatrix} 5\sin\frac{3}{2}t \\ -3\cos\frac{3}{2}t + 3\sin\frac{3}{2}t \end{pmatrix}.$$



5. Setting $\mathbf{x} = \boldsymbol{\xi}\, t^r$ results in the algebraic equations

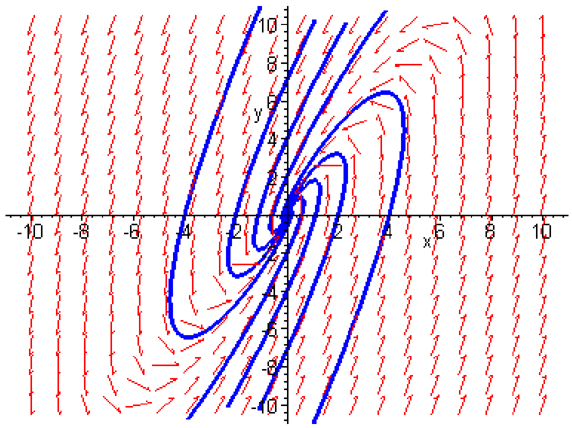
$$\begin{pmatrix} 1-r & -1 \\ 5 & -3-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 + 2r + 2 = 0$, with roots $r = -1 \pm i$. Substituting $r = -1 - i$ reduces the system of equations to $(2+i)\xi_1 - \xi_2 = 0$. The eigenvectors are $\boldsymbol{\xi}^{(1)} = (1, 2+i)^T$ and $\boldsymbol{\xi}^{(2)} = (1, 2-i)^T$. Hence one of the *complex-valued* solutions is given by

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 2+i \end{pmatrix} e^{-(1+i)t}$$
$$= \begin{pmatrix} 1 \\ 2+i \end{pmatrix} e^{-t}(cos\,t - i\,sin\,t)$$
$$= e^{-t}\begin{pmatrix} cos\,t \\ 2\,cos\,t + sin\,t \end{pmatrix} + ie^{-t}\begin{pmatrix} -\,sin\,t \\ cos\,t - 2sin\,t \end{pmatrix}.$$

The general solution is

$$\mathbf{x} = c_1 e^{-t}\begin{pmatrix} cos\,t \\ 2\,cos\,t + sin\,t \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} sin\,t \\ -\,cos\,t + 2sin\,t \end{pmatrix}.$$



6. Solution of the ODEs is based on the analysis of the algebraic equations

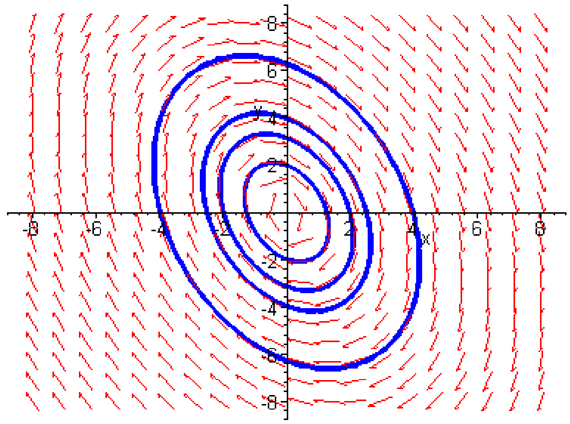
$$\begin{pmatrix} 1-r & 2 \\ -\,5 & -\,1-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we require that $det(\mathbf{A} - r\,\mathbf{I}) = r^2 + 9 = 0$. The roots of the characteristic equation are $r = \pm 3\,i$. Setting $r = 3\,i$, the two equations reduce to $(1-3\,i)\xi_1 + 2\xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (-\,2\,, 1-3i)^T$. Hence one of the *complex-valued* solutions is given by

$$\mathbf{x}^{(1)} = \begin{pmatrix} -\,2 \\ 1-3i \end{pmatrix} e^{3it}$$
$$= \begin{pmatrix} -\,2 \\ 1-3i \end{pmatrix}(cos\,3t + i\,sin\,3t)$$
$$= \begin{pmatrix} -\,2\,cos\,3t \\ cos\,3t + 3\,sin\,3t \end{pmatrix} + i\begin{pmatrix} -\,2\,sin\,3t \\ -\,3cos\,3t + sin\,3t \end{pmatrix}.$$

The general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} -2\,cos\,3t \\ cos\,3t + 3\,sin\,3t \end{pmatrix} + c_2 \begin{pmatrix} 2\,sin\,3t \\ 3cos\,3t - sin\,3t \end{pmatrix}.$$



8. The eigensystem is obtained from analysis of the equation

$$\begin{pmatrix} -3-r & 0 & 2 \\ 1 & -1-r & 0 \\ -2 & -1 & -r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The characteristic equation of the coefficient matrix is $r^3 + 4r^2 + 7r + 6 = 0$, with roots $r_1 = -2$, $r_2 = -1 - \sqrt{2}\,i$ and $r_3 = -1 + \sqrt{2}\,i$. Setting $r = -2$, the equations reduce to

$$\begin{aligned} -\xi_1 + 2\,\xi_3 &= 0 \\ \xi_1 + \xi_2 &= 0\,. \end{aligned}$$

The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (2\,,\,-2\,,1)^T$. With $r = -1 - \sqrt{2}\,i$, the system of equations is equivalent to

$$\begin{aligned} \left(2 - i\sqrt{2}\right)\xi_1 - 2\,\xi_3 &= 0 \\ \xi_1 + i\sqrt{2}\,\xi_2 &= 0\,. \end{aligned}$$

An eigenvector is given by $\boldsymbol{\xi}^{(2)} = \left(-i\sqrt{2}\,,1\,,\,-1-i\sqrt{2}\right)^T$. Hence one of the *complex-valued* solutions is given by

$$\begin{aligned}
\mathbf{x}^{(2)} &= \begin{pmatrix} -i\sqrt{2} \\ 1 \\ -1-i\sqrt{2} \end{pmatrix} e^{-\left(1+i\sqrt{2}\right)it} \\
&= \begin{pmatrix} -i\sqrt{2} \\ 1 \\ -1-i\sqrt{2} \end{pmatrix} e^{-t}\left(\cos\sqrt{2}\,t - i\sin\sqrt{2}\,t\right) \\
&= e^{-t}\begin{pmatrix} -\sqrt{2}\sin\sqrt{2}\,t \\ \cos\sqrt{2}\,t \\ -\cos\sqrt{2}\,t - \sqrt{2}\sin\sqrt{2}\,t \end{pmatrix} + ie^{-t}\begin{pmatrix} -\sqrt{2}\cos\sqrt{2}\,t \\ -\sin\sqrt{2}\,t \\ -\sqrt{2}\cos\sqrt{2}\,t - \sin\sqrt{2}\,t \end{pmatrix}.
\end{aligned}$$

The other complex-valued solution is $\mathbf{x}^{(3)} = \overline{\boldsymbol{\xi}^{(2)}}\, e^{r_3 t}$. The general solution is

$$\begin{aligned}
\mathbf{x} &= c_1\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} e^{-2t} + \\
&+ c_2\, e^{-t}\begin{pmatrix} \sqrt{2}\sin\sqrt{2}\,t \\ -\cos\sqrt{2}\,t \\ \cos\sqrt{2}\,t + \sqrt{2}\sin\sqrt{2}\,t \end{pmatrix} + c_3 e^{-t}\begin{pmatrix} \sqrt{2}\cos\sqrt{2}\,t \\ \sin\sqrt{2}\,t \\ \sqrt{2}\cos\sqrt{2}\,t + \sin\sqrt{2}\,t \end{pmatrix}.
\end{aligned}$$

It is easy to see that all solutions converge to the equilibrium point $(0\,,0\,,0)$.

10. Solution of the system of ODEs requires that

$$\begin{pmatrix} -3-r & 2 \\ -1 & -1-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 + 4\,r + 5 = 0$, with roots $r = -2 \pm i$. Substituting $r = -2 + i$, the equations are equivalent to $\xi_1 - (1-i)\xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1-i\,,1)^T$. One of the *complex-valued* solutions is given by

$$\begin{aligned}
\mathbf{x}^{(1)} &= \begin{pmatrix} 1-i \\ 1 \end{pmatrix} e^{(-2+i)t} \\
&= \begin{pmatrix} 1-i \\ 1 \end{pmatrix} e^{-2t}(\cos t + i\sin t) \\
&= e^{-2t}\begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix} + ie^{-2t}\begin{pmatrix} -\cos t + \sin t \\ \sin t \end{pmatrix}.
\end{aligned}$$

Hence the general solution is

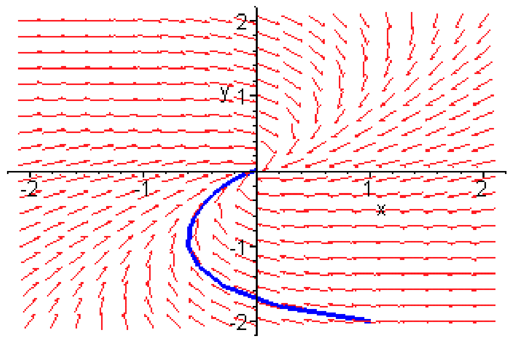
$$\mathbf{x} = c_1\, e^{-2t}\begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix} + c_2\, e^{-2t}\begin{pmatrix} -\cos t + \sin t \\ \sin t \end{pmatrix}.$$

Invoking the initial conditions, we obtain the system of equations

$$\begin{aligned}
c_1 - c_2 &= 1 \\
c_1 &= -2\,.
\end{aligned}$$

Solving for the coefficients, the solution of the initial value problem is

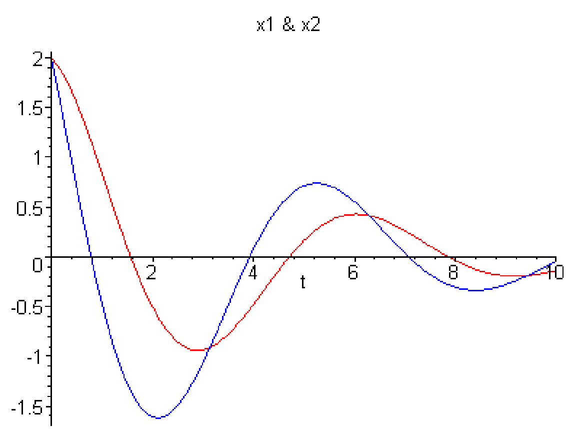
$$\mathbf{x} = -2e^{-2t}\begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix} - 3e^{-2t}\begin{pmatrix} -\cos t + \sin t \\ \sin t \end{pmatrix}$$
$$= e^{-2t}\begin{pmatrix} \cos t - 5\sin t \\ -2\cos t - 3\sin t \end{pmatrix}.$$



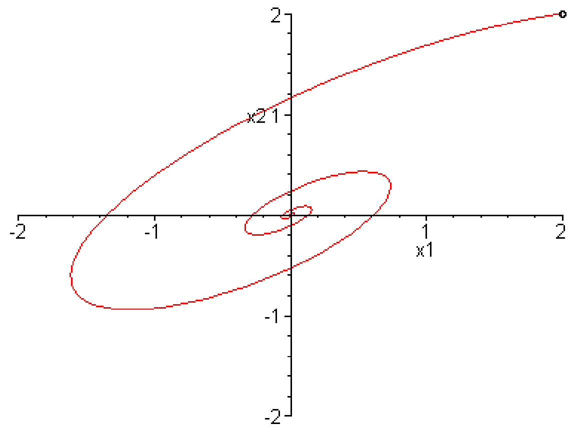
$11(a)$. With $\mathbf{x}(0) = (2\ ,2)^T$, the solution is

$$\mathbf{x} = e^{-t/4}\begin{pmatrix} 2\cos t - 2\sin t \\ 2\cos t \end{pmatrix}.$$

$11(b)$.



$11(c)$.



12. Solution of the ODEs is based on the analysis of the algebraic equations

$$\begin{pmatrix} -\frac{4}{5}-r & 2 \\ -1 & \frac{6}{5}-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $25\,r^2 - 10\,r + 26 = 0$, with roots $r = \frac{1}{5} \pm i$. Setting $r = 1/5 + i$, the two equations reduce to $\xi_1 - (1-i)\xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (1-i\,,1)^T$. One of the *complex-valued* solutions is given by

$$\begin{aligned} \mathbf{x}^{(1)} &= \begin{pmatrix} 1-i \\ 1 \end{pmatrix} e^{(\frac{1}{5}+i)t} \\ &= \begin{pmatrix} 1-i \\ 1 \end{pmatrix} e^{t/5}(\cos t + i \sin t) \\ &= e^{t/5}\begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix} + ie^{t/5}\begin{pmatrix} -\cos t + \sin t \\ \sin t \end{pmatrix}. \end{aligned}$$

Hence the general solution is

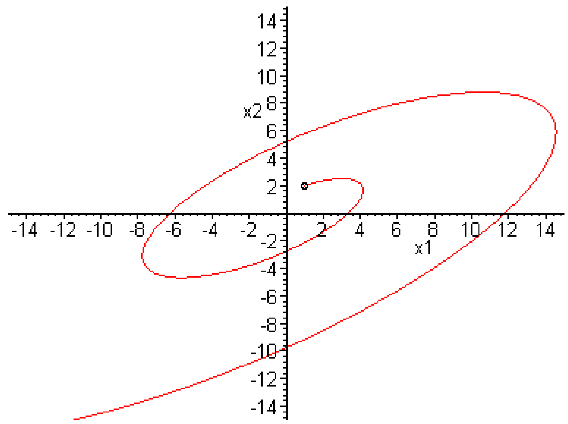
$$\mathbf{x} = c_1 e^{t/5}\begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix} + c_2 e^{t/5}\begin{pmatrix} -\cos t + \sin t \\ \sin t \end{pmatrix}.$$

(b). Let $\mathbf{x}(0) = (x_1^0\,, x_2^0)^T$. The solution of the initial value problem is

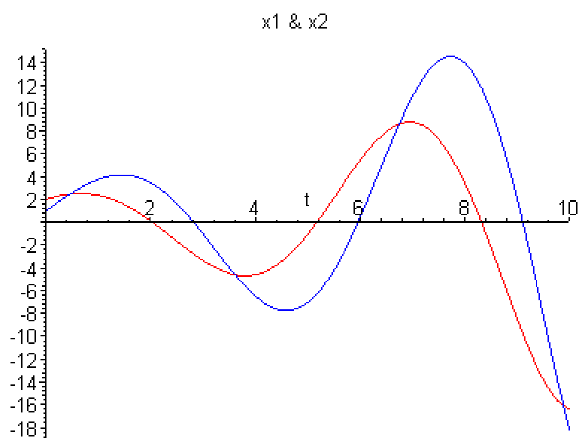
$$\begin{aligned} \mathbf{x} &= x_2^0\, e^{t/5}\begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix} + (x_2^0 - x_1^0)e^{t/5}\begin{pmatrix} -\cos t + \sin t \\ \sin t \end{pmatrix}. \\ &= e^{t/5}\begin{pmatrix} x_1^0 \cos t + (2\,x_2^0 - x_1^0)\sin t \\ x_2^0 \cos t + (x_2^0 - x_1^0)\sin t \end{pmatrix}. \end{aligned}$$

With $\mathbf{x}(0) = (1\;,2)^T$, the solution is

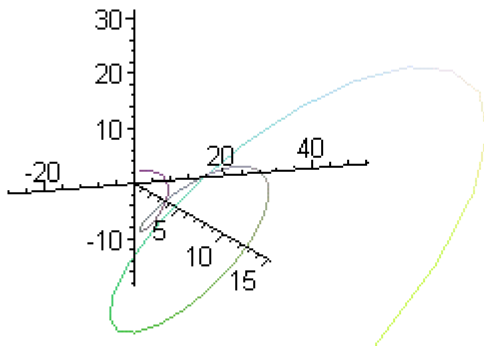
$$\mathbf{x} = e^{t/5}\begin{pmatrix} cos\, t + 3\, sin\, t \\ 2\, cos\, t + sin\, t \end{pmatrix}.$$



(c).



