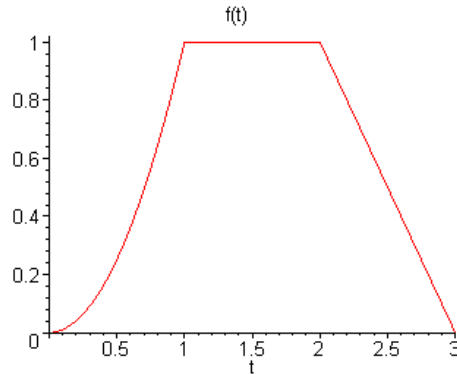


## Chapter Six

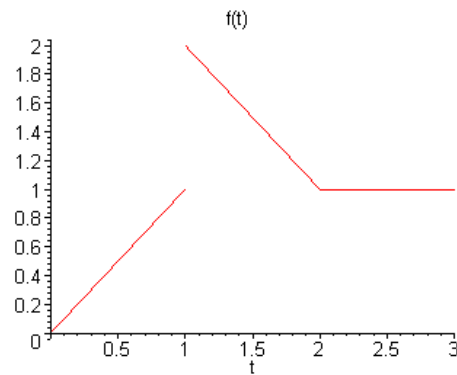
### Section 6.1

3.



The function  $f(t)$  is *continuous*.

4.



The function  $f(t)$  has a *jump discontinuity* at  $t = 1$ .

7. Integration is a linear operation. It follows that

$$\begin{aligned} \int_0^A \cosh bt \cdot e^{-st} dt &= \frac{1}{2} \int_0^A e^{bt} \cdot e^{-st} dt + \frac{1}{2} \int_0^A e^{-bt} \cdot e^{-st} dt \\ &= \frac{1}{2} \int_0^A e^{(b-s)t} dt + \frac{1}{2} \int_0^A e^{-(b+s)t} dt. \end{aligned}$$

Hence

$$\int_0^A \cosh bt \cdot e^{-st} dt = \frac{1}{2} \left[ \frac{1 - e^{(b-s)A}}{s - b} \right] + \frac{1}{2} \left[ \frac{1 - e^{-(b+s)A}}{s + b} \right].$$

Taking a *limit*, as  $A \rightarrow \infty$ ,

$$\begin{aligned}\int_0^\infty \cosh bt \cdot e^{-st} dt &= \frac{1}{2} \left[ \frac{1}{s-b} \right] + \frac{1}{2} \left[ \frac{1}{s+b} \right] \\ &= \frac{s}{s^2 - b^2}.\end{aligned}$$

Note that the above is valid for  $s > |b|$ .

8. Proceeding as in Prob. 7,

$$\int_0^A \sinh bt \cdot e^{-st} dt = \frac{1}{2} \left[ \frac{1 - e^{(b-s)A}}{s-b} \right] - \frac{1}{2} \left[ \frac{1 - e^{-(b+s)A}}{s+b} \right].$$

Taking a *limit*, as  $A \rightarrow \infty$ ,

$$\begin{aligned}\int_0^\infty \sinh bt \cdot e^{-st} dt &= \frac{1}{2} \left[ \frac{1}{s-b} \right] - \frac{1}{2} \left[ \frac{1}{s+b} \right] \\ &= \frac{b}{s^2 - b^2}.\end{aligned}$$

The limit exists as long as  $s > |b|$ .

10. Observe that  $e^{at} \sinh bt = (e^{(a+b)t} - e^{(a-b)t})/2$ . It follows that

$$\int_0^A e^{at} \sinh bt \cdot e^{-st} dt = \frac{1}{2} \left[ \frac{1 - e^{(a+b-s)A}}{s-a+b} \right] - \frac{1}{2} \left[ \frac{1 - e^{-(b-a+s)A}}{s+b-a} \right].$$

Taking a *limit*, as  $A \rightarrow \infty$ ,

$$\begin{aligned}\int_0^\infty e^{at} \sinh bt \cdot e^{-st} dt &= \frac{1}{2} \left[ \frac{1}{s-a+b} \right] - \frac{1}{2} \left[ \frac{1}{s+b-a} \right] \\ &= \frac{b}{(s-a)^2 - b^2}.\end{aligned}$$

The limit exists as long as  $s - a > |b|$ .

11. Using the *linearity* of the Laplace transform,

$$\mathcal{L}[\sin bt] = \frac{1}{2i} \mathcal{L}[e^{ibt}] - \frac{1}{2i} \mathcal{L}[e^{-ibt}].$$

Since

$$\int_0^\infty e^{(a+ib)t} e^{-st} dt = \frac{1}{s-a-ib},$$

we have

$$\int_0^{\infty} e^{\pm ibt} e^{-st} dt = \frac{1}{s \mp ib}.$$

Therefore

$$\begin{aligned} \mathcal{L}[\sin bt] &= \frac{1}{2i} \left[ \frac{1}{s - ib} - \frac{1}{s + ib} \right] \\ &= \frac{b}{s^2 + b^2}. \end{aligned}$$

12. Using the *linearity* of the Laplace transform,

$$\mathcal{L}[\cos bt] = \frac{1}{2} \mathcal{L}[e^{ibt}] + \frac{1}{2} \mathcal{L}[e^{-ibt}].$$

From Prob. 11, we have

$$\int_0^{\infty} e^{\pm ibt} e^{-st} dt = \frac{1}{s \mp ib}.$$

Therefore

$$\begin{aligned} \mathcal{L}[\cos bt] &= \frac{1}{2} \left[ \frac{1}{s - ib} + \frac{1}{s + ib} \right] \\ &= \frac{s}{s^2 + b^2}. \end{aligned}$$

14. Using the *linearity* of the Laplace transform,

$$\mathcal{L}[e^{at} \cos bt] = \frac{1}{2} \mathcal{L}[e^{(a+ib)t}] + \frac{1}{2} \mathcal{L}[e^{(a-ib)t}].$$

Based on the integration in Prob. 11,

$$\int_0^{\infty} e^{(a \pm ib)t} e^{-st} dt = \frac{1}{s - a \mp ib}.$$

Therefore

$$\begin{aligned} \mathcal{L}[e^{at} \cos bt] &= \frac{1}{2} \left[ \frac{1}{s - a - ib} + \frac{1}{s - a + ib} \right] \\ &= \frac{s - a}{(s - a)^2 + b^2}. \end{aligned}$$

The above is valid for  $s > a$ .

15. Integrating *by parts*,

$$\begin{aligned}\int_0^A t e^{at} \cdot e^{-st} dt &= -\left. \frac{t e^{(a-s)t}}{s-a} \right|_0^A + \int_0^A \frac{1}{s-a} e^{(a-s)t} dt \\ &= \frac{1 - e^{A(a-s)} + A(a-s)e^{A(a-s)}}{(s-a)^2}.\end{aligned}$$

Taking a *limit*, as  $A \rightarrow \infty$ ,

$$\int_0^\infty t e^{at} \cdot e^{-st} dt = \frac{1}{(s-a)^2}.$$

Note that the limit exists as long as  $s > a$ .

17. Observe that  $t \cosh at = (t e^{at} + t e^{-at})/2$ . For any value of  $c$ ,

$$\begin{aligned}\int_0^A t e^{ct} \cdot e^{-st} dt &= -\left. \frac{t e^{(c-s)t}}{s-c} \right|_0^A + \int_0^A \frac{1}{s-c} e^{(c-s)t} dt \\ &= \frac{1 - e^{A(c-s)} + A(c-s)e^{A(c-s)}}{(s-c)^2}.\end{aligned}$$

Taking a *limit*, as  $A \rightarrow \infty$ ,

$$\int_0^\infty t e^{ct} \cdot e^{-st} dt = \frac{1}{(s-c)^2}.$$

Note that the limit exists as long as  $s > |c|$ . Therefore,

$$\begin{aligned}\int_0^\infty t \cosh at \cdot e^{-st} dt &= \frac{1}{2} \left[ \frac{1}{(s-a)^2} + \frac{1}{(s+a)^2} \right] \\ &= \frac{s^2 + a^2}{(s-a)^2 (s+a)^2}.\end{aligned}$$

18. Integrating *by parts*,

$$\begin{aligned}\int_0^A t^n e^{at} \cdot e^{-st} dt &= -\left. \frac{t^n e^{(a-s)t}}{s-a} \right|_0^A + \int_0^A \frac{n}{s-a} t^{n-1} e^{(a-s)t} dt \\ &= -\frac{A^n e^{-(s-a)A}}{s-a} + \int_0^A \frac{n}{s-a} t^{n-1} e^{(a-s)t} dt.\end{aligned}$$

Continuing to integrate by parts, it follows that

$$\int_0^A t^n e^{at} \cdot e^{-st} dt = -\frac{A^n e^{(a-s)A}}{s-a} - \frac{nA^{n-1} e^{(a-s)A}}{(s-a)^2} - \dots - \frac{n!(e^{(a-s)A} - 1)}{(s-a)^{n+1}}.$$

That is,

$$\int_0^A t^n e^{at} \cdot e^{-st} dt = p_n(A) \cdot e^{(a-s)A} + \frac{n!}{(s-a)^{n+1}},$$

in which  $p_n(\xi)$  is a *polynomial* of degree  $n$ . For any given polynomial,

$$\lim_{A \rightarrow \infty} p_n(A) \cdot e^{-(s-a)A} = 0,$$

as long as  $s > a$ . Therefore,

$$\int_0^\infty t^n e^{at} \cdot e^{-st} dt = \frac{n!}{(s-a)^{n+1}}.$$

20. Observe that  $t^2 \sinh at = (t^2 e^{at} - t^2 e^{-at})/2$ . Using the result in Prob. 18,

$$\begin{aligned} \int_0^\infty t^2 \sinh at \cdot e^{-st} dt &= \frac{1}{2} \left[ \frac{2!}{(s-a)^3} - \frac{2!}{(s+a)^3} \right] \\ &= \frac{2a(3s^2 + a^2)}{(s^2 - a^2)^3}. \end{aligned}$$

The above is valid for  $s > |a|$ .