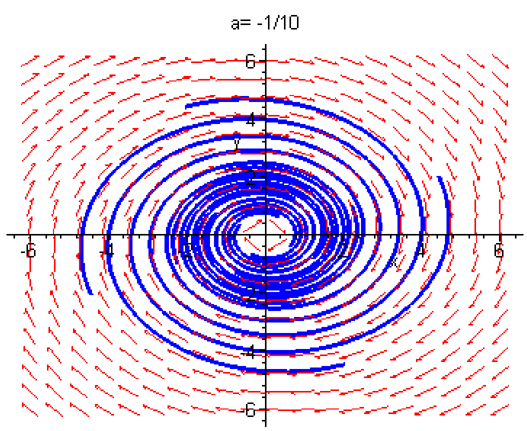
(d).

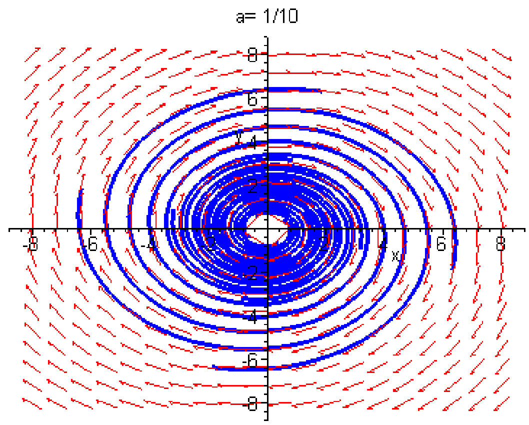


$13(a)$. The characteristic equation of the coefficient matrix is $r^2 - 2\alpha r + 1 + \alpha^2$, with roots $r = \alpha \pm i$.

(b). When $\alpha < 0$ and $\alpha > 0$, the equilibrium point $(0\,,0)$ is a *stable* spiral and an *unstable* spiral, respectively. The equilibrium point is a *center* when $\alpha = 0$.

(c).



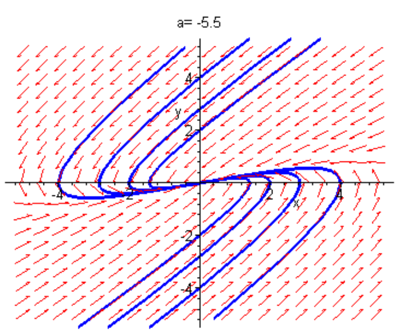


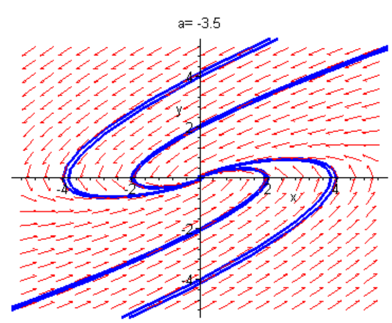
$14(a)$. The roots of the characteristic equation, $r^2 - \alpha\, r + 5 = 0$, are

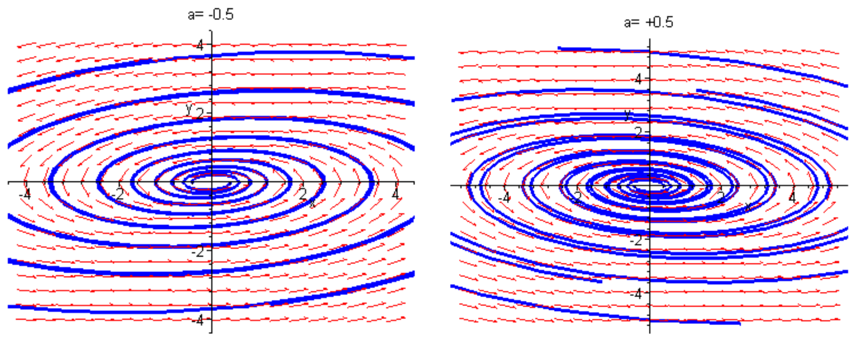
$$r_{1,2} = \frac{\alpha}{2} \pm \frac{1}{2}\sqrt{\alpha^2 - 20}\,.$$

(b). Note that the roots are *complex* when $-\sqrt{20} < \alpha < \sqrt{20}$. For the case when $\alpha \in \left(-\sqrt{20}\,,0\right)$, the equilibrium point $(0\,,0)$ is a *stable* spiral. On the other hand, when $\alpha \in \left(0\,,\sqrt{20}\right)$, the equilibrium point is an *unstable* spiral. For the case $\alpha = 0$, the roots are purely imaginary, so the equilibrium point is a *center*. When $\alpha^2 > 20$, the roots are *real* and *distinct*. The equilibrium point becomes a *node*, with its stability dependent on the sign of α. Finally, the case $\alpha^2 = 20$ marks the transition from spirals to nodes.

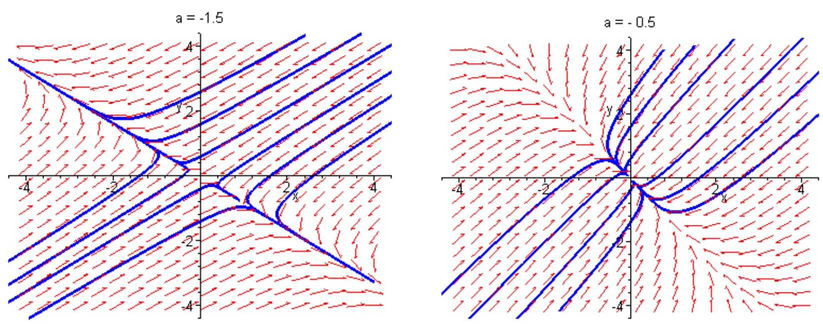
(c).

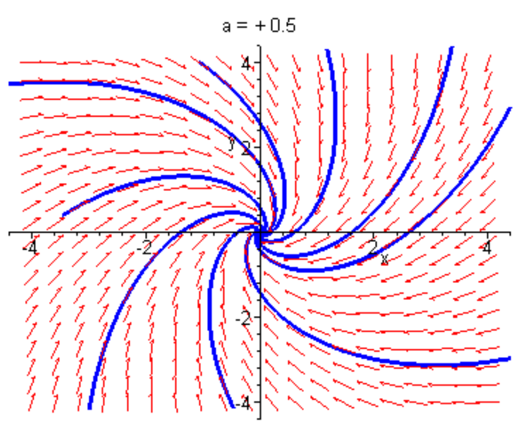






17. The characteristic equation of the coefficient matrix is $r^2 + 2r + 1 + \alpha = 0$, with roots given formally as $r_{1,2} = -1 \pm \sqrt{-\alpha}$. The roots are *real* provided that $\alpha \leq 0$. First note that the *sum* of the roots is -2 and the *product* of the roots is $1 + \alpha$. For *negative* values of α, the roots are distinct, with one always negative. When $\alpha < -1$, the roots have *opposite* signs. Hence the equilibrium point is a *saddle*. For the case $-1 < \alpha < 0$, the roots are both *negative*, and the equilibrium point is a *stable node*. $\alpha = -1$ represents a transition from saddle to node. When $\alpha = 0$, both roots are equal. For the case $\alpha > 0$, the roots are complex conjugates, with negative real part. Hence the equilibrium point is a *stable spiral.*





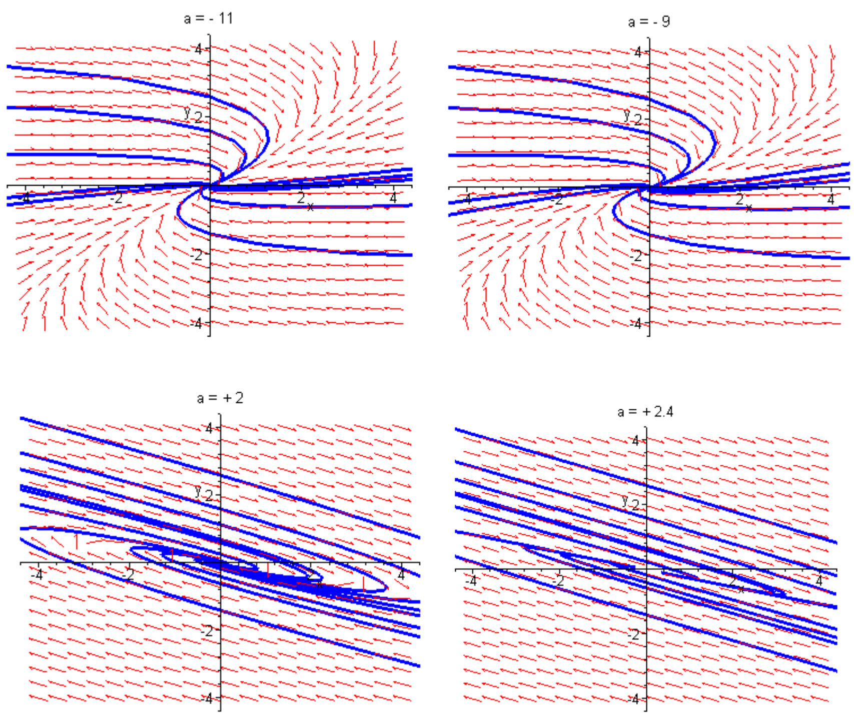
19. The characteristic equation for the system is given by

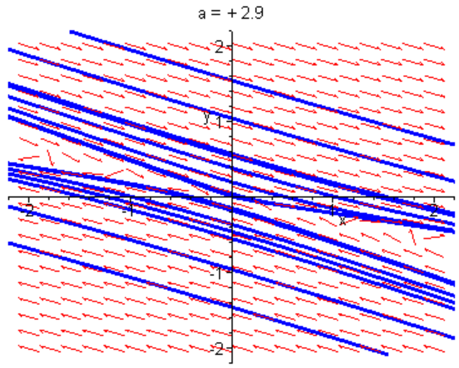
$$r^2 + (4-\alpha)r + 10 - 4\alpha = 0.$$

The roots are

$$r_{1,2} = -2 + \frac{\alpha}{2} \pm \sqrt{\alpha^2 + 8\alpha - 24}\,.$$

First note that the roots are *complex* when $-4-2\sqrt{10} < \alpha < -4+2\sqrt{10}$. We also find that when $-4-2\sqrt{10} < \alpha < 2$, the equilibrium point is a stable *spiral*. For the case $\alpha = 2$, the equilibrium point is a *center*. When $2 < \alpha < -4+2\sqrt{10}$, the equilibrium point is an unstable *spiral*. For all other cases, the roots are *real*. When $\alpha > 2.5$, the roots have *opposite* signs, with the equilibrium point being a *saddle*. For the case $-4+2\sqrt{10} < \alpha < 2.5$, the roots are both *positive*, and the equilibrium point is an unstable *node*. Finally, when $\alpha < -4-2\sqrt{10}$, both roots are negative, with the equilibrium point being a stable *node*.

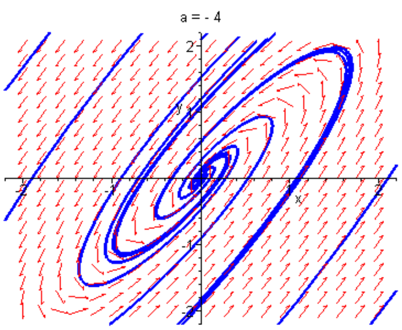


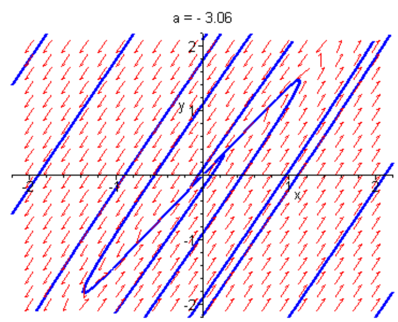


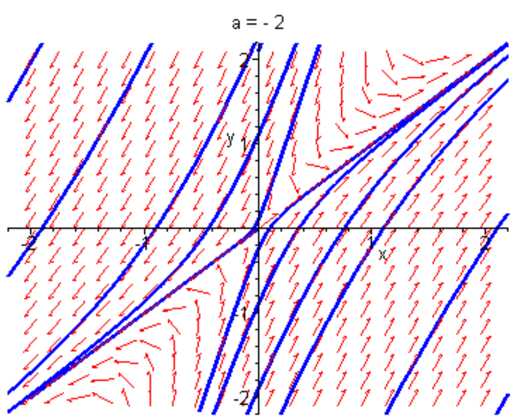
20. The characteristic equation is $r^2 + 2r - (24 + 8\alpha) = 0$, with roots

$$r_{1,2} = -1 \pm \sqrt{25 + 8\alpha}\ .$$

The roots are *complex* when $\alpha < -25/8$. Since the real part is negative, the origin is a stable *spiral*. Otherwise the roots are real. When $-25 < \alpha < -3$, both roots are negative, and hence the equilibrium point is a stable *node*. For $\alpha > -3$, the roots are of opposite sign and the origin is a *saddle*.







22. Based on the method in Prob. 19 of Section 7.5, setting $\mathbf{x} = \boldsymbol{\xi}\, t^r$ results in the

algebraic equations

$$\begin{pmatrix} 2-r & -5 \\ 1 & -2-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation for the system is $r^2+1=0$, with roots $r_{1,2}=\pm i$. With $r=i$, the equations reduce to the single equation $\xi_1-(2+i)\xi_2=0$. A corresponding eigenvector is $\boldsymbol{\xi}^{(1)}=(2+i\,,1)^T$. One *complex-valued* solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} t^i\,.$$

We can write $t^i = e^{i\,ln\,t}$. Hence

$$\begin{aligned} \mathbf{x}^{(1)} &= \begin{pmatrix} 2+i \\ 1 \end{pmatrix} e^{i\,ln\,t} \\ &= \begin{pmatrix} 2+i \\ 1 \end{pmatrix} [cos(ln\,t) + i\,sin(ln\,t)] \\ &= \begin{pmatrix} 2\,cos(ln\,t) - sin(ln\,t) \\ cos(ln\,t) \end{pmatrix} + i\begin{pmatrix} cos(ln\,t) + 2\,sin(ln\,t) \\ sin(ln\,t) \end{pmatrix}. \end{aligned}$$

Therefore the general solution is

$$\mathbf{x} = c_1\begin{pmatrix} 2\,cos(ln\,t) - sin(ln\,t) \\ cos(ln\,t) \end{pmatrix} + c_2\begin{pmatrix} cos(ln\,t) + 2\,sin(ln\,t) \\ sin(ln\,t) \end{pmatrix}.$$

Other combinations are also possible.