22. Integrating by parts,

$$\begin{aligned} \int_0^A t e^{-t} dt &= -t e^{-t} \Big|_0^A + \int_0^A e^{-t} dt \\ &= 1 - e^{-A} - A e^{-A}. \end{aligned}$$

Taking a *limit*, as  $A \rightarrow \infty$ ,

$$\int_0^\infty t e^{-t} dt = 1 - e^{-A}.$$

Hence the integral *converges*.

23. Based on a series expansion, note that for  $t > 0$ ,

$$e^t > 1 + t + t^2/2 > t^2/2.$$

It follows that for  $t > 0$ ,

$$t^{-2}e^t > \frac{1}{2}.$$

Hence for any finite  $A > 1$ ,

$$\int_1^A t^{-2}e^t dt > \frac{A-1}{2}.$$

It is evident that the limit as  $A \rightarrow \infty$  does not exist.

24. Using the fact that  $|\cos t| \leq 1$ , and the fact that

$$\int_0^\infty e^{-t} dt = 1,$$

it follows that the given integral *converges*.

25(a). Let  $p > 0$ . Integrating *by parts*,

$$\begin{aligned} \int_0^A e^{-x} x^p dx &= -e^{-x} x^p \Big|_0^A + p \int_0^A e^{-x} x^{p-1} dx \\ &= -A^p e^{-A} + p \int_0^A e^{-x} x^{p-1} dx. \end{aligned}$$

Taking a *limit*, as  $A \rightarrow \infty$ ,

$$\int_0^\infty e^{-x} x^p dx = p \int_0^\infty e^{-x} x^{p-1} dx.$$

That is,  $\Gamma(p+1) = p\Gamma(p)$ .

(b). Setting  $p = 0$ ,

$$\Gamma(1) = \int_0^\infty e^{-x} dx = 1.$$

(c). Let  $p = n$ . Using the result in Part (b),

$$\begin{aligned} \Gamma(n+1) &= n\Gamma(n) \\ &= n(n-1)\Gamma(n-1) \\ &\quad \vdots \\ &= n(n-1)(n-2)\cdots 2 \cdot 1 \cdot \Gamma(1). \end{aligned}$$

Since  $\Gamma(1) = 1$ ,  $\Gamma(n+1) = n!$ .

(d). Using the result in Part (b),

$$\begin{aligned}
\Gamma(p+n) &= (p+n-1)\Gamma(p+n-1) \\
&= (p+n-1)(p+n-2)\Gamma(p+n-2) \\
&\quad \vdots \\
&= (p+n-1)(p+n-2)\cdots(p+1)p\Gamma(p).
\end{aligned}$$

Hence

$$\frac{\Gamma(p+n)}{\Gamma(p)} = p(p+1)(p+1)\cdots(p+n-1).$$

Given that  $\Gamma(1/2) = \sqrt{\pi}$ , it follows that

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

and

$$\Gamma\left(\frac{11}{2}\right) = \frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \Gamma\left(\frac{3}{2}\right) = \frac{945\sqrt{\pi}}{32}.$$

## Section 6.2

1. Write the function as

$$\frac{3}{s^2 + 4} = \frac{3}{2} \frac{2}{s^2 + 4}.$$

Hence  $\mathcal{L}^{-1}[Y(s)] = \frac{3}{2} \sin 2t$ .

3. Using *partial fractions*,

$$\frac{2}{s^2 + 3s - 4} = \frac{2}{5} \left[ \frac{1}{s - 1} - \frac{1}{s + 4} \right].$$

Hence  $\mathcal{L}^{-1}[Y(s)] = \frac{2}{5}(e^t - e^{-4t})$ .

5. Note that the denominator  $s^2 + 2s + 5$  is *irreducible* over the reals. Completing the square,  $s^2 + 2s + 5 = (s + 1)^2 + 4$ . Now convert the function to a *rational function* of the variable  $\xi = s + 1$ . That is,

$$\frac{2s + 2}{s^2 + 2s + 5} = \frac{2(s + 1)}{(s + 1)^2 + 4}.$$

We know that

$$\mathcal{L}^{-1} \left[ \frac{2\xi}{\xi^2 + 4} \right] = 2 \cos 2t.$$

Using the fact that  $\mathcal{L}[e^{at}f(t)] = \mathcal{L}[f(t)]_{s \rightarrow s-a}$ ,

$$\mathcal{L}^{-1} \left[ \frac{2s + 2}{s^2 + 2s + 5} \right] = 2e^{-t} \cos 2t.$$

6. Using *partial fractions*,

$$\frac{2s - 3}{s^2 - 4} = \frac{1}{4} \left[ \frac{1}{s - 2} + \frac{7}{s + 2} \right].$$

Hence  $\mathcal{L}^{-1}[Y(s)] = \frac{1}{4}(e^{2t} + 7e^{-2t})$ . Note that we can also write

$$\frac{2s - 3}{s^2 - 4} = 2 \frac{s}{s^2 - 4} - \frac{3}{2} \frac{2}{s^2 - 4}.$$

8. Using *partial fractions*,

$$\frac{8s^2 - 4s + 12}{s(s^2 + 4)} = 3 \frac{1}{s} + 5 \frac{s}{s^2 + 4} - 2 \frac{2}{s^2 + 4}.$$



Hence  $\mathcal{L}^{-1}[Y(s)] = 3 + 5 \cos 2t - 2 \sin 2t$ .

9. The denominator  $s^2 + 4s + 5$  is *irreducible* over the reals. Completing the square,  $s^2 + 4s + 5 = (s + 2)^2 + 1$ . Now convert the function to a *rational function* of the variable  $\xi = s + 2$ . That is,

$$\frac{1 - 2s}{s^2 + 4s + 5} = \frac{5 - 2(s + 2)}{(s + 2)^2 + 1}.$$

We find that

$$\mathcal{L}^{-1}\left[\frac{5}{\xi^2 + 1} - \frac{2\xi}{\xi^2 + 1}\right] = 5 \sin t - 2 \cos t.$$

Using the fact that  $\mathcal{L}[e^{at}f(t)] = \mathcal{L}[f(t)]_{s \rightarrow s-a}$ ,

$$\mathcal{L}^{-1}\left[\frac{1 - 2s}{s^2 + 4s + 5}\right] = e^{-2t}(5 \sin t - 2 \cos t).$$

10. Note that the denominator  $s^2 + 2s + 10$  is *irreducible* over the reals. Completing the square,  $s^2 + 2s + 10 = (s + 1)^2 + 9$ . Now convert the function to a *rational function* of the variable  $\xi = s + 1$ . That is,

$$\frac{2s - 3}{s^2 + 2s + 10} = \frac{2(s + 1) - 5}{(s + 1)^2 + 9}.$$

We find that

$$\mathcal{L}^{-1}\left[\frac{2\xi}{\xi^2 + 9} - \frac{5}{\xi^2 + 9}\right] = 2 \cos 3t - \frac{5}{3} \sin 3t.$$

Using the fact that  $\mathcal{L}[e^{at}f(t)] = \mathcal{L}[f(t)]_{s \rightarrow s-a}$ ,

$$\mathcal{L}^{-1}\left[\frac{2s - 3}{s^2 + 2s + 10}\right] = e^{-t}\left(2 \cos 3t - \frac{5}{3} \sin 3t\right).$$

12. Taking the Laplace transform of the ODE, we obtain

$$s^2 Y(s) - s y(0) - y'(0) + 3[s Y(s) - y(0)] + 2 Y(s) = 0.$$

Applying the *initial conditions*,

$$s^2 Y(s) + 3s Y(s) + 2 Y(s) - s - 3 = 0.$$

Solving for  $Y(s)$ , the transform of the solution is

$$Y(s) = \frac{s + 3}{s^2 + 3s + 2}.$$

Using *partial fractions*,

$$\frac{s+3}{s^2+3s+2} = \frac{2}{s+1} - \frac{1}{s+2}.$$

Hence  $y(t) = \mathcal{L}^{-1}[Y(s)] = 2e^{-t} - e^{-2t}$ .

13. Taking the Laplace transform of the ODE, we obtain

$$s^2 Y(s) - s y(0) - y'(0) - 2[s Y(s) - y(0)] + 2 Y(s) = 0.$$

Applying the *initial conditions*,

$$s^2 Y(s) - 2s Y(s) + 2 Y(s) - 1 = 0.$$

Solving for  $Y(s)$ , the transform of the solution is

$$Y(s) = \frac{1}{s^2 - 2s + 2}.$$

Since the denominator is *irreducible*, write the transform as a function of  $\xi = s - 1$ . That is,

$$\frac{1}{s^2 - 2s + 2} = \frac{1}{(s-1)^2 + 1}.$$

First note that

$$\mathcal{L}^{-1}\left[\frac{1}{\xi^2 + 1}\right] = \sin t.$$

Using the fact that  $\mathcal{L}[e^{at}f(t)] = \mathcal{L}[f(t)]_{s \rightarrow s-a}$ ,

$$\mathcal{L}^{-1}\left[\frac{1}{s^2 - 2s + 2}\right] = e^t \sin t.$$

Hence  $y(t) = e^t \sin t$ .

15. Taking the Laplace transform of the ODE, we obtain

$$s^2 Y(s) - s y(0) - y'(0) - 2[s Y(s) - y(0)] - 2 Y(s) = 0.$$

Applying the *initial conditions*,

$$s^2 Y(s) - 2s Y(s) - 2 Y(s) - 2s + 4 = 0.$$

Solving for  $Y(s)$ , the transform of the solution is

$$Y(s) = \frac{2s-4}{s^2-2s-2}.$$

Since the denominator is *irreducible*, write the transform as a function of  $\xi = s - 1$ . Completing the square,

$$\frac{2s - 4}{s^2 - 2s - 2} = \frac{2(s - 1) - 2}{(s - 1)^2 - 3}.$$

First note that

$$\mathcal{L}^{-1} \left[ \frac{2\xi}{\xi^2 - 3} - \frac{2}{\xi^2 - 3} \right] = 2 \cosh \sqrt{3} t - \frac{2}{\sqrt{3}} \sinh \sqrt{3} t.$$

Using the fact that  $\mathcal{L}[e^{at}f(t)] = \mathcal{L}[f(t)]_{s \rightarrow s-a}$ , the solution of the IVP is

$$y(t) = \mathcal{L}^{-1} \left[ \frac{2s - 4}{s^2 - 2s - 2} \right] = e^t \left( 2 \cosh \sqrt{3} t - \frac{2}{\sqrt{3}} \sinh \sqrt{3} t \right).$$

16. Taking the Laplace transform of the ODE, we obtain

$$s^2 Y(s) - s y(0) - y'(0) + 2[s Y(s) - y(0)] + 5 Y(s) = 0.$$

Applying the *initial conditions*,

$$s^2 Y(s) + 2s Y(s) + 5 Y(s) - 2s - 3 = 0.$$

Solving for  $Y(s)$ , the transform of the solution is

$$Y(s) = \frac{2s + 3}{s^2 + 2s + 5}.$$

Since the denominator is *irreducible*, write the transform as a function of  $\xi = s + 1$ . That is,

$$\frac{2s + 3}{s^2 + 2s + 5} = \frac{2(s + 1) + 1}{(s + 1)^2 + 4}.$$

We know that

$$\mathcal{L}^{-1} \left[ \frac{2\xi}{\xi^2 + 4} + \frac{1}{\xi^2 + 4} \right] = 2 \cos 2t + \frac{1}{2} \sin 2t.$$

Using the fact that  $\mathcal{L}[e^{at}f(t)] = \mathcal{L}[f(t)]_{s \rightarrow s-a}$ , the solution of the IVP is

$$y(t) = \mathcal{L}^{-1} \left[ \frac{2s + 3}{s^2 + 2s + 5} \right] = e^{-t} \left( 2 \cos 2t + \frac{1}{2} \sin 2t \right).$$

17. Taking the Laplace transform of the ODE, we obtain

$$s^4 Y(s) - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) - 4[s^3 Y(s) - s^2 y(0) - s y'(0) - y''(0)] + 6[s^2 Y(s) - s y(0) - y'(0)] - 4[s Y(s) - y(0)] + Y(s) = 0$$

Applying the *initial conditions*,

$$s^4 Y(s) - 4s^3 Y(s) + 6s^2 Y(s) - 4s Y(s) + Y(s) - s^2 + 4s - 7 = 0.$$

Solving for the transform of the solution,

$$Y(s) = \frac{s^2 - 4s + 7}{s^4 - 4s^3 + 6s^2 - 4s + 1} = \frac{s^2 - 4s + 7}{(s - 1)^4}.$$

Using *partial fractions*,

$$\frac{s^2 - 4s + 7}{(s - 1)^4} = \frac{4}{(s - 1)^4} - \frac{2}{(s - 1)^3} + \frac{1}{(s - 1)^2}.$$

Note that  $\mathcal{L}[t^n] = (n!)/s^{n+1}$  and  $\mathcal{L}[e^{at} f(t)] = \mathcal{L}[f(t)]_{s \rightarrow s-a}$ . Hence the solution of the IVP is

$$y(t) = \mathcal{L}^{-1} \left[ \frac{s^2 - 4s + 7}{(s - 1)^4} \right] = \frac{2}{3} t^3 e^t - t^2 e^t + t e^t.$$

18. Taking the Laplace transform of the ODE, we obtain

$$s^4 Y(s) - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) - Y(s) = 0.$$

Applying the *initial conditions*,

$$s^4 Y(s) - Y(s) - s^3 - s = 0.$$

Solving for the transform of the solution,

$$Y(s) = \frac{s}{s^2 - 1}.$$

By inspection, it follows that  $y(t) = \mathcal{L}^{-1} \left[ \frac{s}{s^2 - 1} \right] = \cosh t$ .

19. Taking the Laplace transform of the ODE, we obtain

$$s^4 Y(s) - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) - 4Y(s) = 0.$$

Applying the *initial conditions*,

$$s^4 Y(s) - 4Y(s) - s^3 + 2s = 0.$$

Solving for the transform of the solution,

$$Y(s) = \frac{s}{s^2 + 2}.$$

It follows that  $y(t) = \mathcal{L}^{-1} \left[ \frac{s}{s^2 + 2} \right] = \cos \sqrt{2} t$ .

20. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2 Y(s) - s y(0) - y'(0) + \omega^2 Y(s) = \frac{s}{s^2 + 4}.$$

Applying the *initial conditions*,

$$s^2 Y(s) + \omega^2 Y(s) - s = \frac{s}{s^2 + 4}.$$

Solving for  $Y(s)$ , the transform of the solution is

$$Y(s) = \frac{s}{(s^2 + \omega^2)(s^2 + 4)} + \frac{s}{s^2 + \omega^2}.$$

Using *partial fractions* on the first term,

$$\frac{s}{(s^2 + \omega^2)(s^2 + 4)} = \frac{1}{4 - \omega^2} \left[ \frac{s}{s^2 + \omega^2} - \frac{s}{s^2 + 4} \right].$$

First note that

$$\mathcal{L}^{-1} \left[ \frac{s}{s^2 + \omega^2} \right] = \cos \omega t \quad \text{and} \quad \mathcal{L}^{-1} \left[ \frac{s}{s^2 + 4} \right] = \cos 2t.$$

Hence the solution of the IVP is

$$\begin{aligned} y(t) &= \frac{1}{4 - \omega^2} \cos \omega t - \frac{1}{4 - \omega^2} \cos 2t + \cos \omega t \\ &= \frac{5 - \omega^2}{4 - \omega^2} \cos \omega t - \frac{1}{4 - \omega^2} \cos 2t. \end{aligned}$$

21. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2 Y(s) - s y(0) - y'(0) - 2[s Y(s) - y(0)] + 2Y(s) = \frac{s}{s^2 + 1}.$$

Applying the *initial conditions*,

$$s^2 Y(s) - 2s Y(s) + 2Y(s) - s + 2 = \frac{s}{s^2 + 1}.$$

Solving for  $Y(s)$ , the transform of the solution is

$$Y(s) = \frac{s}{(s^2 - 2s + 2)(s^2 + 1)} + \frac{s - 2}{s^2 - 2s + 2}.$$

Using *partial fractions* on the first term,

$$\frac{s}{(s^2 - 2s + 2)(s^2 + 1)} = \frac{1}{5} \left[ \frac{s - 2}{s^2 + 1} - \frac{s - 4}{s^2 - 2s + 2} \right].$$

Thus we can write

$$Y(s) = \frac{1}{5} \frac{s}{s^2 + 1} - \frac{2}{5} \frac{1}{s^2 + 1} + \frac{2}{5} \frac{2s - 3}{s^2 - 2s + 2}.$$