$24(a)$. The characteristic equation of the system is

$$r^3 + \frac{2}{5}r^2 + \frac{81}{80}r - \frac{17}{160} = 0\,,$$

with eigenvalues $r_1 = 1/10$, and $r_{2,3} = -1/4 \pm i$. For $r = 1/10$, simple calculations reveal that a corresponding eigenvector is $\boldsymbol{\xi}^{(1)} = (0\,,0\,,1)^T$. Setting $r = -1/4 - i$, we obtain the system of equations

$$\begin{aligned} \xi_1 - i\,\xi_2 &= 0 \\ \xi_3 &= 0\,. \end{aligned}$$

A corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (i\,,1\,,0)^T$. Hence one solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{t/10}.$$

Another solution, which is *complex-valued*, is given by

$$\mathbf{x}^{(2)} = \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} e^{-(\frac{1}{4}+i)t}$$
$$= \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} e^{-t/4}(\cos t - i\,\sin t)$$
$$= e^{-t/4}\begin{pmatrix} \sin t \\ \cos t \\ 0 \end{pmatrix} + ie^{-t/4}\begin{pmatrix} \cos t \\ -\sin t \\ 0 \end{pmatrix}.$$

Using the real and imaginary parts of $\mathbf{x}^{(2)}$, the general solution is constructed as

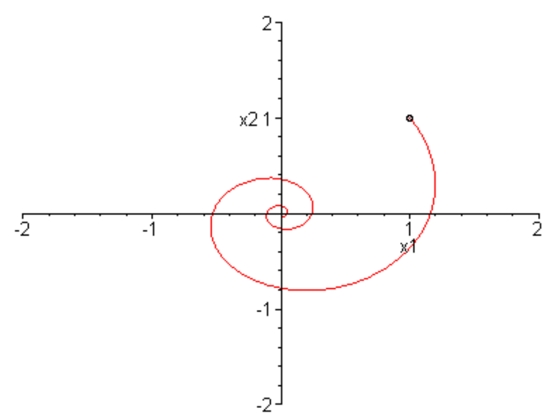
$$\mathbf{x} = c_1\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{t/10} + c_2\, e^{-t/4}\begin{pmatrix} \sin t \\ \cos t \\ 0 \end{pmatrix} + c_3\, e^{-t/4}\begin{pmatrix} \cos t \\ -\sin t \\ 0 \end{pmatrix}.$$

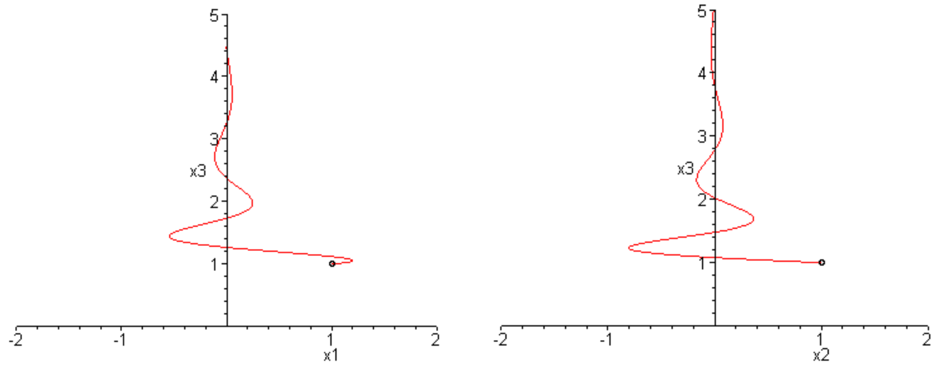
(b). Let $\mathbf{x}(0) = (x_1^0\,, x_2^0\,, x_3^0)$. The solution can be written as

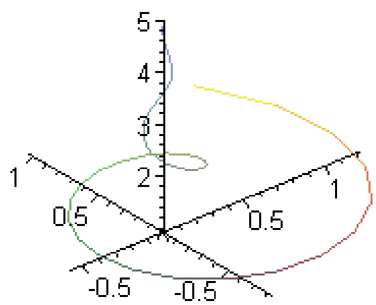
$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ x_3^0\, e^{t/10} \end{pmatrix} + e^{-t/4}\begin{pmatrix} x_2^0 \sin t + x_1^0 \cos t \\ x_2^0 \cos t - x_1^0 \sin t \\ 0 \end{pmatrix}.$$

With $\mathbf{x}(0) = (1\,, 1\,, 1)$, the solution of the initial value problem is

$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ e^{t/10} \end{pmatrix} + e^{-t/4}\begin{pmatrix} \sin t + \cos t \\ \cos t - \sin t \\ 0 \end{pmatrix}.$$







$25(a)$. Based on Probs. $18-20$ of Section 7.1, the system of differential equations is

$$\frac{d}{dt}\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} -\frac{R_1}{L} & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{CR_2} \end{pmatrix}\begin{pmatrix} I \\ V \end{pmatrix}.$$

With $R_1 = R_2 = 4\,ohms$, $C = \frac{1}{2}\,farads$ and $L = 8\,henrys$, the eigenvalue problem is

$$\begin{pmatrix} -\frac{1}{2}-r & -\frac{1}{8} \\ 2 & -\frac{1}{2}-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

(b). The characteristic equation of the system is $r^2 + r + \frac{1}{2} = 0$, with eigenvalues

$$r_{1,2} = -\frac{1}{2} \pm \frac{1}{2}i\,.$$

Setting $r = -1/2 + i/2$, the algebraic equations reduce to $4i\xi_1 + \xi_2 = 0$. It follows that $\boldsymbol{\xi}^{(1)} = (1\,,\,-4i)^T$. Hence one *complex-valued* solution is

$$\begin{pmatrix} I \\ V \end{pmatrix}^{(1)} = \begin{pmatrix} 1 \\ -4i \end{pmatrix} e^{(-1+i)t/2}$$
$$= \begin{pmatrix} 1 \\ -4i \end{pmatrix} e^{-t/2}[cos(t/2) + i\,sin(t/2)]$$
$$= e^{-t/2}\begin{pmatrix} cos(t/2) \\ 4\,sin(t/2) \end{pmatrix} + ie^{-t/2}\begin{pmatrix} sin(t/2) \\ -4\,cos(t/2) \end{pmatrix}.$$

Therefore the general solution is

$$\begin{pmatrix} I \\ V \end{pmatrix} = c_1 e^{-t/2}\begin{pmatrix} cos(t/2) \\ 4\,sin(t/2) \end{pmatrix} + c_2 e^{-t/2}\begin{pmatrix} sin(t/2) \\ -4\,cos(t/2) \end{pmatrix}.$$

(c). Imposing the initial conditions, we arrive at the equations $c_1 = 2$ and $c_2 = -\frac{3}{4}$, and

$$\begin{pmatrix} I \\ V \end{pmatrix} = e^{-t/2}\begin{pmatrix} 2\,cos(t/2) - \frac{3}{4}\,sin(t/2) \\ 8\,sin(t/2) + 3\,cos(t/2) \end{pmatrix}.$$

(d). Since the eigenvalues have *negative* real parts, all solutions converge to the origin.

26(a). The characteristic equation of the system is

$$r^2 + \frac{1}{RC}r + \frac{1}{CL} = 0\,,$$

with eigenvalues

$$r_{1,2} = -\frac{1}{2RC} \pm \frac{1}{2RC}\sqrt{1 - \frac{4R^2C}{L}}\;.$$

The eigenvalues are real and different provided that

$$1 - \frac{4R^2C}{L} > 0\,.$$

The eigenvalues are complex conjugates as long as

$$1 - \frac{4R^2C}{L} < 0\,.$$

(b). With the specified values, the eigenvalues are $r_{1,2} = -1 \pm i$. The eigenvector corresponding to $r = -1 + i$ is $\boldsymbol{\xi}^{(1)} = (1\,,\,-4i)^T$. Hence one *complex-valued* solution is

$$\begin{pmatrix} I \\ V \end{pmatrix}^{(1)} = \begin{pmatrix} 1 \\ -1+i \end{pmatrix} e^{(-1+i)t}$$
$$= \begin{pmatrix} 1 \\ -1+i \end{pmatrix} e^{-t}(\cos t + i \sin t)$$
$$= e^{-t}\begin{pmatrix} \cos t \\ -\cos t - \sin t \end{pmatrix} + ie^{-t}\begin{pmatrix} \sin t \\ \cos t - \sin t \end{pmatrix}.$$

Therefore the general solution is

$$\begin{pmatrix} I \\ V \end{pmatrix} = c_1 e^{-t}\begin{pmatrix} \cos t \\ -\cos t - \sin t \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} \sin t \\ \cos t - \sin t \end{pmatrix}.$$

(c). Imposing the initial conditions, we arrive at the equations

$$c_1 = 2$$
$$-c_1 + c_2 = 1\,,$$

with $c_1 = 2$ and $c_2 = 3$. Therefore the solution of the IVP is

$$\begin{pmatrix} I \\ V \end{pmatrix} = e^{-t}\begin{pmatrix} 2\cos t + 3\sin t \\ \cos t - 5\sin t \end{pmatrix}.$$

(d). Since $Re(r_{1,2}) = -1$, all solutions converge to the origin.

$27(a)$. Suppose that $c_1\mathbf{a} + c_2\mathbf{b} = \mathbf{0}$. Since **a** and **b** are the real and imaginary parts of the vector $\boldsymbol{\xi}^{(1)}$, respectively, $\mathbf{a} = \left(\boldsymbol{\xi}^{(1)} + \overline{\boldsymbol{\xi}^{(1)}}\right)/2$ and $\mathbf{b} = \left(\boldsymbol{\xi}^{(1)} - \overline{\boldsymbol{\xi}^{(1)}}\right)/2i$. Hence

$$c_1\left(\boldsymbol{\xi}^{(1)} + \overline{\boldsymbol{\xi}^{(1)}}\right) - ic_2\left(\boldsymbol{\xi}^{(1)} - \overline{\boldsymbol{\xi}^{(1)}}\right) = \mathbf{0}\,,$$

which leads to

$$(c_1 - ic_2)\boldsymbol{\xi}^{(1)} + (c_1 + ic_2)\overline{\boldsymbol{\xi}^{(1)}} = \mathbf{0}\,.$$

Now since $\boldsymbol{\xi}^{(1)}$ and $\overline{\boldsymbol{\xi}^{(1)}}$ are *linearly independent*, we must have

$$c_1 - ic_2 = 0$$
$$c_1 + ic_2 = 0\,.$$

It follows that $c_1 = c_2 = 0$.

(c). Recall that

$$\mathbf{u}(t) = e^{\lambda t}(\mathbf{a}\cos\mu t - \mathbf{b}\sin\mu t)$$
$$\mathbf{v}(t) = e^{\lambda t}(\mathbf{a}\cos\mu t + \mathbf{b}\sin\mu t)\,.$$

Consider the equation $c_1\mathbf{u}(t_0) + c_2\mathbf{v}(t_0) = \mathbf{0}$, for some t_0. We can then write

$$c_1 e^{\lambda t_0}(\mathbf{a}\,cos\,\mu t_0 - \mathbf{b}\,sin\,\mu t_0) + c_2 e^{\lambda t_0}(\mathbf{a}\,cos\,\mu t_0 + \mathbf{b}\,sin\,\mu t_0) = \mathbf{0}\,. \quad (*)$$

Rearranging the terms, and dividing by the exponential,

$$(c_1 + c_2)cos\,\mu t_0\,\mathbf{a} + (c_2 - c_1)sin\,\mu t_0\,\mathbf{b} = \mathbf{0}\,.$$

From Part (b), since **a** and **b** are *linearly independent*, it follows that

$$(c_1 + c_2)cos\,\mu t_0 = (c_2 - c_1)sin\,\mu t_0 = 0\,.$$

Without loss of generality, assume that the trigonometric factors are *nonzero*. Otherwise proceed again from Equation $(*)$, above. We then conclude that

$$c_1 + c_2 = 0 \text{ and } c_2 - c_1 = 0\,,$$

which leads to $c_1 = c_2 = 0$. Thus $\mathbf{u}(t_0)$ and $\mathbf{v}(t_0)$ are linearly independent for some t_0, and hence the functions are linearly independent at every point.

28(a). Let $x_1 = u$ and $x_2 = u'$. It follows that $x_1' = x_2$ and

$$\begin{aligned} x_2' &= u'' \\ &= -\frac{k}{m}\,u\,. \end{aligned}$$

In terms of the new variables, we obtain the system of two first order ODEs

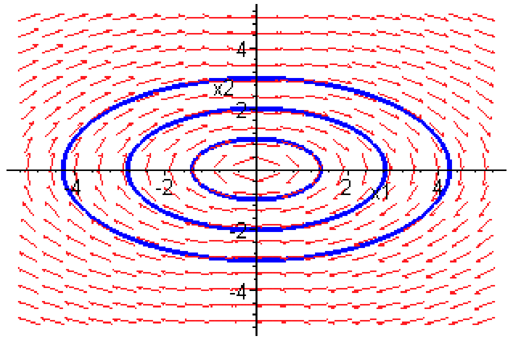
$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -\frac{k}{m}\,x_1\,. \end{aligned}$$

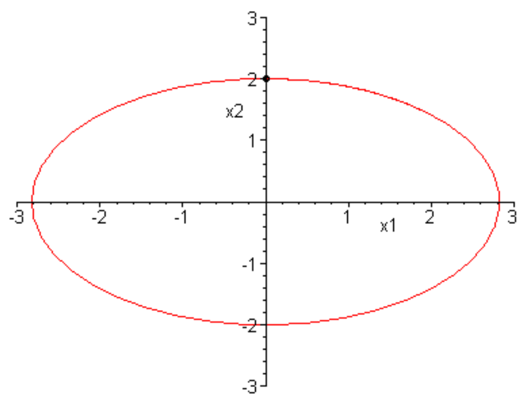
(b). The associated eigenvalue problem is

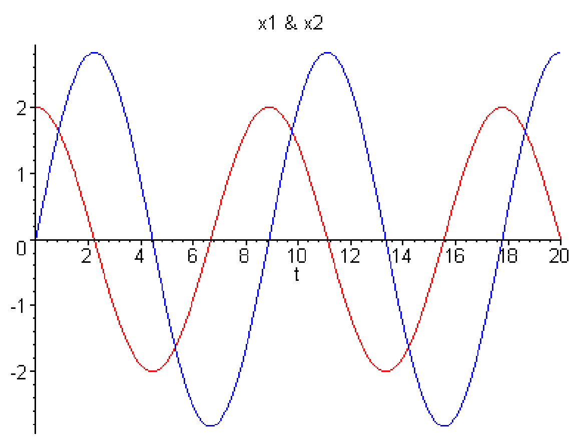
$$\begin{pmatrix} -r & 1 \\ -k/m & -r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 + k/m = 0$, with roots $r_{1,2} = \pm\, i\sqrt{k/m}$.

(c). Since the eigenvalues are purely imaginary, the origin is a *center*. Hence the phase curves are *ellipses*, with a *clockwise* flow. For computational purposes, let $k = 1$ and $m = 2$.







(d). The general solution of the second order equation is

$$u(t) = c_1 cos\sqrt{\frac{k}{m}}\,t + c_2 sin\sqrt{\frac{k}{m}}\,t\,.$$

The general solution of the system of ODEs is given by

$$\mathbf{x} = c_1 \begin{pmatrix} \sqrt{\frac{m}{k}}\, sin\sqrt{\frac{k}{m}}\, t \\ cos\sqrt{\frac{k}{m}}\, t \end{pmatrix} + c_2 \begin{pmatrix} \sqrt{\frac{m}{k}}\, cos\sqrt{\frac{k}{m}}\, t \\ -\, sin\sqrt{\frac{k}{m}}\, t \end{pmatrix}.$$

It is evident that the natural frequency of the system is equal to $Im(r_{1,2})$.