For the *last term*, we note that  $s^2 - 2s + 2 = (s - 1)^2 + 1$ . So that

$$\frac{2s - 3}{s^2 - 2s + 2} = \frac{2(s - 1) - 1}{(s - 1)^2 + 1}.$$

We know that

$$\mathcal{L}^{-1} \left[ \frac{2\xi}{\xi^2 + 1} - \frac{1}{\xi^2 + 1} \right] = 2 \cos t - \sin t.$$

Based on the *translation property* of the Laplace transform,

$$\mathcal{L}^{-1} \left[ \frac{2s - 3}{s^2 - 2s + 2} \right] = e^t (2 \cos t - \sin t).$$

Combining the above, the solution of the IVP is

$$y(t) = \frac{1}{5} \cos t - \frac{2}{5} \sin t + \frac{2}{5} e^t (2 \cos t - \sin t).$$

23. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2 Y(s) - s y(0) - y'(0) + 2[s Y(s) - y(0)] + Y(s) = \frac{4}{s + 1}.$$

Applying the *initial conditions*,

$$s^2 Y(s) + 2s Y(s) + Y(s) - 2s - 3 = \frac{4}{s + 1}.$$

Solving for  $Y(s)$ , the transform of the solution is

$$Y(s) = \frac{4}{(s + 1)^3} + \frac{2s + 3}{(s + 1)^2}.$$

First write

$$\frac{2s + 3}{(s + 1)^2} = \frac{2(s + 1) + 1}{(s + 1)^2} = \frac{2}{s + 1} + \frac{1}{(s + 1)^2}.$$

We note that

$$\mathcal{L}^{-1} \left[ \frac{4}{\xi^3} + \frac{2}{\xi} + \frac{1}{\xi^2} \right] = 2t^2 + 2 + t.$$

So based on the *translation property* of the Laplace transform, the solution of the IVP is

$$y(t) = 2t^2e^{-t} + te^{-t} + 2e^{-t}.$$

25. Let  $f(t)$  be the *forcing function* on the right-hand-side. Taking the Laplace transform

of both sides of the ODE, we obtain

$$s^2 Y(s) - sy(0) - y'(0) + Y(s) = \mathcal{L}[f(t)].$$

Applying the *initial conditions*,

$$s^2 Y(s) + Y(s) = \mathcal{L}[f(t)].$$

Based on the definition of the Laplace transform,

$$\begin{aligned} \mathcal{L}[f(t)] &= \int_0^{\infty} f(t) e^{-st} dt \\ &= \int_0^1 t e^{-st} dt \\ &= \frac{1}{s^2} - \frac{e^{-s}}{s} - \frac{e^{-s}}{s^2}. \end{aligned}$$

Solving for the transform,

$$Y(s) = \frac{1}{s^2(s^2 + 1)} - e^{-s} \frac{s + 1}{s^2(s^2 + 1)}.$$

Using *partial fractions*,

$$\frac{1}{s^2(s^2 + 1)} = \frac{1}{s^2} - \frac{1}{s^2 + 1}$$

and

$$\frac{s}{s^2(s^2 + 1)} = \frac{1}{s} - \frac{s}{s^2 + 1}.$$

We find, by inspection, that

$$\mathcal{L}^{-1} \left[ \frac{1}{s^2(s^2 + 1)} \right] = t - \sin t.$$

Referring to *Line 13*, in Table 6.2.1,

$$\mathcal{L}[u_c(t)f(t - c)] = e^{-cs} \mathcal{L}[f(t)].$$

Let

$$\mathcal{L}[g(t)] = \frac{s + 1}{s^2(s^2 + 1)} = \frac{1}{s} + \frac{1}{s^2} - \frac{s}{s^2 + 1} - \frac{1}{s^2 + 1}.$$

Then  $g(t) = 1 + t - \cos t - \sin t$ . It follows, therefore, that

$$\mathcal{L}^{-1}\left[e^{-s} \cdot \frac{s+1}{s^2(s^2+1)}\right] = u_1(t)[1 + (t-1) - \cos(t-1) - \sin(t-1)].$$

Combining the above, the solution of the IVP is

$$y(t) = t - \sin t - u_1(t)[1 + (t-1) - \cos(t-1) - \sin(t-1)].$$

26. Let  $f(t)$  be the *forcing function* on the right-hand-side. Taking the Laplace transform

of both sides of the ODE, we obtain

$$s^2 Y(s) - s y(0) - y'(0) + 4 Y(s) = \mathcal{L}[f(t)].$$

Applying the *initial conditions*,

$$s^2 Y(s) + 4 Y(s) = \mathcal{L}[f(t)].$$

Based on the definition of the Laplace transform,

$$\begin{aligned} \mathcal{L}[f(t)] &= \int_0^{\infty} f(t) e^{-st} dt \\ &= \int_0^1 t e^{-st} dt + \int_1^{\infty} e^{-st} dt \\ &= \frac{1}{s^2} - \frac{e^{-s}}{s^2}. \end{aligned}$$

Solving for the transform,

$$Y(s) = \frac{1}{s^2(s^2+4)} - e^{-s} \frac{1}{s^2(s^2+4)}.$$

Using *partial fractions*,

$$\frac{1}{s^2(s^2+4)} = \frac{1}{4} \left[ \frac{1}{s^2} - \frac{1}{s^2+4} \right].$$

We find that

$$\mathcal{L}^{-1}\left[\frac{1}{s^2(s^2+4)}\right] = \frac{1}{4}t - \frac{1}{8}\sin t.$$

Referring to *Line 13*, in Table 6.2.1,

$$\mathcal{L}[u_c(t)f(t-c)] = e^{-cs}\mathcal{L}[f(t)].$$

It follows that



$$\mathcal{L}^{-1}\left[e^{-s} \cdot \frac{1}{s^2(s^2+4)}\right] = u_1(t) \left[ \frac{1}{4}(t-1) - \frac{1}{8} \sin(t-1) \right].$$

Combining the above, the solution of the IVP is

$$y(t) = \frac{1}{4}t - \frac{1}{8} \sin t - u_1(t) \left[ \frac{1}{4}(t-1) - \frac{1}{8} \sin(t-1) \right].$$

28(a). Assuming that the conditions of Theorem 6.2.1 are satisfied,

$$\begin{aligned} F'(s) &= \frac{d}{ds} \int_0^{\infty} e^{-st} f(t) dt \\ &= \int_0^{\infty} \frac{\partial}{\partial s} [e^{-st} f(t)] dt \\ &= \int_0^{\infty} [-t e^{-st} f(t)] dt \\ &= \int_0^{\infty} e^{-st} [-t f(t)] dt. \end{aligned}$$

(b). Using *mathematical induction*, suppose that for some  $k \geq 1$ ,

$$F^{(k)}(s) = \int_0^{\infty} e^{-st} [(-t)^k f(t)] dt.$$

Differentiating both sides,

$$\begin{aligned} F^{(k+1)}(s) &= \frac{d}{ds} \int_0^{\infty} e^{-st} [(-t)^k f(t)] dt \\ &= \int_0^{\infty} \frac{\partial}{\partial s} [e^{-st} (-t)^k f(t)] dt \\ &= \int_0^{\infty} [-t e^{-st} (-t)^k f(t)] dt \\ &= \int_0^{\infty} e^{-st} [(-t)^{k+1} f(t)] dt. \end{aligned}$$

29. We know that

$$\mathcal{L}[e^{at}] = \frac{1}{s-a}.$$

Based on Prob. 28,

$$\mathcal{L}[-t e^{at}] = \frac{d}{ds} \left[ \frac{1}{s-a} \right].$$

Therefore,

$$\mathcal{L}[t e^{at}] = \frac{1}{(s - a)^2}.$$

31. Based on Prob. 28,

$$\begin{aligned} \mathcal{L}[(-t)^n] &= \frac{d^n}{ds^n} \mathcal{L}[1] \\ &= \frac{d^n}{ds^n} \left[ \frac{1}{s} \right]. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathcal{L}[t^n] &= (-1)^n \frac{(-1)^n n!}{s^{n+1}} \\ &= \frac{n!}{s^{n+1}}. \end{aligned}$$

33. Using the *translation property* of the Laplace transform,

$$\mathcal{L}[e^{at} \sin bt] = \frac{b}{(s - a)^2 + b^2}.$$

Therefore,

$$\begin{aligned} \mathcal{L}[t e^{at} \sin bt] &= -\frac{d}{ds} \left[ \frac{b}{(s - a)^2 + b^2} \right] \\ &= \frac{2b(s - a)}{(s^2 - 2as + a^2 + b^2)^2}. \end{aligned}$$

34. Using the *translation property* of the Laplace transform,

$$\mathcal{L}[e^{at} \cos bt] = \frac{s - a}{(s - a)^2 + b^2}.$$

Therefore,

$$\begin{aligned} \mathcal{L}[t e^{at} \cos bt] &= -\frac{d}{ds} \left[ \frac{s - a}{(s - a)^2 + b^2} \right] \\ &= \frac{(s - a)^2 - b^2}{(s^2 - 2as + a^2 + b^2)^2}. \end{aligned}$$

35(a). Taking the Laplace transform of the given *Bessel equation*,

$$\mathcal{L}[ty''] + \mathcal{L}[y'] + \mathcal{L}[ty] = 0.$$

Using the *differentiation property* of the transform,

$$-\frac{d}{ds}\mathcal{L}[y''] + \mathcal{L}[y'] - \frac{d}{ds}\mathcal{L}[y] = 0.$$

That is,

$$-\frac{d}{ds}[s^2Y(s) - sy(0) - y'(0)] + sY(s) - y(0) - \frac{d}{ds}Y(s) = 0.$$

It follows that

$$(1 + s^2)Y'(s) + sY(s) = 0.$$

(b). We obtain a *first-order linear* ODE in  $Y(s)$ :

$$Y'(s) + \frac{s}{s^2 + 1}Y(s) = 0,$$

with *integrating factor*

$$\mu(s) = \exp\left(\int \frac{s}{s^2 + 1} ds\right) = \sqrt{s^2 + 1}.$$

The first-order ODE can be written as

$$\frac{d}{ds}[\sqrt{s^2 + 1} \cdot Y(s)] = 0,$$

with solution

$$Y(s) = \frac{c}{\sqrt{s^2 + 1}}.$$

(c). In order to obtain *negative* powers of  $s$ , first write

$$\frac{1}{\sqrt{s^2 + 1}} = \frac{1}{s} \left[1 + \frac{1}{s^2}\right]^{-1/2}.$$

Expanding  $\left(1 + \frac{1}{s^2}\right)^{-1/2}$  in a *binomial series*,

$$\frac{1}{\sqrt{1 + (1/s^2)}} = 1 - \frac{1}{2}s^{-2} + \frac{1 \cdot 3}{2 \cdot 4}s^{-4} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}s^{-6} + \dots,$$

valid for  $s^{-2} < 1$ . Hence, we can formally express  $Y(s)$  as

$$Y(s) = c \left[ \frac{1}{s} - \frac{1}{2} \frac{1}{s^3} + \frac{1 \cdot 3}{2 \cdot 4} \frac{1}{s^5} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \frac{1}{s^7} + \dots \right].$$

Assuming that *term-by-term* inversion is valid,

$$\begin{aligned} y(t) &= c \left[ 1 - \frac{1}{2} \frac{t^2}{2!} + \frac{1 \cdot 3}{2 \cdot 4} \frac{t^4}{4!} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \frac{t^6}{6!} + \dots \right] \\ &= c \left[ 1 - \frac{2!}{2^2} \frac{t^2}{2!} + \frac{4!}{2^2 \cdot 4^2} \frac{t^4}{4!} - \frac{6!}{2^2 \cdot 4^2 \cdot 6^2} \frac{t^6}{6!} + \dots \right]. \end{aligned}$$

It follows that

$$\begin{aligned} y(t) &= c \left[ 1 - \frac{1}{2^2} t^2 + \frac{1}{2^2 \cdot 4^2} t^4 - \frac{1}{2^2 \cdot 4^2 \cdot 6^2} t^6 + \dots \right] \\ &= c \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n} (n!)^2} t^{2n}. \end{aligned}$$

The series is evidently the expansion, about  $x = 0$ , of  $J_0(t)$ .

36(b). Taking the Laplace transform of the given *Legendre equation*,

$$\mathcal{L}[y''] - \mathcal{L}[t^2 y''] - 2\mathcal{L}[t y'] + \alpha(\alpha + 1)\mathcal{L}[y] = 0.$$

Using the *differentiation property* of the transform,

$$\mathcal{L}[y''] - \frac{d^2}{ds^2} \mathcal{L}[y''] + 2 \frac{d}{ds} \mathcal{L}[y'] + \alpha(\alpha + 1)\mathcal{L}[y] = 0.$$

That is,

$$\begin{aligned} [s^2 Y(s) - s y(0) - y'(0)] - \frac{d^2}{ds^2} [s^2 Y(s) - s y(0) - y'(0)] + \\ + 2 \frac{d}{ds} [s Y(s) - y(0)] + \alpha(\alpha + 1)Y(s) = 0. \end{aligned}$$

Invoking the *initial conditions*, we have

$$s^2 Y(s) - 1 - \frac{d^2}{ds^2} [s^2 Y(s) - 1] + 2 \frac{d}{ds} [s Y(s)] + \alpha(\alpha + 1)Y(s) = 0.$$

After carrying out the differentiation, the equation simplifies to

$$\frac{d^2}{ds^2} [s^2 Y(s)] - 2 \frac{d}{ds} [s Y(s)] - [s^2 + \alpha(\alpha + 1)]Y(s) = -1.$$

That is,

$$s^2 \frac{d^2}{ds^2} Y(s) + 2s \frac{d}{ds} Y(s) - [s^2 + \alpha(\alpha + 1)]Y(s) = -1.$$

37. By definition of the Laplace transform, given the appropriate conditions,

$$\begin{aligned}\mathcal{L}[g(t)] &= \int_0^{\infty} e^{-st} \left[ \int_0^t f(\tau) d\tau \right] dt \\ &= \int_0^{\infty} \int_0^t e^{-st} f(\tau) d\tau dt.\end{aligned}$$

Assuming that the order of integration can be exchanged,

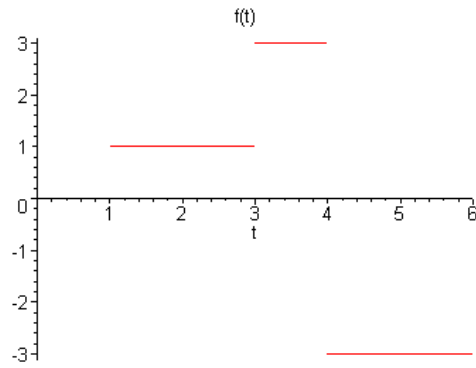
$$\begin{aligned}\mathcal{L}[g(t)] &= \int_0^{\infty} f(\tau) \left[ \int_{\tau}^{\infty} e^{-st} dt \right] d\tau \\ &= \int_0^{\infty} f(\tau) \left[ \frac{e^{-s\tau}}{s} \right] d\tau.\end{aligned}$$

[Note the *region* of integration is the area between the lines  $\tau(t) = t$  and  $\tau(t) = 0$ .]  
Hence

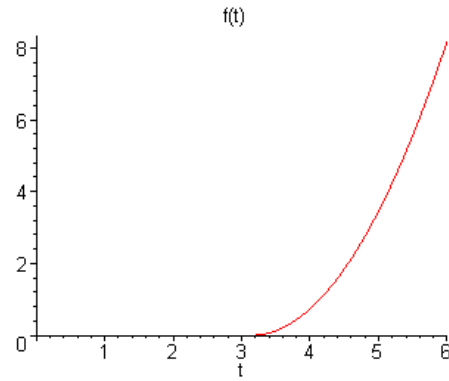
$$\begin{aligned}\mathcal{L}[g(t)] &= \frac{1}{s} \int_0^{\infty} f(\tau) e^{-s\tau} d\tau \\ &= \frac{1}{s} \mathcal{L}[f(t)].\end{aligned}$$

**Section 6.3**

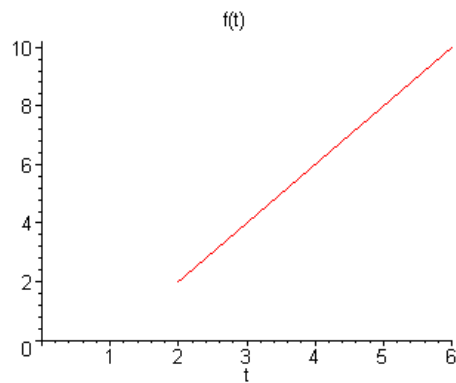
1.



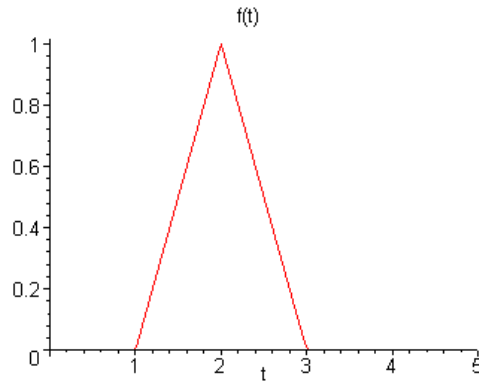
3.



5.



6.