Section 7.7

1. The eigenvalues and eigenvectors were found in Prob. 1, Section 7.5.

$$r_1 = -1,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix};\ r_2 = 2,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

The general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} e^{-t} \\ 2\,e^{-t} \end{pmatrix} + c_2 \begin{pmatrix} 2\,e^{2t} \\ e^{2t} \end{pmatrix}.$$

Hence a fundamental matrix is given by

$$\boldsymbol{\Psi}(t) = \begin{pmatrix} e^{-t} & 2\,e^{2t} \\ 2\,e^{-t} & e^{2t} \end{pmatrix}.$$

We now have

$$\boldsymbol{\Psi}(0) = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \text{ and } \boldsymbol{\Psi}^{-1}(0) = \frac{1}{3}\begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix},$$

So that

$$\boldsymbol{\Phi}(t) = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}^{-1}(0) = \frac{1}{3}\begin{pmatrix} -e^{-t} + 4e^{2t} & 2e^{-t} - 2e^{2t} \\ -2e^{-t} + 2e^{2t} & 4e^{-t} - e^{2t} \end{pmatrix}.$$

3. The eigenvalues and eigenvectors were found in Prob. 3, Section 7.5. The general solution of the system is

$$\mathbf{x} = c_1 \begin{pmatrix} e^{t} \\ e^{t} \end{pmatrix} + c_2 \begin{pmatrix} e^{-t} \\ 3e^{-t} \end{pmatrix}.$$

Given the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(1)}$, we solve the equations

$$\begin{aligned} c_1 + c_2 &= 1 \\ c_1 + 3c_2 &= 0\,, \end{aligned}$$

to obtain $c_1 = 3/2\,, c_2 = -1/2$. The corresponding solution is

$$\mathbf{x} = \begin{pmatrix} \frac{3}{2}e^{t} - \frac{1}{2}e^{-t} \\ \frac{3}{2}e^{t} - \frac{3}{2}e^{-t} \end{pmatrix}.$$

Given the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(2)}$, we solve the equations

$$\begin{aligned} c_1 + c_2 &= 0 \\ c_1 + 3c_2 &= 1\,, \end{aligned}$$

to obtain $c_1 = -1/2\,, c_2 = 1/2$. The corresponding solution is

$$\mathbf{x} = \begin{pmatrix} -\frac{1}{2}e^{t} + \frac{1}{2}e^{-t} \\ -\frac{1}{2}e^{t} + \frac{3}{2}e^{-t} \end{pmatrix}.$$

Therefore the fundamental matrix is

$$\mathbf{\Phi}(t) = \frac{1}{2}\begin{pmatrix} 3e^{t} - e^{-t} & -e^{t} + e^{-t} \\ 3e^{t} - 3e^{-t} & -e^{t} + 3e^{-t} \end{pmatrix}.$$

5. The general solution, found in Prob. 3 , Section 7.6 , is given by

$$\mathbf{x} = c_1\begin{pmatrix} 5\,cos\,t \\ 2\,cos\,t + sin\,t \end{pmatrix} + c_2\begin{pmatrix} 5\,sin\,t \\ -\,cos\,t + 2\,sin\,t \end{pmatrix}.$$

Given the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(1)}$, we solve the equations

$$\begin{aligned} 5c_1 &= 1 \\ 2c_1 - c_2 &= 0\,, \end{aligned}$$

resulting in $c_1 = 1/5\,, c_2 = 2/5$. The corresponding solution is

$$\mathbf{x} = \begin{pmatrix} cos\,t + 2\,sin\,t \\ sin\,t \end{pmatrix}.$$

Given the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(2)}$, we solve the equations

$$\begin{aligned} 5c_1 &= 0 \\ 2c_1 - c_2 &= 1\,, \end{aligned}$$

resulting in $c_1 = 0\,, c_2 = -1$. The corresponding solution is

$$\mathbf{x} = \begin{pmatrix} -\,5\,sin\,t \\ cos\,t - 2\,sin\,t \end{pmatrix}.$$

Therefore the fundamental matrix is

$$\mathbf{\Phi}(t) = \begin{pmatrix} cos\,t + 2\,sin\,t & -\,5\,sin\,t \\ sin\,t & cos\,t - 2\,sin\,t \end{pmatrix}.$$

7. The general solution, found in Prob. 15 , Section 7.5 , is given by

$$\mathbf{x} = c_1\begin{pmatrix} e^{2t} \\ 3e^{2t} \end{pmatrix} + c_2\begin{pmatrix} e^{4t} \\ e^{4t} \end{pmatrix}.$$

Given the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(1)}$, we solve the equations

$$\begin{aligned} c_1 + c_2 &= 1 \\ 3c_1 + c_2 &= 0\,, \end{aligned}$$

resulting in $c_1 = -1/2\,, c_2 = 3/2$. The corresponding solution is

$$\mathbf{x} = \frac{1}{2}\begin{pmatrix} -e^{2t} + 3e^{4t} \\ -3e^{2t} + 3e^{4t} \end{pmatrix}.$$

The initial conditions $\mathbf{x}(0) = \mathbf{e}^{(2)}$ require that

$$c_1 + c_2 = 0$$
$$3c_1 + c_2 = 1\,,$$

resulting in $c_1 = 1/2\,, c_2 = -1/2$. The corresponding solution is

$$\mathbf{x} = \frac{1}{2}\begin{pmatrix} e^{2t} - e^{4t} \\ 3e^{2t} - e^{4t} \end{pmatrix}.$$

Therefore the fundamental matrix is

$$\mathbf{\Phi}(t) = \frac{1}{2}\begin{pmatrix} -e^{2t} + 3e^{4t} & e^{2t} - e^{4t} \\ -3e^{2t} + 3e^{4t} & 3e^{2t} - e^{4t} \end{pmatrix}.$$

8. The general solution, found in Prob. 5, Section 7.6, is given by

$$\mathbf{x} = c_1 e^{-t}\begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} \sin t \\ -\cos t + 2\sin t \end{pmatrix}.$$

The specific solution corresponding to the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(1)}$ is

$$\mathbf{x} = e^{-t}\begin{pmatrix} \cos t + 2\sin t \\ 5\sin t \end{pmatrix}.$$

For the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(2)}$, the solution is

$$\mathbf{x} = e^{-t}\begin{pmatrix} -\sin t \\ \cos t - 2\sin t \end{pmatrix}.$$

Therefore the fundamental matrix is

$$\mathbf{\Phi}(t) = e^{-t}\begin{pmatrix} \cos t + 2\sin t & -\sin t \\ 5\sin t & \cos t - 2\sin t \end{pmatrix}.$$

9. The general solution, found in Prob. 13, Section 7.5, is given by

$$\mathbf{x} = c_1\begin{pmatrix} 4e^{-2t} \\ -5e^{-2t} \\ -7e^{-2t} \end{pmatrix} + c_2\begin{pmatrix} 3e^{-t} \\ -4e^{-t} \\ -2e^{-t} \end{pmatrix} + c_3\begin{pmatrix} 0 \\ e^{2t} \\ -e^{2t} \end{pmatrix}.$$

Given the initial conditions $\mathbf{x}(0) = \mathbf{e}^{(1)}$, we solve the equations

$$\begin{aligned} 4c_1 + 3c_2 &= 1 \\ -5c_1 - 4c_2 + c_3 &= 0 \\ -7c_1 - 2c_2 - c_3 &= 0\,, \end{aligned}$$

resulting in $c_1 = -1/2\,, c_2 = 1\,, c_3 = 3/2\,.$ The corresponding solution is

$$\mathbf{x} = \begin{pmatrix} -2e^{-2t} + 3e^{-t} \\ \frac{5}{2}e^{-2t} - 4e^{-t} + \frac{3}{2}e^{2t} \\ \frac{7}{2}e^{-2t} - 2e^{-t} - \frac{3}{2}e^{2t} \end{pmatrix}.$$

The initial conditions $\mathbf{x}(0) = \mathbf{e}^{(2)}$, we solve the equations

$$\begin{aligned} 4c_1 + 3c_2 &= 0 \\ -5c_1 - 4c_2 + c_3 &= 1 \\ -7c_1 - 2c_2 - c_3 &= 0\,, \end{aligned}$$

resulting in $c_1 = -1/4\,, c_2 = 1/3\,, c_3 = 13/12\,.$ The corresponding solution is

$$\mathbf{x} = \begin{pmatrix} -e^{-2t} + e^{-t} \\ \frac{5}{4}e^{-2t} - \frac{4}{3}e^{-t} + \frac{13}{12}e^{2t} \\ \frac{7}{4}e^{-2t} - \frac{2}{3}e^{-t} - \frac{13}{12}e^{2t} \end{pmatrix}.$$

The initial conditions $\mathbf{x}(0) = \mathbf{e}^{(3)}$, we solve the equations

$$\begin{aligned} 4c_1 + 3c_2 &= 0 \\ -5c_1 - 4c_2 + c_3 &= 0 \\ -7c_1 - 2c_2 - c_3 &= 1\,, \end{aligned}$$

resulting in $c_1 = -1/4\,, c_2 = 1/3\,, c_3 = 1/12\,.$ The corresponding solution is

$$\mathbf{x} = \begin{pmatrix} -e^{-2t} + e^{-t} \\ \frac{5}{4}e^{-2t} - \frac{4}{3}e^{-t} + \frac{1}{12}e^{2t} \\ \frac{7}{4}e^{-2t} - \frac{2}{3}e^{-t} - \frac{1}{12}e^{2t} \end{pmatrix}.$$

Therefore the fundamental matrix is

$$\mathbf{\Phi}(t) = \begin{pmatrix} -2e^{-2t} + 3e^{-t} & -e^{-2t} + e^{-t} & -e^{-2t} + e^{-t} \\ \frac{5}{2}e^{-2t} - 4e^{-t} + \frac{3}{2}e^{2t} & \frac{5}{4}e^{-2t} - \frac{4}{3}e^{-t} + \frac{13}{12}e^{2t} & \frac{5}{4}e^{-2t} - \frac{4}{3}e^{-t} + \frac{1}{12}e^{2t} \\ \frac{7}{2}e^{-2t} - 2e^{-t} - \frac{3}{2}e^{2t} & \frac{7}{4}e^{-2t} - \frac{2}{3}e^{-t} - \frac{13}{12}e^{2t} & \frac{7}{4}e^{-2t} - \frac{2}{3}e^{-t} - \frac{1}{12}e^{2t} \end{pmatrix}.$$

12. The solution of the initial value problem is given by

$$\begin{aligned}\mathbf{x} &= \mathbf{\Phi}(t)\mathbf{x}(0)\\ &= \begin{pmatrix} e^{-t}cos\,2t & -2e^{-t}sin\,2t \\ \frac{1}{2}e^{-t}sin\,2t & e^{-t}cos\,2t \end{pmatrix}\begin{pmatrix}3\\1\end{pmatrix}\\ &= e^{-t}\begin{pmatrix} 3cos\,2t - 2sin\,2t \\ \frac{3}{2}sin\,2t + cos\,2t \end{pmatrix}.\end{aligned}$$

13. Let

$$\mathbf{\Psi}(t) = \begin{pmatrix} x_1^{(1)}(t) & \cdots & x_1^{(n)}(t) \\ \vdots & & \vdots \\ x_n^{(1)}(t) & \cdots & x_n^{(n)}(t) \end{pmatrix}.$$

It follows that

$$\mathbf{\Psi}(t_0) = \begin{pmatrix} x_1^{(1)}(t_0) & \cdots & x_1^{(n)}(t_0) \\ \vdots & & \vdots \\ x_n^{(1)}(t_0) & \cdots & x_n^{(n)}(t_0) \end{pmatrix}$$

is a *scalar* matrix, which is invertible, since the solutions are linearly independent. Let $\mathbf{\Psi}^{-1}(t_0) = (c_{ij})$. Then

$$\mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(t_0) = \begin{pmatrix} x_1^{(1)}(t) & \cdots & x_1^{(n)}(t) \\ \vdots & & \vdots \\ x_n^{(1)}(t) & \cdots & x_n^{(n)}(t) \end{pmatrix}\begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix}.$$

The *j-th* column of the product matrix is

$$[\mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(t_0)]^{(j)} = \sum_{k=1}^{n} c_{kj}\,\mathbf{x}^{(k)},$$

which is a solution vector, since it is a linear combination of solutions. Furthermore, the columns are all linearly independent, since the vectors $\mathbf{x}^{(k)}$ are. Hence the product is a fundamental matrix. Finally, setting $t = t_0$, $\mathbf{\Psi}(t_0)\mathbf{\Psi}^{-1}(t_0) = \mathbf{I}$. This is precisely the definition of $\mathbf{\Phi}(t)$.

14. The fundamental matrix $\mathbf{\Phi}(t)$ for the system

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\mathbf{x}$$

is given by

$$\mathbf{\Phi}(t) = \frac{1}{4}\begin{pmatrix} 2e^{3t} + 2e^{-t} & e^{3t} - e^{-t} \\ 4e^{3t} - 4e^{-t} & 2e^{3t} + 2e^{-t} \end{pmatrix}.$$

Direct multiplication results in

$$\begin{aligned}\mathbf{\Phi}(t)\mathbf{\Phi}(s) &= \frac{1}{16}\begin{pmatrix} 2e^{3t}+2e^{-t} & e^{3t}-e^{-t} \\ 4e^{3t}-4e^{-t} & 2e^{3t}+2e^{-t}\end{pmatrix}\begin{pmatrix} 2e^{3s}+2e^{-s} & e^{3s}-e^{-s} \\ 4e^{3s}-4e^{-s} & 2e^{3s}+2e^{-s}\end{pmatrix} \\ &= \frac{1}{16}\begin{pmatrix} 8(e^{3t+3s}+e^{-t-s}) & 4(e^{3t+3s}-e^{-t-s}) \\ 16(e^{3t+3s}-e^{-t-s}) & 8(e^{3t+3s}+e^{-t-s})\end{pmatrix}.\end{aligned}$$

Hence

$$\mathbf{\Phi}(t)\mathbf{\Phi}(s) = \frac{1}{4}\begin{pmatrix} 2e^{3(t+s)}+2e^{-(t+s)} & e^{3(t+s)}-e^{-(t+s)} \\ 4e^{3(t+s)}-4e^{-(t+s)} & 2e^{3(t+s)}+2e^{-(t+s)}\end{pmatrix}.$$

$15(a)$. Let s be arbitrary, but *fixed*, and t variable. Similar to the argument in Prob. 13, the *columns* of the matrix $\mathbf{\Phi}(t)\mathbf{\Phi}(s)$ are linear combinations of fundamental solutions. Hence the columns of $\mathbf{\Phi}(t)\mathbf{\Phi}(s)$ are also solution of the system of equations. Further, setting $t=0$, $\mathbf{\Phi}(0)\mathbf{\Phi}(s) = \mathbf{I}\,\mathbf{\Phi}(s) = \mathbf{\Phi}(s)$. That is, $\mathbf{\Phi}(t)\mathbf{\Phi}(s)$ is a solution of the initial value problem $\mathbf{Z}' = \mathbf{AZ}$, with $\mathbf{Z}(0) = \mathbf{\Phi}(s)$. Now consider the change of variable $\tau = t+s$. Let $\mathbf{W}(\tau) = \mathbf{Z}(\tau - s)$. The given initial value problem can be reformulated as

$$\frac{d}{d\tau}\mathbf{W} = \mathbf{AW}\,,\text{ with } \mathbf{W}(s) = \mathbf{\Phi}(s)\,.$$

Since $\mathbf{\Phi}(t)$ is a fundamental matrix satisfying $\mathbf{\Phi}' = \mathbf{A\Phi}$, with $\mathbf{\Phi}(0) = \mathbf{I}$, it follows that

$$\begin{aligned}\mathbf{W}(\tau) &= \left[\mathbf{\Phi}(\tau)\mathbf{\Phi}^{-1}(s)\right]\mathbf{\Phi}(s) \\ &= \mathbf{\Phi}(\tau).\end{aligned}$$

That is, $\mathbf{\Phi}(t+s) = \mathbf{\Phi}(\tau) = \mathbf{W}(\tau) = \mathbf{Z}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}(s)$.