7. Using the Heaviside function, we can write

$$f(t) = (t - 2)^2 u_2(t).$$

The Laplace transform has the property that

$$\mathcal{L}[u_c(t)f(t - c)] = e^{-cs} \mathcal{L}[f(t)].$$

Hence

$$\mathcal{L}[(t - 2)^2 u_2(t)] = \frac{2e^{-2s}}{s^2}.$$

9. The function can be expressed as

$$f(t) = (t - \pi)[u_\pi(t) - u_{2\pi}(t)].$$

Before invoking the *translation property* of the transform, write the function as

$$f(t) = (t - \pi) u_\pi(t) - (t - 2\pi) u_{2\pi}(t) - \pi u_{2\pi}(t).$$

It follows that

$$\mathcal{L}[f(t)] = \frac{e^{-\pi s}}{s^2} - \frac{e^{-2\pi s}}{s^2} - \frac{\pi e^{-2\pi s}}{s}.$$

10. It follows directly from the *translation property* of the transform that

$$\mathcal{L}[f(t)] = \frac{e^{-s}}{s} + 2\frac{e^{-3s}}{s} - 6\frac{e^{-4s}}{s}.$$

11. Before invoking the *translation property* of the transform, write the function as

$$f(t) = (t - 2) u_2(t) - u_2(t) - (t - 3) u_3(t) - u_3(t).$$

It follows that

$$\mathcal{L}[f(t)] = \frac{e^{-2s}}{s^2} - \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s^2} - \frac{e^{-3s}}{s}.$$

12. It follows directly from the *translation property* of the transform that

$$\mathcal{L}[f(t)] = \frac{1}{s^2} - \frac{e^{-s}}{s^2}.$$

13. Using the fact that  $\mathcal{L}[e^{at}f(t)] = \mathcal{L}[f(t)]_{s \rightarrow s-a}$ ,

$$\mathcal{L}^{-1}\left[\frac{3!}{(s-2)^4}\right] = t^3 e^{2t}.$$

15. First consider the function

$$G(s) = \frac{2(s-1)}{s^2 - 2s + 2}.$$

Completing the square in the denominator,

$$G(s) = \frac{2(s-1)}{(s-1)^2 + 1}.$$

It follows that

$$\mathcal{L}^{-1}[G(s)] = 2e^t \cos t.$$

Hence

$$\mathcal{L}^{-1}[e^{-2s}G(s)] = 2e^{(t-2)} \cos(t-2) u_2(t).$$

16. The *inverse transform* of the function  $2/(s^2 - 4)$  is  $f(t) = \sinh 2t$ . Using the *translation property* of the transform,

$$\mathcal{L}^{-1}\left[\frac{2e^{-2s}}{s^2 - 4}\right] = \sinh 2(t-2) \cdot u_2(t).$$

17. First consider the function

$$G(s) = \frac{(s-2)}{s^2 - 4s + 3}.$$

Completing the square in the denominator,



$$G(s) = \frac{(s-2)}{(s-2)^2 - 1}.$$

It follows that

$$\mathcal{L}^{-1}[G(s)] = e^{2t} \cosh t.$$

Hence

$$\mathcal{L}^{-1}\left[\frac{(s-2)e^{-s}}{s^2 - 4s + 3}\right] = e^{2(t-1)} \cosh(t-1) u_1(t).$$

18. Write the function as

$$F(s) = \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} - \frac{e^{-4s}}{s}.$$

It follows from the *translation property* of the transform, that

$$\mathcal{L}^{-1}\left[\frac{e^{-s} + e^{-2s} - e^{-3s} - e^{-4s}}{s}\right] = u_1(t) + u_2(t) - u_3(t) - u_4(t).$$

19(a). By definition of the Laplace transform,

$$\mathcal{L}[f(ct)] = \int_0^{\infty} e^{-st} f(ct) dt.$$

Making a change of variable,  $\tau = ct$ , we have

$$\begin{aligned} \mathcal{L}[f(ct)] &= \frac{1}{c} \int_0^{\infty} e^{-s(\tau/c)} f(\tau) d\tau \\ &= \frac{1}{c} \int_0^{\infty} e^{-(s/c)\tau} f(\tau) d\tau. \end{aligned}$$

Hence  $\mathcal{L}[f(ct)] = \frac{1}{c} F\left(\frac{s}{c}\right)$ , where  $s/c > a$ .

(b). Using the result in Part (a),

$$\mathcal{L}\left[f\left(\frac{t}{k}\right)\right] = k F(ks).$$

Hence

$$\mathcal{L}^{-1}[F(ks)] = \frac{1}{k} f\left(\frac{t}{k}\right).$$

(c). From Part (b),

$$\mathcal{L}^{-1}[F(as)] = \frac{1}{a} f\left(\frac{t}{a}\right).$$

Note that  $as + b = a(s + b/a)$ . Using the fact that  $\mathcal{L}[e^{ct}f(t)] = \mathcal{L}[f(t)]_{s \rightarrow s-c}$ ,

$$\mathcal{L}^{-1}[F(as + b)] = e^{-bt/a} \frac{1}{a} f\left(\frac{t}{a}\right).$$

20. First write

$$F(s) = \frac{n!}{\left(\frac{s}{2}\right)^{n+1}}.$$

Let  $G(s) = n!/s^{n+1}$ . Based on the results in Prob. 19,

$$\frac{1}{2} \mathcal{L}^{-1}\left[G\left(\frac{s}{2}\right)\right] = g(2t),$$

in which  $g(t) = t^n$ . Hence

$$\mathcal{L}^{-1}[F(s)] = 2(2t)^n = 2^{n+1}t^n.$$

23. First write

$$F(s) = \frac{e^{-4(s-1/2)}}{2(s-1/2)}.$$

Now consider

$$G(s) = \frac{e^{-2s}}{s}.$$

Using the result in Prob. 19(b),

$$\mathcal{L}^{-1}[G(2s)] = \frac{1}{2} g\left(\frac{t}{2}\right),$$

in which  $g(t) = u_2(t)$ . Hence  $\mathcal{L}^{-1}[G(2s)] = \frac{1}{2} u_2(t/2) = \frac{1}{2} u_4(t)$ . It follows that

$$\mathcal{L}^{-1}[F(s)] = \frac{1}{2} e^{t/2} u_4(t).$$

24. By definition of the Laplace transform,

$$\mathcal{L}[f(t)] = \int_0^{\infty} e^{-st} u_1(t) dt.$$

That is,

$$\begin{aligned} \mathcal{L}[f(t)] &= \int_0^1 e^{-st} dt \\ &= \frac{1 - e^{-s}}{s}. \end{aligned}$$

25. First write the function as  $f(t) = u_0(t) - u_1(t) + u_2(t) - u_3(t)$ . It follows that

$$\mathcal{L}[f(t)] = \int_0^1 e^{-st} dt + \int_2^3 e^{-st} dt.$$

That is,

$$\begin{aligned} \mathcal{L}[f(t)] &= \frac{1 - e^{-s}}{s} + \frac{e^{-2s} - e^{-3s}}{s} \\ &= \frac{1 - e^{-s} + e^{-2s} - e^{-3s}}{s}. \end{aligned}$$

26. The transform may be computed directly. On the other hand, using the *translation property* of the transform,

$$\begin{aligned} \mathcal{L}[f(t)] &= \frac{1}{s} + \sum_{k=1}^{2n+1} (-1)^k \frac{e^{-ks}}{s} \\ &= \frac{1}{s} \left[ \sum_{k=0}^{2n+1} (-e^{-s})^k \right] \\ &= \frac{1}{s} \frac{1 - (-e^{-s})^{2n+2}}{1 + e^{-s}}. \end{aligned}$$

That is,

$$\mathcal{L}[f(t)] = \frac{1 - (e^{-2s})^{n+1}}{s(1 + e^{-s})}.$$

29. The given function is *periodic*, with  $T = 2$ . Using the result of Prob. 28,

$$\mathcal{L}[f(t)] = \frac{1}{1 - e^{-2s}} \int_0^2 e^{-st} f(t) dt = \frac{1}{1 - e^{-2s}} \int_0^1 e^{-st} dt.$$

That is,

$$\begin{aligned}\mathcal{L}[f(t)] &= \frac{1 - e^{-s}}{s(1 - e^{-2s})} \\ &= \frac{1}{s(1 + e^{-s})}.\end{aligned}$$

31. The function is *periodic*, with  $T = 1$ . Using the result of Prob. 28,

$$\mathcal{L}[f(t)] = \frac{1}{1 - e^{-s}} \int_0^1 t e^{-st} dt.$$

It follows that

$$\mathcal{L}[f(t)] = \frac{1 - e^{-s}(1 + s)}{s^2(1 - e^{-s})}.$$

32. The function is *periodic*, with  $T = \pi$ . Using the result of Prob. 28,

$$\mathcal{L}[f(t)] = \frac{1}{1 - e^{-\pi s}} \int_0^\pi \sin t \cdot e^{-st} dt.$$