(b). Based on Part (a), $\mathbf{\Phi}(t)\mathbf{\Phi}(-t) = \mathbf{\Phi}(t+(-t)) = \mathbf{\Phi}(0) = \mathbf{I}$. Hence

$$\mathbf{\Phi}(-t) = \mathbf{\Phi}^{-1}(t).$$

(c). It also follows that $\mathbf{\Phi}(t-s) = \mathbf{\Phi}(t+(-s)) = \mathbf{\Phi}(t)\mathbf{\Phi}(-s) = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(s)$.

16. Let $\mathbf{A}$ be a *diagonal matrix*, with $\mathbf{A} = [a_1\mathbf{e}^{(1)}, a_2\mathbf{e}^{(2)}, \cdots, a_n\mathbf{e}^{(n)}]$. Note that for any positive integer, k,

$$\mathbf{A}^k = \left[a_1^k\,\mathbf{e}^{(1)}, a_2^k\,\mathbf{e}^{(2)}, \cdots, a_n^k\,\mathbf{e}^{(n)}\right].$$

It follows, from basic matrix algebra, that

$$\mathbf{I}+\sum_{k=1}^{m}\mathbf{A}^k\frac{t^k}{k!}=\begin{pmatrix}\sum\limits_{k=0}^{m}a_1^k\frac{t^k}{k!} & 0 & \cdots & 0\\ 0 & \sum\limits_{k=0}^{m}a_2^k\frac{t^k}{k!} & \cdots & 0\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \cdots & \sum\limits_{k=0}^{m}a_n^k\frac{t^k}{k!}\end{pmatrix}.$$

It can be shown that the partial sums on the left hand side converge for all t. Taking the limit (as $m\to\infty$) on both sides of the equation, we obtain

$$exp(\mathbf{A}t)=\begin{pmatrix}e^{a_1t} & 0 & \cdots & 0\\ 0 & e^{a_2t} & \cdots & 0\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \cdots & e^{a_nt}\end{pmatrix}.$$

Alternatively, consider the system $\mathbf{x}'=\mathbf{A}\mathbf{x}$. Since ODEs are *uncoupled*, the vectors $\mathbf{x}^{(j)}=exp(a_jt)\,\mathbf{e}^{(j)},\ j=1,2,\cdots n$, are a set of linearly independent solutions. Hence the matrix

$$\mathbf{X}=\left[exp(a_1t)\,\mathbf{e}^{(1)},exp(a_2t)\,\mathbf{e}^{(2)},\cdots,exp(a_nt)\,\mathbf{e}^{(n)}\right]$$

is a *fundamental matrix*. Finally, since $\mathbf{X}(0)=\mathbf{I}$, it follows that

$$\left[exp(a_1t)\,\mathbf{e}^{(1)},exp(a_2t)\,\mathbf{e}^{(2)},\cdots,exp(a_nt)\,\mathbf{e}^{(n)}\right]=\mathbf{\Phi}(t)=exp(\mathbf{A}t)\,.$$

$17(a)$. Assuming that $\mathbf{x}=\phi(t)$ is a solution, then $\phi'=\mathbf{A}\phi$, with $\phi(0)=\mathbf{x}^0$. Integrate both sides of the equation to obtain

$$\phi(t)-\phi(0)=\int_0^t\mathbf{A}\phi(s)ds\,.$$

Hence

$$\phi(t)=\mathbf{x}^0+\int_0^t\mathbf{A}\phi(s)ds\,.$$

(b). Proceed with the iteration

$$\phi^{(i+1)}(t)=\mathbf{x}^0+\int_0^t\mathbf{A}\phi^{(i)}(s)ds\,.$$

With $\phi^{(0)}(t)=\mathbf{x}^0$, and noting that $\mathbf{A}$ is a *constant* matrix,

$$\begin{aligned}\phi^{(1)}(t) &= \mathbf{x}^0 + \int_0^t \mathbf{A}\mathbf{x}^0 ds \\ &= \mathbf{x}^0 + \mathbf{A}\mathbf{x}^0 t\,.\end{aligned}$$

That is, $\phi^{(1)}(t) = (\mathbf{I} + \mathbf{A}t)\mathbf{x}^0$.

(c). We then have

$$\begin{aligned}\phi^{(2)}(t) &= \mathbf{x}^0 + \int_0^t \mathbf{A}(\mathbf{I} + \mathbf{A}t)\mathbf{x}^0 ds \\ &= \mathbf{x}^0 + \mathbf{A}\mathbf{x}^0 t + \mathbf{A}^2\mathbf{x}^0\frac{t^2}{2} \\ &= \left(\mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2}\right)\mathbf{x}^0.\end{aligned}$$

Now suppose that

$$\phi^{(n)}(t) = \left(\mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2} + \cdots + \mathbf{A}^n\frac{t^n}{n!}\right)\mathbf{x}^0.$$

It follows that

$$\int_0^t \mathbf{A}\left(\mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2} + \cdots + \mathbf{A}^n\frac{t^n}{n!}\right)\mathbf{x}^0 ds =$$

$$\begin{aligned}&= \mathbf{A}\left(\mathbf{I}t + \mathbf{A}\frac{t^2}{2} + \mathbf{A}^2\frac{t^3}{3!} + \cdots + \mathbf{A}^n\frac{t^{n+1}}{(n+1)!}\right)\mathbf{x}^0 \\ &= \left(\mathbf{A}t + \mathbf{A}^2\frac{t^2}{2} + \mathbf{A}^3\frac{t^3}{3!} + \cdots + \mathbf{A}^{n+1}\frac{t^n}{n!}\right)\mathbf{x}^0.\end{aligned}$$

Therefore

$$\phi^{(n+1)}(t) = \left(\mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2} + \cdots + \mathbf{A}^{n+1}\frac{t^{n+1}}{(n+1)!}\right)\mathbf{x}^0.$$

By induction, the asserted form of $\phi^{(n)}(t)$ is valid for all $n \geq 0$.

(d). Define $\phi^{(\infty)}(t) = \lim_{n\to\infty}\phi^{(n)}(t)$. It can be shown that the limit does exist. In fact,

$$\phi^{(\infty)}(t) = exp(\mathbf{A}t)\mathbf{x}^0.$$

Term-by-term differentiation results in

$$
\begin{aligned}
\frac{d}{dt}\phi^{(\infty)}(t) &= \frac{d}{dt}\left(\mathbf{I}+\mathbf{A}t+\mathbf{A}^2\frac{t^2}{2}+\cdots+\mathbf{A}^n\frac{t^n}{n!}+\right)\mathbf{x}^0 \\
&= \left(\mathbf{A}+\mathbf{A}^2t+\cdots+\mathbf{A}^n\frac{t^{n-1}}{(n-1)!}+\right)\mathbf{x}^0 \\
&= \mathbf{A}\left(\mathbf{I}+\mathbf{A}t+\mathbf{A}^2\frac{t^2}{2}+\cdots+\mathbf{A}^{n-1}\frac{t^{n-1}}{(n-1)!}+\right)\mathbf{x}^0.
\end{aligned}
$$

That is,

$$\frac{d}{dt}\phi^{(\infty)}(t) = \mathbf{A}\phi^{(\infty)}(t).$$

Furthermore, $\phi^{(\infty)}(0) = \mathbf{x}^0$. Based on *uniqueness* of solutions, $\phi(t) = \phi^{(\infty)}(t)$.

Section 7.8

2. Setting $\mathbf{x} = \boldsymbol{\xi}\, t^r$ results in the algebraic equations

$$\begin{pmatrix} 4-r & -2 \\ 8 & -4-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 = 0$, with the *single* root $r = 0$. Substituting $r = 0$ reduces the system of equations to $2\xi_1 - \xi_2 = 0$. Therefore the only eigenvector is $\boldsymbol{\xi} = (1\,,2)^T$. One solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

which is a *constant* vector. In order to generate a second linearly independent solution, we must search for a *generalized eigenvector*. This leads to the system of equations

$$\begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

This system also reduces to a single equation, $2\eta_1 - \eta_2 = 1/2$. Setting $\eta_1 = k$, some arbitrary constant, we obtain $\eta_2 = 2k - 1/2$. A second solution is

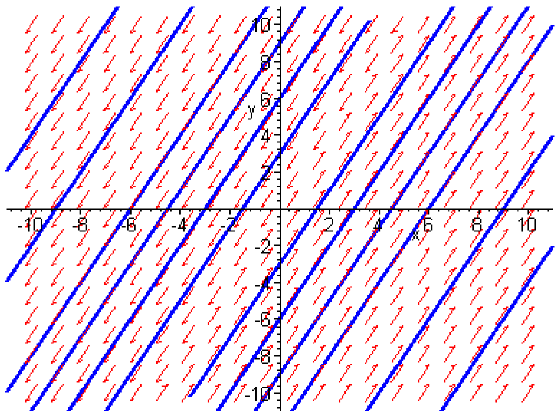
$$\begin{aligned} \mathbf{x}^{(2)} &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} k \\ 2k - 1/2 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1/2 \end{pmatrix} + k\begin{pmatrix} 1 \\ 2 \end{pmatrix}. \end{aligned}$$

Note that the *last* term is a multiple of $\mathbf{x}^{(1)}$ and may be dropped. Hence

$$\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1/2 \end{pmatrix}.$$

The general solution is

$$\mathbf{x} = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2\left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1/2 \end{pmatrix}\right].$$



All of the points on the line $x_2 = 2x_1$ are equilibrium points. Solutions starting at all other points become unbounded.

3. Solution of the ODEs is based on the analysis of the algebraic equations

$$\begin{pmatrix} -\frac{3}{2} - r & 1 \\ -\frac{1}{4} & -\frac{1}{2} - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 + 2\,r + 1 = 0$, with a single root $r = -1$. Setting $r = -1$, the two equations reduce to $\xi_1 - 2\xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi} = (2\,,1)^T$. One solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t}.$$

A second linearly independent solution is obtained by finding a *generalized eigenvector*. We therefore analyze the system

$$\begin{pmatrix} -\frac{1}{2} & 1 \\ -\frac{1}{4} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

The equations reduce to the single equation $-\eta_1 + 2\eta_2 = 2$. Let $\eta_1 = 2k$. We obtain $\eta_2 = 1 + k$, and a second linearly independent solution is

$$\begin{aligned} \mathbf{x}^{(2)} &= \begin{pmatrix} 2 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} 2k \\ 1+k \end{pmatrix} e^{-t} \\ &= \begin{pmatrix} 2 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t} + k \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t}. \end{aligned}$$

Dropping the last term, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t} + c_2 \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t} \right].$$

4. Solution of the ODE requires analysis of the algebraic equations

$$\begin{pmatrix} -3-r & \frac{5}{2} \\ -\frac{5}{2} & 2-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $det(\mathbf{A} - r\,\mathbf{I}) = r^2 + r + \frac{1}{4} = 0$. The only root is $r = -1/2$, which is an eigenvalue of multiplicity *two*. Setting $r = -1/2$ is the coefficient matrix reduces the system to the single equation $-\xi_1 + \xi_2 = 0$. Hence the corresponding eigenvector is $\boldsymbol{\xi} = (1\,,1)^T$. One solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t/2}.$$

In order to obtain a second linearly independent solution, we find a solution of the system

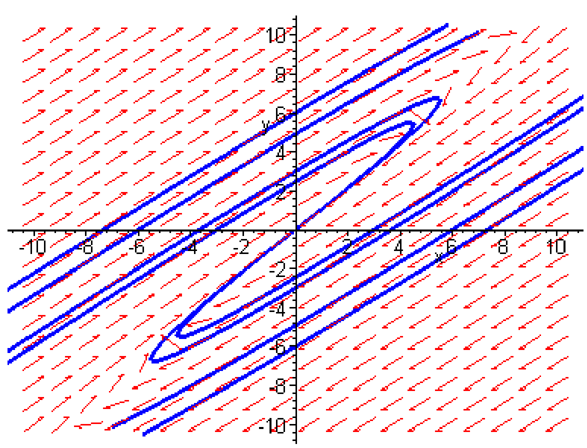
$$\begin{pmatrix} -\frac{5}{2} & \frac{5}{2} \\ -\frac{5}{2} & \frac{5}{2} \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

There equations reduce to $-5\eta_1 + 5\eta_2 = 2$. Set $\eta_1 = k$, some arbitrary constant. Then $\eta_2 = k + 2/5$. A second solution is

$$\begin{aligned}\mathbf{x}^{(2)} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t/2} + \begin{pmatrix} k \\ k+2/5 \end{pmatrix} e^{-t/2} \\ &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t/2} + \begin{pmatrix} 0 \\ 2/5 \end{pmatrix} e^{-t/2} + k\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t/2}.\end{aligned}$$

Dropping the *last* term, the general solution is

$$\mathbf{x} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t/2} + c_2\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t/2} + \begin{pmatrix} 0 \\ 2/5 \end{pmatrix} e^{-t/2}\right].$$



6. The eigensystem is obtained from analysis of the equation

$$\begin{pmatrix} -r & 1 & 1 \\ 1 & -r & 1 \\ 1 & 1 & -r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The characteristic equation of the coefficient matrix is $r^3 - 3r - 2 = 0$, with roots $r_1 = 2$ and $r_{2,3} = -1$. Setting $r = 2$, we have

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This system is reduced to the equations

$$\xi_1 - \xi_3 = 0$$
$$\xi_2 - \xi_3 = 0 .$$

A corresponding eigenvector vector is given by $\boldsymbol{\xi}^{(1)} = (1, 1, 1)^T$. Setting $r = -1$, the system of equations is reduced to the *single* equation

$$\xi_1 + \xi_2 + \xi_3 = 0 .$$

An eigenvector vector is given by $\boldsymbol{\xi}^{(2)} = (1, 0, -1)^T$. Since the last equation has two free variables, a third linearly independent eigenvector (associated with $r = -1$) is $\boldsymbol{\xi}^{(3)} = (0, 1, -1)^T$. Therefore the general solution may be written as

$$\mathbf{x} = c_1\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}e^{2t} + c_2\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}e^{-t} + c_3\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}e^{-t}.$$

7. Solution of the ODE requires analysis of the algebraic equations

$$\begin{pmatrix} 1-r & -4 \\ 4 & -7-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a nonzero solution, we must have $det(\mathbf{A} - r\,\mathbf{I}) = r^2 + 6r + 9 = 0$. The only root is $r = -3$, which is an eigenvalue of multiplicity *two*. Substituting $r = 3$ into the coefficient matrix, the system reduces to the single equation $\xi_1 - \xi_2 = 0$. Hence the corresponding eigenvector is $\boldsymbol{\xi} = (1, 1)^T$. One solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-3t}.$$

For a second linearly independent solution, we search for a *generalized eigenvector*. Its components satisfy

$$\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

that is, $4\eta_1 - 4\eta_2 = 1$. Let $\eta_2 = k$, some arbitrary constant. Then $\eta_1 = k + 1/4$. It follows that a second solution is given by

$$\begin{aligned}\mathbf{x}^{(2)} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-3t} + \begin{pmatrix} k+1/4 \\ k \end{pmatrix} e^{-3t} \\ &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-3t} + \begin{pmatrix} 1/4 \\ 0 \end{pmatrix} e^{-3t} + k\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t}.\end{aligned}$$