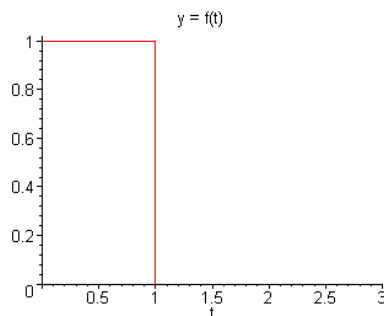
We first calculate

$$\int_0^\pi \sin t \cdot e^{-st} dt = \frac{1 + e^{-\pi s}}{1 + s^2}.$$

Hence

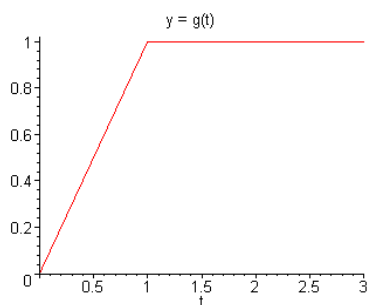
$$\mathcal{L}[f(t)] = \frac{1 + e^{-\pi s}}{(1 - e^{-\pi s})(1 + s^2)}.$$

33(a).



$$\begin{aligned}\mathcal{L}[f(t)] &= \mathcal{L}[1] - \mathcal{L}[u_1(t)] \\ &= \frac{1}{s} - \frac{e^{-s}}{s}.\end{aligned}$$

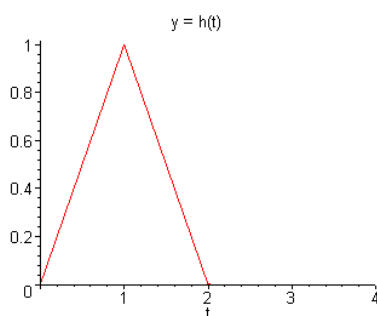
(b).



Let  $F(s) = \mathcal{L}[1 - u_1(t)]$ . Then

$$\mathcal{L}\left[\int_0^t [1 - u_1(\tau)] d\tau\right] = \frac{1}{s} F(s) = \frac{1 - e^{-s}}{s^2}.$$

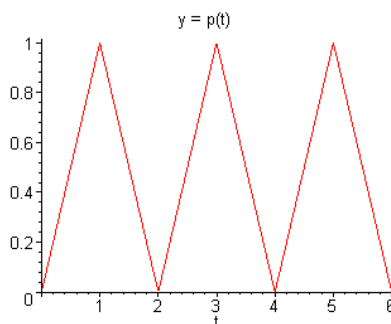
(c).



Let  $G(s) = \mathcal{L}[g(t)]$ . Then

$$\begin{aligned} \mathcal{L}[h(t)] &= G(s) - e^{-s} G(s) \\ &= \frac{1 - e^{-s}}{s^2} - e^{-s} \frac{1 - e^{-s}}{s^2} \\ &= \frac{(1 - e^{-s})^2}{s^2}. \end{aligned}$$

34(a).



(b). The given function is *periodic*, with  $T = 2$ . Using the result of Prob. 28,

$$\mathcal{L}[f(t)] = \frac{1}{1 - e^{-2s}} \int_0^2 e^{-st} p(t) dt.$$

Based on the piecewise definition of  $p(t)$ ,

$$\begin{aligned} \int_0^2 e^{-st} p(t) dt &= \int_0^1 t e^{-st} dt + \int_1^2 (2-t) e^{-st} dt \\ &= \frac{1}{s^2} (1 - e^{-s})^2. \end{aligned}$$

Hence

$$\mathcal{L}[p(t)] = \frac{(1 - e^{-s})}{s^2(1 + e^{-s})}.$$

(c). Since  $p(t)$  satisfies the hypotheses of Theorem 6.2.1,

$$\mathcal{L}[p'(t)] = s \mathcal{L}[p(t)] - p(0).$$

Using the result of Prob. 30,

$$\mathcal{L}[p'(t)] = \frac{(1 - e^{-s})}{s(1 + e^{-s})}.$$

We note the  $p(0) = 0$ , hence

$$\mathcal{L}[p(t)] = \frac{1}{s} \left[ \frac{(1 - e^{-s})}{s(1 + e^{-s})} \right].$$

## Section 6.4

2. Let  $h(t)$  be the *forcing function* on the right-hand-side. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2 Y(s) - s y(0) - y'(0) + 2[s Y(s) - y(0)] + 2 Y(s) = \mathcal{L}[h(t)].$$

Applying the initial conditions,

$$s^2 Y(s) + 2s Y(s) + 2 Y(s) - 1 = \mathcal{L}[h(t)].$$

The forcing function can be written as  $h(t) = u_\pi(t) - u_{2\pi}(t)$ . Its transform is

$$\mathcal{L}[h(t)] = \frac{e^{-\pi s} - e^{-2\pi s}}{s}.$$

Solving for  $Y(s)$ , the transform of the solution is

$$Y(s) = \frac{1}{s^2 + 2s + 2} + \frac{e^{-\pi s} - e^{-2\pi s}}{s(s^2 + 2s + 2)}.$$

First note that

$$\frac{1}{s^2 + 2s + 2} = \frac{1}{(s + 1)^2 + 1}.$$

Using partial fractions,

$$\frac{1}{s(s^2 + 2s + 2)} = \frac{1}{2} \frac{1}{s} - \frac{1}{2} \frac{(s + 1) + 1}{(s + 1)^2 + 1}.$$

Taking the inverse transform, term-by-term,

$$\mathcal{L}^{-1}\left[\frac{1}{s^2 + 2s + 2}\right] = \mathcal{L}^{-1}\left[\frac{1}{(s + 1)^2 + 1}\right] = e^{-t} \sin t.$$

Now let

$$G(s) = \frac{1}{s(s^2 + 2s + 2)}.$$

Then

$$\mathcal{L}^{-1}[G(s)] = \frac{1}{2} - \frac{1}{2} e^{-t} \cos t - \frac{1}{2} e^{-t} \sin t.$$

Using Theorem 6.3.1,

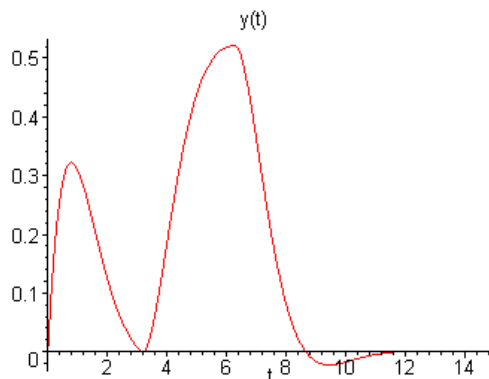
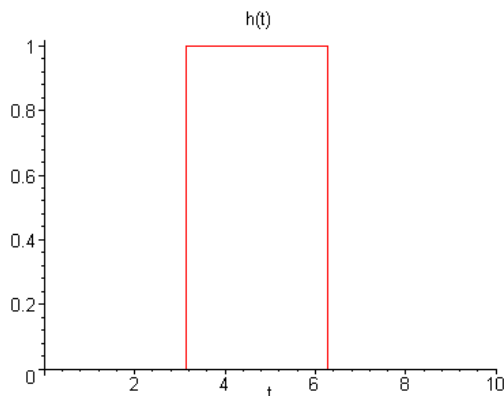
$$\mathcal{L}^{-1}[e^{-cs} G(s)] = \frac{1}{2} u_c(t) - \frac{1}{2} e^{-(t-c)} [\cos(t-c) + \sin(t-c)] u_c(t).$$

Hence the solution of the IVP is

$$y(t) = e^{-t} \sin t + \frac{1}{2} u_{\pi}(t) - \frac{1}{2} e^{-(t-\pi)} [\cos(t-\pi) + \sin(t-\pi)] u_{\pi}(t) - \frac{1}{2} u_{2\pi}(t) + \frac{1}{2} e^{-(t-2\pi)} [\cos(t-2\pi) + \sin(t-2\pi)] u_{2\pi}(t).$$

That is,

$$y(t) = e^{-t} \sin t + \frac{1}{2} [u_{\pi}(t) - u_{2\pi}(t)] + \frac{1}{2} e^{-(t-\pi)} [\cos t + \sin t] u_{\pi}(t) + \frac{1}{2} e^{-(t-2\pi)} [\cos t + \sin t] u_{2\pi}(t).$$



The solution starts out as free oscillation, due to the initial conditions. The amplitude increases, as long as the forcing is present. Thereafter, the solution rapidly decays.