Dropping the last term, the general solution is

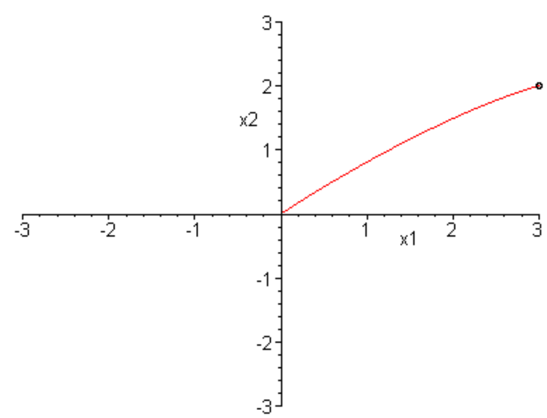
$$\mathbf{x} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t} + c_2\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-3t} + \begin{pmatrix} 1/4 \\ 0 \end{pmatrix} e^{-3t}\right].$$

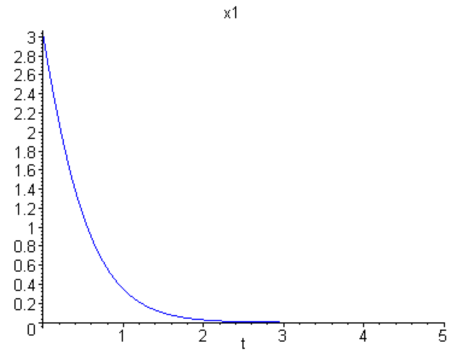
Imposing the initial conditions, we require that

$$\begin{aligned} c_1 + \frac{1}{4}c_2 &= 3 \\ c_1 &= 2\,, \end{aligned}$$

which results in $c_1 = 2$ and $c_2 = 4$. Therefore the solution of the IVP is

$$\mathbf{x} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{-3t} + \begin{pmatrix} 4 \\ 4 \end{pmatrix} te^{-3t}.$$





8. Solution of the ODEs is based on the analysis of the algebraic equations

$$\begin{pmatrix} -\frac{5}{2}-r & \frac{3}{2} \\ -\frac{3}{2} & \frac{1}{2}-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 + 2\,r + 1 = 0$, with a single root $r = -1$. Setting $r = -1$, the two equations reduce to $-\xi_1 + \xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi} = (1\,,1)^T$. One solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}.$$

A second linearly independent solution is obtained by solving the system

$$\begin{pmatrix} -\frac{3}{2} & \frac{3}{2} \\ -\frac{3}{2} & \frac{3}{2} \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The equations reduce to the single equation $-3\eta_1 + 3\eta_2 = 2$. Let $\eta_1 = k$. We obtain $\eta_2 = 2/3 + k$, and a second linearly independent solution is

$$\begin{aligned} \mathbf{x}^{(2)} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} k \\ 2/3+k \end{pmatrix} e^{-t} \\ &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} 0 \\ 2/3 \end{pmatrix} e^{-t} + k\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}. \end{aligned}$$

Dropping the last term, the general solution is

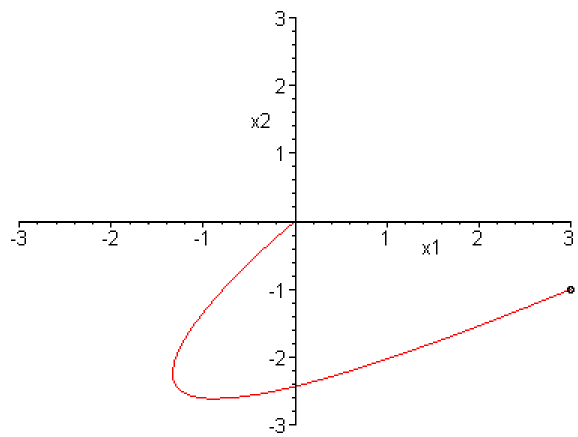
$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} 0 \\ 2/3 \end{pmatrix} e^{-t} \right].$$

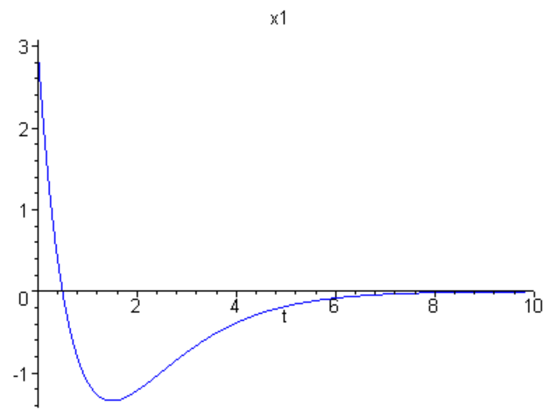
Imposing the initial conditions, find that

$$c_1 = 3$$
$$c_1 + \frac{2}{3}c_2 = -1\,,$$

so that $c_1 = 3$ and $c_2 = -6$. Therefore the solution of the IVP is

$$\mathbf{x} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} e^{-t} - \begin{pmatrix} 6 \\ 6 \end{pmatrix} te^{-t}.$$





10. The eigensystem is obtained from analysis of the equation

$$\begin{pmatrix} 3-r & 9 \\ -1 & -3-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 = 0$, with a single root $r = 0$. Setting $r = 0$, the two equations reduce to $\xi_1 + 3\xi_2 = 0$. The corresponding eigenvector is $\boldsymbol{\xi} = (-3\,,1)^T$. Hence one solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} -3 \\ 1 \end{pmatrix},$$

which is a constant vector. A second linearly independent solution is obtained from the system

$$\begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}.$$

The equations reduce to the single equation $\eta_1 + 3\eta_2 = -1$. Let $\eta_2 = k$. We obtain $\eta_1 = -1 - 3k$, and a second linearly independent solution is

$$\begin{aligned} \mathbf{x}^{(2)} &= \begin{pmatrix} -3 \\ 1 \end{pmatrix} t + \begin{pmatrix} -1-3k \\ k \end{pmatrix} \\ &= \begin{pmatrix} -3 \\ 1 \end{pmatrix} t + \begin{pmatrix} -1 \\ 0 \end{pmatrix} + k\begin{pmatrix} -3 \\ 1 \end{pmatrix}. \end{aligned}$$

Dropping the last term, the general solution is

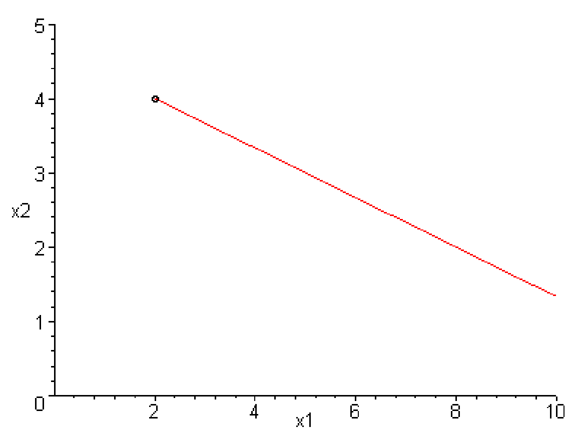
$$\mathbf{x} = c_1\begin{pmatrix} -3 \\ 1 \end{pmatrix} + c_2\left[\begin{pmatrix} -3 \\ 1 \end{pmatrix} t + \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right].$$

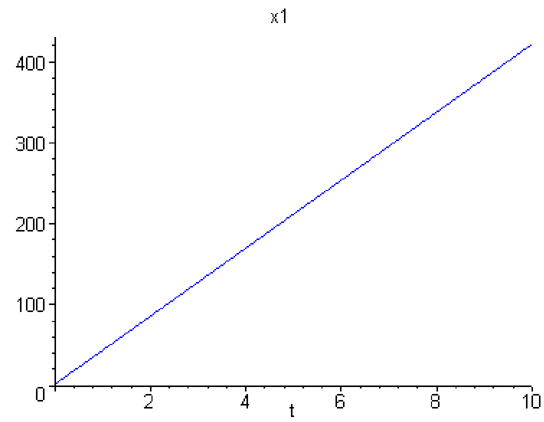
Imposing the initial conditions, we require that

$$\begin{aligned} -3c_1 - c_2 &= 2 \\ c_1 &= 4\,, \end{aligned}$$

which results in $c_1 = 4$ and $c_2 = -14$. Therefore the solution of the IVP is

$$\mathbf{x} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} - 14\begin{pmatrix} -3 \\ 1 \end{pmatrix} t.$$





12. The characteristic equation of the system is $8\,r^3 + 60\,r^2 + 126\,r + 49 = 0$. The eigenvalues are $r_1 = -1/2$ and $r_{2,3} = -7/2$. The eigenvector associated with r_1 is $\boldsymbol{\xi}^{(1)} = (1\,,1\,,1)^T$. Setting $r = -7/2$, the components of the eigenvectors must satisfy the relation

$$\xi_1 + \xi_2 + \xi_3 = 0\,.$$

An eigenvector vector is given by $\boldsymbol{\xi}^{(2)} = (1\,,0\,,\,-1)^T$. Since the last equation has two free variables, a third linearly independent eigenvector (associated with $r = -7/2$) is $\boldsymbol{\xi}^{(3)} = (0\,,1\,,\,-1)^T$. Therefore the general solution may be written as

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{-t/2} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-7t/2} + c_3 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-7t/2}.$$

Invoking the initial conditions, we require that

$$\begin{aligned} c_1 + c_2 &= 2 \\ c_1 + c_3 &= 3 \\ c_1 - c_2 - c_3 &= -1\,. \end{aligned}$$

Hence the solution of the IVP is

$$\mathbf{x} = \frac{4}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{-t/2} + \frac{2}{3} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-7t/2} + \frac{5}{3} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-7t/2}.$$

13. Setting $\mathbf{x} = \boldsymbol{\xi}\, t^r$ results in the algebraic equations

$$\begin{pmatrix} 3-r & -4 \\ 1 & -1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 - 2r + 1 = 0$, with a single root of $r_{1,2} = 1$. With

$r = 1$, the system reduces to a single equation $\xi_1 - 2\xi_2 = 0$. An eigenvector is given by $\boldsymbol{\xi} = (2, 1)^T$. Hence one solution is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} t \,.$$

In order to find a second linearly independent solution, we search for a *generalized eigenvector* whose components satisfy

$$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

These equations reduce to $\eta_1 - 2\eta_2 = 1$. Let $\eta_2 = k$, some arbitrary constant. Then $\eta_1 = 1 + 2k$. [Before proceeding, note that if we set $u = ln\, t$, the original equation is transformed into a constant coefficient equation with independent variable u. Recall that a second solution is obtained by multiplication of the first solution by the factor u. This implies that we must multiply first solution by a factor of $ln\, t$.] Hence a second linearly independent solution is

$$\begin{aligned} \mathbf{x}^{(2)} &= \begin{pmatrix} 2 \\ 1 \end{pmatrix} t\, ln\, t + \begin{pmatrix} 1 + 2k \\ k \end{pmatrix} t \\ &= \begin{pmatrix} 2 \\ 1 \end{pmatrix} t\, ln\, t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} t + k \begin{pmatrix} 2 \\ 1 \end{pmatrix} t. \end{aligned}$$

Dropping the last term, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} t + c_2 \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} t\, ln\, t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} t \right].$$

15. The characteristic equation is

$$r^2 - (a + d)r + ad - bc = 0 \,.$$

Hence the eigenvalues are

$$r_{1,2} = \frac{a + d}{2} \pm \frac{1}{2} \sqrt{(a + d)^2 - 4(ad - bc)} \,.$$

16(a). Using the result in Prob. 15, the eigenvalues are

$$r_{1,2} = -\frac{1}{2RC} \pm \frac{\sqrt{L^2 - 4R^2CL}}{2RCL} \,.$$

The discriminant vanishes when $L = 4R^2CL$.

(b). The system of differential equations is

$$\frac{d}{dt}\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{4} \\ -1 & -1 \end{pmatrix}\begin{pmatrix} I \\ V \end{pmatrix}.$$

The associated eigenvalue problem is

$$\begin{pmatrix} -r & \frac{1}{4} \\ -1 & -1-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The characteristic equation is $r^2 + r + 1/4 = 0$, with a single root of $r_{1,2} = -1/2$. Setting $r = -1/2$, the algebraic equations reduce to $2\xi_1 + \xi_2 = 0$. An eigenvector is given by $\boldsymbol{\xi} = (1, -2)^T$. Hence one solution is

$$\begin{pmatrix} I \\ V \end{pmatrix}^{(1)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t/2}.$$

A second solution is obtained from a generalized eigenvector whose components satisfy

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ -1 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

It follows that $\eta_1 = k$ and $\eta_2 = 4 - 2k$. A second linearly independent solution is

$$\begin{aligned} \begin{pmatrix} I \\ V \end{pmatrix}^{(2)} &= \begin{pmatrix} 1 \\ -2 \end{pmatrix} t\, e^{-t/2} + \begin{pmatrix} k \\ 4-2k \end{pmatrix} e^{-t/2} \\ &= \begin{pmatrix} 1 \\ -2 \end{pmatrix} t\, e^{-t/2} + \begin{pmatrix} 0 \\ 4 \end{pmatrix} e^{-t/2} + k\begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t/2}. \end{aligned}$$

Dropping the last term, the general solution is

$$\begin{pmatrix} I \\ V \end{pmatrix} = c_1\begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t/2} + c_2\left[\begin{pmatrix} 1 \\ -2 \end{pmatrix} t\, e^{-t/2} + \begin{pmatrix} 0 \\ 4 \end{pmatrix} e^{-t/2}\right].$$

Imposing the initial conditions, we require that

$$\begin{aligned} c_1 &= 1 \\ -2c_1 + 4c_2 &= 2, \end{aligned}$$

which results in $c_1 = 1$ and $c_2 = 1$. Therefore the solution of the IVP is

$$\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t/2} + \begin{pmatrix} 1 \\ -2 \end{pmatrix} t\, e^{-t/2}.$$

$18(a)$. The eigensystem is obtained from analysis of the equation

$$\begin{pmatrix} 5-r & -3 & -2 \\ 8 & -5-r & -4 \\ -4 & 3 & 3-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The characteristic equation of the coefficient matrix is $r^3 - 3r^2 + 3r - 1 = 0$, with a single root of *multiplicity three*, $r = 1$. Setting $r = 1$, we have

$$\begin{pmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The system of algebraic equations reduce to a single equation

$$4\xi_1 - 3\xi_2 - 2\xi_3 = 0\,.$$

An eigenvector vector is given by $\boldsymbol{\xi}^{(1)} = (1\,,0\,,2)^T$. Since the last equation has two free variables, a second linearly independent eigenvector (associated with $r = 1$) is $\boldsymbol{\xi}^{(2)} = (0\,,2\,,\,-3)^T$. Therefore two solutions are obtained as

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} e^t \text{ and } \mathbf{x}^{(2)} = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix} e^t.$$

(b). It follows directly that $\mathbf{x}' = \boldsymbol{\xi} te^t + \boldsymbol{\xi} e^t + \boldsymbol{\eta} e^t$. Hence the coefficient vectors must satisfy $\boldsymbol{\xi} te^t + \boldsymbol{\xi} e^t + \boldsymbol{\eta} e^t = \mathbf{A}\boldsymbol{\xi} te^t + \mathbf{A}\boldsymbol{\eta} e^t$. Rearranging the terms, we have

$$\boldsymbol{\xi} e^t = (\mathbf{A} - \mathbf{I})\boldsymbol{\xi} te^t + (\mathbf{A} - \mathbf{I})\boldsymbol{\eta} e^t.$$

Given an eigenvector $\boldsymbol{\xi}$, it follows that $(\mathbf{A} - \mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}$.

(c). Note that a linear combination of two eigenvectors, associated with the *same* eigenvalue, is also an eigenvector. Consider the equation $(\mathbf{A} - \mathbf{I})\boldsymbol{\eta} = c_1\boldsymbol{\xi}^{(1)} + c_2\boldsymbol{\xi}^{(2)}$. The *augmented* matrix is

$$\begin{pmatrix} 4 & -3 & -2 & | & c_1 \\ 8 & -6 & -4 & | & 2c_2 \\ -4 & 3 & 2 & | & 2c_1 - 3c_2 \end{pmatrix}.$$

Using elementary row operations, we obtain

$$\begin{pmatrix} 4 & -3 & -2 & | & c_1 \\ 0 & 0 & 0 & | & -2c_1 + 2c_2 \\ 0 & 0 & 0 & | & 3c_1 - 3c_2 \end{pmatrix}.$$

It is evident that a solution exists provided $c_1 = c_2$.

(d). Let $c_1 = c_2 = 2$. The components of the generalized eigenvector must satisfy

$$\begin{pmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix}.$$

Based on Part (c), the equations reduce to the single equation $4\eta_1 - 3\eta_2 - 2\eta_3 = 2$. Let $\eta_1 = \alpha$ and $\eta_2 = 2\beta$, where α and β are arbitrary constants. We then have

$$\eta_3 = -1 + 2\alpha - 3\beta,$$

so that

$$\boldsymbol{\eta} = \begin{pmatrix} \alpha \\ 2\beta \\ -1 + 2\alpha - 3\beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} + \alpha\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \beta\begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}.$$

Observe that $\boldsymbol{\eta} = \alpha\,\boldsymbol{\xi}^{(1)} + \beta\,\boldsymbol{\xi}^{(2)}$. Hence a third linearly independent solution is

$$\mathbf{x}^{(3)} = \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} te^t + \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} e^t.$$

(e). Given the three linearly independent solutions, a fundamental matrix is given by

$$\boldsymbol{\Psi}(t) = \begin{pmatrix} e^t & 0 & 2t\,e^t \\ 0 & 2e^t & 4t\,e^t \\ 2e^t & -3e^t & -2t\,e^t - e^t \end{pmatrix}.$$

(f). We construct the transformation matrix

$$\mathbf{T} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 4 & 0 \\ 2 & -2 & -1 \end{pmatrix},$$

with inverse

$$\mathbf{T}^{-1} = \begin{pmatrix} 1 & -1/2 & 0 \\ 0 & 1/4 & 0 \\ 2 & -3/2 & -1 \end{pmatrix}.$$

The *Jordan form* of the matrix **A** is

$$\mathbf{J} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

20(a). Direct multiplication results in

$$\mathbf{J}^2 = \begin{pmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda^2 & 2\lambda \\ 0 & 0 & \lambda^2 \end{pmatrix}, \mathbf{J}^3 = \begin{pmatrix} \lambda^3 & 0 & 0 \\ 0 & \lambda^3 & 3\lambda^2 \\ 0 & 0 & \lambda^3 \end{pmatrix}, \mathbf{J}^4 = \begin{pmatrix} \lambda^4 & 0 & 0 \\ 0 & \lambda^4 & 4\lambda^3 \\ 0 & 0 & \lambda^4 \end{pmatrix}.$$

(b). Suppose that

$$\mathbf{J}^n = \begin{pmatrix} \lambda^n & 0 & 0 \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix}.$$

Then

$$\begin{aligned} \mathbf{J}^{n+1} &= \begin{pmatrix} \lambda^n & 0 & 0 \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix} \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \\ &= \begin{pmatrix} \lambda \cdot \lambda^n & 0 & 0 \\ 0 & \lambda \cdot \lambda^n & \lambda^n + n\lambda \cdot \lambda^{n-1} \\ 0 & 0 & \lambda \cdot \lambda^n \end{pmatrix}. \end{aligned}$$

Hence the result follows by mathematical induction.

(c). Note that **J** is *block diagonal*. Hence each *block* may be *exponentiated*. Using the result in Prob. (19),

$$exp(\mathbf{J}t) = \begin{pmatrix} e^{\lambda t} & 0 & 0 \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{pmatrix}.$$

(d). Setting $\lambda = 1$, and using the transformation matrix **T** in Prob. (18),

$$\begin{aligned} \mathbf{T}exp(\mathbf{J}t) &= \begin{pmatrix} 1 & 2 & 0 \\ 0 & 4 & 0 \\ 2 & -2 & -1 \end{pmatrix} \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{pmatrix} \\ &= \begin{pmatrix} e^t & 2e^t & 2t\,e^t \\ 0 & 4e^t & 4t\,e^t \\ 2e^t & -2e^t & -2t\,e^t - e^t \end{pmatrix}. \end{aligned}$$

Based on the form of $\mathbf{J}$, $exp(\mathbf{J}t)$ is the fundamental matrix associated with the solutions

$$\mathbf{y}^{(1)} = \boldsymbol{\xi}^{(1)}e^t,\ \mathbf{y}^{(2)} = \left(2\boldsymbol{\xi}^{(1)} + 2\boldsymbol{\xi}^{(2)}\right)e^t \text{ and } \mathbf{y}^{(3)} = \left(2\boldsymbol{\xi}^{(1)} + 2\boldsymbol{\xi}^{(2)}\right)te^t + \boldsymbol{\eta}e^t.$$

Hence the resulting matrix is the fundamental matrix associated with the solution set

$$\{\boldsymbol{\xi}^{(1)}e^t\,,\left(2\boldsymbol{\xi}^{(1)}+2\boldsymbol{\xi}^{(2)}\right)e^t,\left(2\boldsymbol{\xi}^{(1)}+2\boldsymbol{\xi}^{(2)}\right)te^t+\boldsymbol{\eta}e^t\},$$

as opposed to the solution set in Prob. (18), given by

$$\{\boldsymbol{\xi}^{(1)}e^t\,,\boldsymbol{\xi}^{(2)}e^t,\left(2\boldsymbol{\xi}^{(1)}+2\boldsymbol{\xi}^{(2)}\right)te^t+\boldsymbol{\eta}e^t\}.$$

$21(a)$. Direct multiplication results in

$$\mathbf{J}^2=\begin{pmatrix}\lambda^2 & 2\lambda & 1\\ 0 & \lambda^2 & 2\lambda\\ 0 & 0 & \lambda^2\end{pmatrix},\ \mathbf{J}^3=\begin{pmatrix}\lambda^3 & 3\lambda^2 & 3\lambda\\ 0 & \lambda^3 & 3\lambda^2\\ 0 & 0 & \lambda^3\end{pmatrix},\ \mathbf{J}^4=\begin{pmatrix}\lambda^4 & 4\lambda^3 & 6\lambda^2\\ 0 & \lambda^4 & 4\lambda^3\\ 0 & 0 & \lambda^4\end{pmatrix}.$$

(b). Suppose that

$$\mathbf{J}^n=\begin{pmatrix}\lambda^n & n\lambda^{n-1} & \frac{n(n-1)}{2}\lambda^{n-2}\\ 0 & \lambda^n & n\lambda^{n-1}\\ 0 & 0 & \lambda^n\end{pmatrix}.$$

Then

$$\begin{aligned}\mathbf{J}^{n+1}&=\begin{pmatrix}\lambda^n & n\lambda^{n-1} & \frac{n(n-1)}{2}\lambda^{n-2}\\ 0 & \lambda^n & n\lambda^{n-1}\\ 0 & 0 & \lambda^n\end{pmatrix}\begin{pmatrix}\lambda & 1 & 0\\ 0 & \lambda & 1\\ 0 & 0 & \lambda\end{pmatrix}\\ &=\begin{pmatrix}\lambda\cdot\lambda^n & \lambda^n+n\lambda\cdot\lambda^{n-1} & n\lambda^{n-1}+\frac{n(n-1)}{2}\lambda\cdot\lambda^{n-2}\\ 0 & \lambda\cdot\lambda^n & \lambda^n+n\lambda\cdot\lambda^{n-1}\\ 0 & 0 & \lambda\cdot\lambda^n\end{pmatrix}.\end{aligned}$$

The result follows by noting that

$$\begin{aligned}n\lambda^{n-1}+\frac{n(n-1)}{2}\lambda\cdot\lambda^{n-2}&=\left[n+\frac{n(n-1)}{2}\right]\lambda^{n-1}\\ &=\frac{n^2+n}{2}\lambda^{n-1}.\end{aligned}$$

(c). We first observe that

$$\sum_{n=0}^{\infty} \lambda^n \frac{t^n}{n!} = e^{\lambda t}$$

$$\sum_{n=0}^{\infty} n\lambda^{n-1} \frac{t^n}{n!} = t \sum_{n=1}^{\infty} \lambda^{n-1} \frac{t^{n-1}}{(n-1)!} = t\, e^{\lambda t}$$

$$\sum_{n=0}^{\infty} \frac{n(n-1)}{2} \lambda^{n-2} \frac{t^n}{n!} = \frac{t^2}{2} \sum_{n=2}^{\infty} \lambda^{n-2} \frac{t^{n-2}}{(n-2)!} = \frac{t^2}{2} e^{\lambda t}.$$

Therefore

$$exp(\mathbf{J}t) = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2} e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{pmatrix}.$$

(d). Setting $\lambda = 2$, and using the transformation matrix **T** in Prob. (17),

$$\begin{aligned} \mathbf{T}exp(\mathbf{J}t) &= \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ -1 & 0 & 3 \end{pmatrix} \begin{pmatrix} e^{2t} & te^{2t} & \frac{t^2}{2} e^{2t} \\ 0 & e^{2t} & te^{2t} \\ 0 & 0 & e^{2t} \end{pmatrix} \\ &= \begin{pmatrix} 0 & e^{2t} & te^{2t} + 2e^{2t} \\ e^{2t} & te^{2t} + e^{2t} & \frac{t^2}{2} e^{2t} + te^{2t} \\ -e^{2t} & -te^{2t} & -\frac{t^2}{2} e^{2t} + 3e^{2t} \end{pmatrix}. \end{aligned}$$

Section 7.9

5. As shown in Prob. 2 , Section 7.8 , the general solution of the homogeneous equation is

$$\mathbf{x}_c = c_1\begin{pmatrix}1\\2\end{pmatrix} + c_2\begin{pmatrix}t\\2t-\frac{1}{2}\end{pmatrix}.$$

An associated fundamental matrix is

$$\boldsymbol{\Psi}(t) = \begin{pmatrix}1 & t\\2 & 2t-\frac{1}{2}\end{pmatrix}.$$

The inverse of the fundamental matrix is easily determined as

$$\boldsymbol{\Psi}^{-1}(t) = \begin{pmatrix}4t-3 & -2t+2\\8t-8 & -4t+5\end{pmatrix}.$$

We can now compute

$$\boldsymbol{\Psi}^{-1}(t)\mathbf{g}(t) = -\frac{1}{t^3}\begin{pmatrix}2t^2+4t-1\\-2t-4\end{pmatrix},$$

and

$$\int\boldsymbol{\Psi}^{-1}(t)\mathbf{g}(t)\,dt = \begin{pmatrix}-\frac{1}{2}t^{-2}+4t^{-1}-2ln\,t\\-2t^{-2}-2t^{-1}\end{pmatrix}.$$

Finally,

$$\mathbf{v}(t) = \boldsymbol{\Psi}(t)\int\boldsymbol{\Psi}^{-1}(t)\mathbf{g}(t)\,dt,$$

where

$$v_1(t) = -\frac{1}{2}t^{-2}+2t^{-1}-2\,ln\,t-2$$
$$v_2(t) = 5t^{-1}-4\,ln\,t-4\,.$$

Note that the vector $(2\,,4)^T$ is a multiple of one of the fundamental solutions. Hence we can write the general solution as

$$\mathbf{x} = c_1\begin{pmatrix}1\\2\end{pmatrix} + c_2\begin{pmatrix}t\\2t-\frac{1}{2}\end{pmatrix} - \frac{1}{t^2}\begin{pmatrix}1/2\\0\end{pmatrix} + \frac{1}{t}\begin{pmatrix}2\\5\end{pmatrix} - 2ln\,t\begin{pmatrix}1\\2\end{pmatrix}.$$

6. The eigenvalues of the coefficient matrix are $r_1 = 0$ and $r_2 = -5$. It follows that the solution of the homogeneous equation is

$$\mathbf{x}_c = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} -2e^{-5t} \\ e^{-5t} \end{pmatrix}.$$

The coefficient matrix is *symmetric*. Hence the system is diagonalizable. Using the *normalized* eigenvectors as columns, the transformation matrix, and its inverse, are

$$\mathbf{T} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}, \ \mathbf{T}^{-1} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}.$$

Setting $\mathbf{x} = \mathbf{T}\mathbf{y}$, and $\mathbf{h}(t) = \mathbf{T}^{-1}\mathbf{g}(t)$, the transformed system is given, in scalar form, as

$$y_1' = \frac{5+8t}{\sqrt{5}\ t}$$
$$y_2' = -5y_2 + \frac{4}{\sqrt{5}}.$$

The solutions are readily obtained as

$$y_1(t) = \sqrt{5}\,ln\,t + \frac{4}{\sqrt{5}}\,t + c_1 \quad \text{and} \quad y_2(t) = c_2\,e^{-5t} + \frac{4}{5\sqrt{5}}.$$

Transforming back to the original variables, we have $\mathbf{x} = \mathbf{T}\mathbf{y}$, with

$$\begin{aligned}\mathbf{x} &= \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \\ &= \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}y_1(t) + \frac{1}{\sqrt{5}}\begin{pmatrix} -2 \\ 1 \end{pmatrix}y_2(t).\end{aligned}$$

Hence the general solution is,

$$\mathbf{x} = k_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + k_2\begin{pmatrix} -2e^{-5t} \\ e^{-5t} \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}ln\,t + \frac{4}{5}\begin{pmatrix} 1 \\ 2 \end{pmatrix}t + \frac{4}{25}\begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

7. The solution of the homogeneous equation is

$$\mathbf{x}_c = c_1\begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix} + c_2\begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix}.$$

Based on the simple form of the right hand side, we use the method of *undetermined coefficients*. Set $\mathbf{v} = \mathbf{a}\,e^t$. Substitution into the ODE yields

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}e^t = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}e^t + \begin{pmatrix} 2 \\ -1 \end{pmatrix}e^t.$$

In scalar form, after canceling the exponential, we have

$$a_1 = a_1 + a_2 + 2$$
$$a_2 = 4a_1 + a_2 - 1\,,$$

with $a_1 = 1/4$ and $a_2 = -2$. Hence the particular solution is

$$\mathbf{v} = \begin{pmatrix} 1/4 \\ -2 \end{pmatrix} e^t,$$

so that the general solution is

$$\mathbf{x} = c_1\begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix} + c_2\begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix} + \frac{1}{4}\begin{pmatrix} e^t \\ -8e^t \end{pmatrix}.$$

8. The eigenvalues of the coefficient matrix are $r_1 = 1$ and $r_2 = -1$. It follows that the solution of the homogeneous equation is

$$\mathbf{x}_c = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2\begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}.$$

Use the method of *undetermined coefficients*. Since the right hand side is related to one of the fundamental solutions, set $\mathbf{v} = \mathbf{a}\,te^t + \mathbf{b}\,e^t$. Substitution into the ODE yields

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\left(e^t + te^t\right) + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} e^t = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} te^t + \\ + \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} e^t + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t.$$

In scalar form, we have

$$(a_1 + b_1)e^t + a_1 te^t = (2a_1 - a_2)te^t + (2b_1 - b_2)e^t + e^t$$
$$(a_2 + b_2)e^t + a_2 te^t = (3a_1 - 2a_2)te^t + (3b_1 - 2b_2)e^t - e^t\,.$$

Equating the coefficients in these two equations, we find that

$$a_1 = 2a_1 - a_2$$
$$a_1 + b_1 = 2b_1 - b_2 + 1$$
$$a_2 = 3a_1 - 2a_2$$
$$a_2 + b_2 = 3b_1 - 2b_2 - 1\,.$$

It follows that $a_1 = a_2$. Setting $a_1 = a_2 = a$, the equations reduce to

$$b_1 - b_2 = a - 1$$
$$3b_1 - 3b_2 = 1 + a\,.$$

Combining these equations, it is necessary that $a = 2$. As a result, $b_1 = b_2 + 1$. Choosing $a_1 = a_2 = 2$, and $b_2 = k$, some arbitrary constant, a particular solution is

$$\mathbf{v} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} te^t + \begin{pmatrix} k+1 \\ k \end{pmatrix} e^t = \begin{pmatrix} 2 \\ 2 \end{pmatrix} te^t + k \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t.$$

Since the *second* vector is a fundamental solution, the general solution can be written as

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t} + \begin{pmatrix} 2 \\ 2 \end{pmatrix} te^t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t.$$

9. Note that the coefficient matrix is *symmetric*. Hence the system is diagonalizable. The eigenvalues and eigenvectors are given by

$$r_1 = -\frac{1}{2}\,,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } r_2 = -2\,,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Using the *normalized* eigenvectors as columns, the transformation matrix, and its inverse, are

$$\mathbf{T} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},\ \mathbf{T}^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Setting $\mathbf{x} = \mathbf{T}\mathbf{y}$, and $\mathbf{h}(t) = \mathbf{T}^{-1}\mathbf{g}(t)$, the transformed system is given, in scalar form, as

$$y_1' = -\frac{1}{2}y_1 + \sqrt{2}\,t + \frac{1}{\sqrt{2}}e^t$$
$$y_2' = -2y_2 + \sqrt{2}\,t - \frac{1}{\sqrt{2}}e^t\,.$$

Using any elementary method for first order linear equations, the solutions are

$$y_1(t) = k_1 e^{-t/2} + \frac{\sqrt{2}}{3}e^t - 4\sqrt{2} + 2\sqrt{2}\,t$$
$$y_2(t) = k_2 e^{-2t} - \frac{1}{3\sqrt{2}}e^t - \frac{1}{2\sqrt{2}} + \frac{1}{\sqrt{2}}\,t\,.$$

Transforming back to the original variables, $\mathbf{x} = \mathbf{T}\mathbf{y}$, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t/2} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} - \frac{1}{4}\begin{pmatrix} 17 \\ 15 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 5 \\ 3 \end{pmatrix} t + \frac{1}{6}\begin{pmatrix} 1 \\ 3 \end{pmatrix} e^t.$$

10. Since the coefficient matrix is *symmetric*, the differential equations can be decoupled.
The eigenvalues and eigenvectors are given by

$$r_1 = -4\ ,\ \boldsymbol{\xi}^{(1)} = \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix} \text{ and } r_2 = -1\ ,\ \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}.$$

Using the *normalized* eigenvectors as columns, the transformation matrix, and its inverse, are

$$\mathbf{T} = \frac{1}{\sqrt{3}}\begin{pmatrix} \sqrt{2} & 1 \\ -1 & \sqrt{2} \end{pmatrix},\ \mathbf{T}^{-1} = \frac{1}{\sqrt{3}}\begin{pmatrix} \sqrt{2} & -1 \\ 1 & \sqrt{2} \end{pmatrix}.$$

Setting $\mathbf{x} = \mathbf{Ty}$, and $\mathbf{h}(t) = \mathbf{T}^{-1}\mathbf{g}(t)$, the transformed system is given, in scalar form, as

$$y_1' = -4y_1 + \frac{1}{\sqrt{3}}\left(1 + \sqrt{2}\right)e^{-t}$$
$$y_2' = -y_2 + \frac{1}{\sqrt{3}}\left(1 - \sqrt{2}\right)e^{-t}.$$

The solutions are easily obtained as

$$y_1(t) = k_1 e^{-4t} + \frac{1}{3\sqrt{3}}\left(1 + \sqrt{2}\right)e^{-t}$$
$$y_2(t) = k_2 e^{-t} + \frac{1}{\sqrt{3}}\left(1 - \sqrt{2}\right)te^{-t}.$$

Transforming back to the original variables, the general solution is

$$\mathbf{x} = c_1\begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix}e^{-4t} + c_2\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}e^{-t} + \frac{1}{9}\begin{pmatrix} 2+\sqrt{2}+3\sqrt{3} \\ 3\sqrt{6}-\sqrt{2}-1 \end{pmatrix}e^{-t} + \frac{1}{3}\begin{pmatrix} 1-\sqrt{2} \\ \sqrt{2}-2 \end{pmatrix}te^{-t}.$$

Note that

$$\begin{pmatrix} 2+\sqrt{2}+3\sqrt{3} \\ 3\sqrt{6}-\sqrt{2}-1 \end{pmatrix} = \begin{pmatrix} 2+\sqrt{2} \\ -\sqrt{2}-1 \end{pmatrix} + 3\sqrt{3}\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}.$$

The *second* vector is an *eigenvector*, hence the solution may be written as

$$\mathbf{x} = c_1\begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix}e^{-4t} + c_2\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}e^{-t} + \frac{1}{9}\begin{pmatrix} 2+\sqrt{2} \\ -\sqrt{2}-1 \end{pmatrix}e^{-t} + \frac{1}{3}\begin{pmatrix} 1-\sqrt{2} \\ \sqrt{2}-2 \end{pmatrix}te^{-t}.$$

11. Based on the solution of Prob. 3 of Section 7.6, a fundamental matrix is given by

$$\boldsymbol{\Psi}(t) = \begin{pmatrix} 5\cos t & 5\sin t \\ 2\cos t + \sin t & -\cos t + 2\sin t \end{pmatrix}.$$

The inverse of the fundamental matrix is easily determined as

$$\mathbf{\Psi}^{-1}(t) = \frac{1}{5}\begin{pmatrix} cos\,t - 2sin\,t & 5sin\,t \\ 2cos\,t + sin\,t & -5cos\,t \end{pmatrix}.$$

It follows that

$$\mathbf{\Psi}^{-1}(t)\mathbf{g}(t) = \begin{pmatrix} cos\,t\,sin\,t \\ -cos^2 t \end{pmatrix},$$

and

$$\int \mathbf{\Psi}^{-1}(t)\mathbf{g}(t)\,dt = \begin{pmatrix} \frac{1}{2}sin^2 t \\ -\frac{1}{2}cos\,t\,sin\,t - \frac{1}{2}t \end{pmatrix}.$$

A particular solution is constructed as

$$\mathbf{v}(t) = \mathbf{\Psi}(t)\int \mathbf{\Psi}^{-1}(t)\mathbf{g}(t)\,dt,$$

where

$$v_1(t) = \frac{5}{2}cos\,t\,sin\,t - cos^2 t + \frac{5}{2}t + 1$$
$$v_2(t) = cos\,t\,sin\,t - \frac{1}{2}cos^2 t + t + \frac{1}{2}\,.$$

Hence the general solution is

$$\mathbf{x} = c_1\begin{pmatrix} 5\,cos\,t \\ 2\,cos\,t + sin\,t \end{pmatrix} + c_2\begin{pmatrix} 5\,sin\,t \\ -cos\,t + 2\,sin\,t \end{pmatrix} - $$
$$-\,t\,sin\,t\begin{pmatrix} 5/2 \\ 1 \end{pmatrix} + (t\,cos\,t + sin\,t)\begin{pmatrix} 0 \\ 1/2 \end{pmatrix}.$$

$13(a)$. As shown in Prob. 25 of Section 7.6 , the solution of the homogeneous system is

$$\begin{pmatrix} x_1^{(c)} \\ x_2^{(c)} \end{pmatrix} = c_1 e^{-t/2}\begin{pmatrix} cos(t/2) \\ 4\,sin(t/2) \end{pmatrix} + c_2 e^{-t/2}\begin{pmatrix} sin(t/2) \\ -4\,cos(t/2) \end{pmatrix}.$$

Therefore the associated fundamental matrix is given by

$$\mathbf{\Psi}(t) = e^{-t/2}\begin{pmatrix} cos(t/2) & sin(t/2) \\ 4\,sin(t/2) & -4\,cos(t/2) \end{pmatrix}.$$

(b). The inverse of the fundamental matrix is

$$\mathbf{\Psi}^{-1}(t) = \frac{e^{t/2}}{4}\begin{pmatrix} 4\,cos(t/2) & sin(t/2) \\ 4\,sin(t/2) & -cos(t/2) \end{pmatrix}.$$

It follows that

$$\mathbf{\Psi}^{-1}(t)\mathbf{g}(t) = \frac{1}{2}\begin{pmatrix} cos(t/2) \\ sin(t/2) \end{pmatrix},$$

and

$$\int \mathbf{\Psi}^{-1}(t)\mathbf{g}(t)\,dt = \begin{pmatrix} sin(t/2) \\ -\,cos(t/2) \end{pmatrix}.$$

A particular solution is constructed as

$$\mathbf{v}(t) = \mathbf{\Psi}(t)\int \mathbf{\Psi}^{-1}(t)\mathbf{g}(t)\,dt,$$

where

$$v_1(t) = 0$$
$$v_2(t) = 4\,e^{-t/2}\,.$$

Hence the general solution is

$$\mathbf{x} = c_1 e^{-t/2}\begin{pmatrix} cos(t/2) \\ 4\,sin(t/2) \end{pmatrix} + c_2 e^{-t/2}\begin{pmatrix} sin(t/2) \\ -\,4\,cos(t/2) \end{pmatrix} + 4\,e^{-t/2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Imposing the initial conditions, we require that

$$c_1 = 0$$
$$-\,4c_2 + 4 = 0\,,$$

which results in $c_1 = 0$ and $c_2 = 1$. Therefore the solution of the IVP is

$$\mathbf{x} = e^{-t/2}\begin{pmatrix} sin(t/2) \\ 4 - 4\,cos(t/2) \end{pmatrix}.$$

15. The general solution of the homogeneous problem is

$$\begin{pmatrix} x_1^{(c)} \\ x_2^{(c)} \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}t^{-1} + c_2\begin{pmatrix} 2 \\ 1 \end{pmatrix}t^2,$$

which can be verified by substitution into the system of ODEs. Since the vectors are linearly independent, a fundamental matrix is given by

$$\mathbf{\Psi}(t) = \begin{pmatrix} t^{-1} & 2t^2 \\ 2t^{-1} & t^2 \end{pmatrix}.$$

The inverse of the fundamental matrix is

$$\boldsymbol{\Psi}^{-1}(t) = \frac{1}{3}\begin{pmatrix} -t & 2t \\ 2t^{-2} & -t^{-2} \end{pmatrix}.$$

Dividing both equations by t, we obtain

$$\mathbf{g}(t) = \begin{pmatrix} -2 \\ t^3 - t^{-1} \end{pmatrix}.$$

Proceeding with the method of *variation of parameters*,

$$\boldsymbol{\Psi}^{-1}(t)\mathbf{g}(t) = \begin{pmatrix} \frac{2}{3}t^4 + \frac{2}{3}t - \frac{2}{3} \\ -\frac{1}{3}t - \frac{4}{3}t^{-2} + \frac{1}{3}t^{-3} \end{pmatrix},$$

and

$$\int \boldsymbol{\Psi}^{-1}(t)\mathbf{g}(t)\,dt = \begin{pmatrix} \frac{2}{15}t^5 + \frac{1}{3}t^2 - \frac{2}{3}t \\ -\frac{1}{6}t^2 + \frac{4}{3}t^{-1} - \frac{1}{6}t^{-2} \end{pmatrix}.$$

Hence a particular solution is obtained as

$$\mathbf{v} = \begin{pmatrix} -\frac{1}{5}t^4 + 3t - 1 \\ \frac{1}{10}t^4 + 2t - \frac{3}{2} \end{pmatrix}.$$

The general solution is

$$\mathbf{x} = \; = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}t^{-1} + c_2\begin{pmatrix} 2 \\ 1 \end{pmatrix}t^2 + \frac{1}{10}\begin{pmatrix} -2 \\ 1 \end{pmatrix}t^4 + \begin{pmatrix} 3 \\ 2 \end{pmatrix}t - \begin{pmatrix} 1 \\ 3/2 \end{pmatrix}.$$

16. Based on the hypotheses,

$$\phi'(t) = \mathbf{P}(t)\phi(t) + \mathbf{g}(t) \text{ and } \mathbf{v}'(t) = \mathbf{P}(t)\mathbf{v}(t) + \mathbf{g}(t)\,.$$

Subtracting the two equations results in

$$\phi'(t) - \mathbf{v}'(t) = \mathbf{P}(t)\phi(t) - \mathbf{P}(t)\mathbf{v}(t)\,,$$

that is,

$$[\phi(t) - \mathbf{v}(t)]' = \mathbf{P}(t)[\phi(t) - \mathbf{v}(t)].$$

It follows that $\phi(t) - \mathbf{v}(t)$ is a solution of the *homogeneous equation*. According to Theorem 7.4.2,

$$\phi(t) - \mathbf{v}(t) = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t).$$

Hence

$$\phi(t) = \mathbf{u}(t) + \mathbf{v}(t),$$

in which $\mathbf{u}(t)$ is the general solution of the homogeneous problem.

$17(a)$. Setting $t_0 = 0$ in Eq. (34),

$$\begin{aligned}\mathbf{x} &= \mathbf{\Phi}(t)\mathbf{x}^0 + \mathbf{\Phi}(t)\int_0^t \mathbf{\Phi}^{-1}(s)\mathbf{g}(s)ds \\ &= \mathbf{\Phi}(t)\mathbf{x}^0 + \int_0^t \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(s)\mathbf{g}(s)ds\,.\end{aligned}$$

It was shown in Prob. $15(c)$ in Section 7.7 that $\mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(s) = \mathbf{\Phi}(t-s)$. Therefore

$$\mathbf{x} = \mathbf{\Phi}(t)\mathbf{x}^0 + \int_0^t \mathbf{\Phi}(t-s)\mathbf{g}(s)ds\,.$$

(b). The *principal* fundamental matrix is identified as $\mathbf{\Phi}(t) = exp(\mathbf{A}t)$. Hence

$$\mathbf{x} = exp(\mathbf{A}t)\mathbf{x}^0 + \int_0^t exp[\mathbf{A}(t-s)]\mathbf{g}(s)ds\,.$$

In Prob. 26 of Section 3.7, the particular solution is given as

$$Y(t) = \int_{t_0}^t K(t-s)g(s)ds\,,$$

in which the kernel $K(t)$ depends on the nature of the fundamental solutions.