4. Let  $h(t)$  be the *forcing function* on the right-hand-side. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2 Y(s) - s y(0) - y'(0) + 4Y(s) = \mathcal{L}[h(t)].$$

Applying the initial conditions,



$$s^2 Y(s) + 4Y(s) = \mathcal{L}[h(t)].$$

The transform of the forcing function is

$$\mathcal{L}[h(t)] = \frac{1}{s^2 + 1} + \frac{e^{-\pi s}}{s^2 + 1}.$$

Solving for  $Y(s)$ , the transform of the solution is

$$Y(s) = \frac{1}{(s^2 + 4)(s^2 + 1)} + \frac{e^{-\pi s}}{(s^2 + 4)(s^2 + 1)}.$$

Using partial fractions,

$$\frac{1}{(s^2 + 4)(s^2 + 1)} = \frac{1}{3} \left[ \frac{1}{s^2 + 1} - \frac{1}{s^2 + 4} \right].$$

It follows that

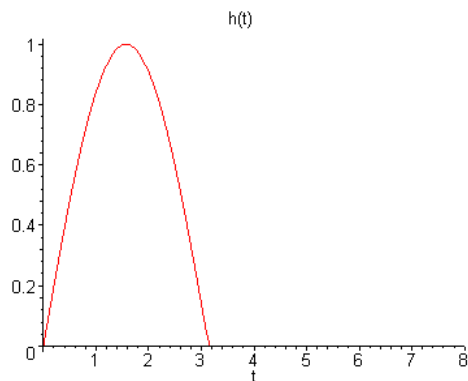
$$\mathcal{L}^{-1} \left[ \frac{1}{(s^2 + 4)(s^2 + 1)} \right] = \frac{1}{3} \left[ \sin t - \frac{1}{2} \sin 2t \right].$$

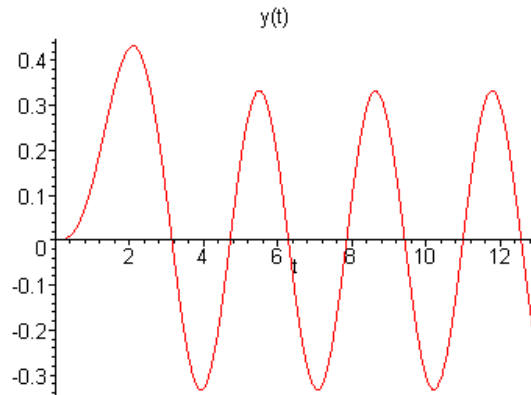
Based on Theorem 6.3.1,

$$\mathcal{L}^{-1} \left[ \frac{e^{-\pi s}}{(s^2 + 4)(s^2 + 1)} \right] = \frac{1}{3} \left[ \sin(t - \pi) - \frac{1}{2} \sin(2t - 2\pi) \right] u_\pi(t).$$

Hence the solution of the IVP is

$$y(t) = \frac{1}{3} \left[ \sin t - \frac{1}{2} \sin 2t \right] - \frac{1}{3} \left[ \sin t + \frac{1}{2} \sin 2t \right] u_\pi(t).$$





Since there is no *damping term*, the solution follows the forcing function, after which the response is a steady oscillation about  $y = 0$ .

5. Let  $f(t)$  be the *forcing function* on the right-hand-side. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2 Y(s) - s y(0) - y'(0) + 3[s Y(s) - y(0)] + 2 Y(s) = \mathcal{L}[f(t)].$$

Applying the initial conditions,

$$s^2 Y(s) + 3s Y(s) + 2 Y(s) = \mathcal{L}[f(t)].$$

The transform of the forcing function is

$$\mathcal{L}[f(t)] = \frac{1}{s} - \frac{e^{-10s}}{s}.$$

Solving for the transform,

$$Y(s) = \frac{1}{s(s^2 + 3s + 2)} - \frac{e^{-10s}}{s(s^2 + 3s + 2)}.$$

Using partial fractions,

$$\frac{1}{s(s^2 + 3s + 2)} = \frac{1}{2} \left[ \frac{1}{s} + \frac{1}{s+2} - \frac{2}{s+1} \right].$$

Hence

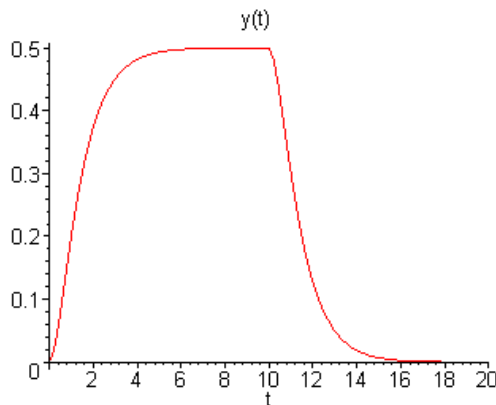
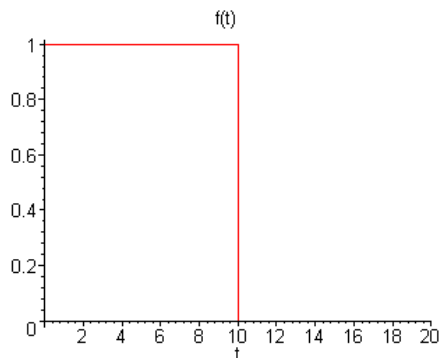
$$\mathcal{L}^{-1} \left[ \frac{1}{s(s^2 + 3s + 2)} \right] = \frac{1}{2} + \frac{e^{-2t}}{2} - e^{-t}.$$

Based on Theorem 6.3.1,

$$\mathcal{L}^{-1} \left[ \frac{e^{-10s}}{s(s^2 + 3s + 2)} \right] = \frac{1}{2} [1 + e^{-2(t-10)} - 2e^{-(t-10)}] u_{10}(t).$$

Hence the solution of the IVP is

$$y(t) = \frac{1}{2}[1 - u_{10}(t)] + \frac{e^{-2t}}{2} - e^{-t} - \frac{1}{2}[e^{-(2t-20)} - 2e^{-(t-10)}]u_{10}(t).$$



The solution increases to a *temporary* steady value of  $y = 1/2$ . After the forcing ceases, the response decays exponentially to  $y = 0$ .

6. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2 Y(s) - s y(0) - y'(0) + 3[s Y(s) - y(0)] + 2 Y(s) = \frac{e^{-2s}}{s}.$$

Applying the initial conditions,

$$s^2 Y(s) + 3s Y(s) + 2 Y(s) - 1 = \frac{e^{-2s}}{s}.$$

Solving for the transform,

$$Y(s) = \frac{1}{s^2 + 3s + 2} + \frac{e^{-2s}}{s(s^2 + 3s + 2)}.$$

Using partial fractions,

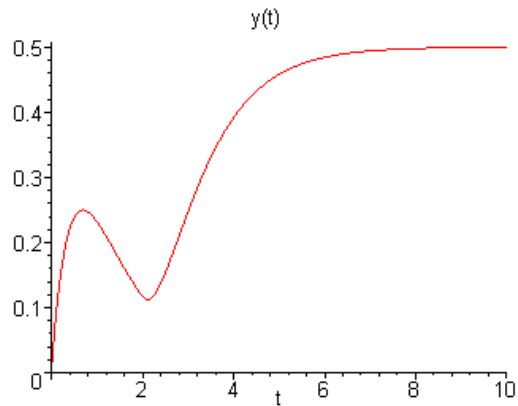
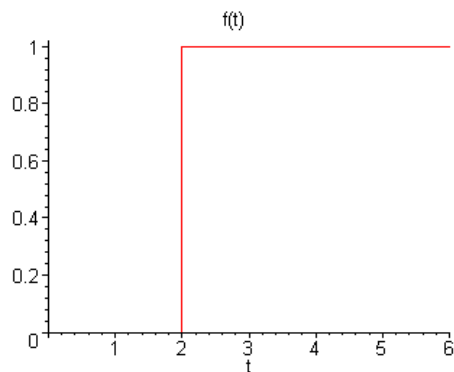
$$\frac{1}{s^2 + 3s + 2} = \frac{1}{s + 1} - \frac{1}{s + 2}$$

and

$$\frac{1}{s(s^2 + 3s + 2)} = \frac{1}{2} \left[ \frac{1}{s} + \frac{1}{s + 2} - \frac{2}{s + 1} \right].$$

Taking the inverse transform, term-by-term, the solution of the IVP is

$$y(t) = e^{-t} - e^{-2t} + \left[ \frac{1}{2} - e^{-(t-2)} + \frac{1}{2}e^{-2(t-2)} \right] u_2(t).$$



Due to the initial conditions, the response has a transient *overshoot*, followed by an exponential convergence to a steady value of  $y_s = 1/2$ .

7. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2 Y(s) - s y(0) - y'(0) + Y(s) = \frac{e^{-3\pi s}}{s}.$$

Applying the initial conditions,

$$s^2 Y(s) + Y(s) - s = \frac{e^{-3\pi s}}{s}.$$

Solving for the transform,

$$Y(s) = \frac{s}{s^2 + 1} + \frac{e^{-3\pi s}}{s(s^2 + 1)}.$$

Using partial fractions,

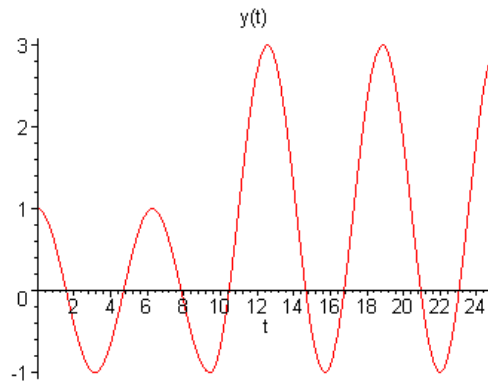
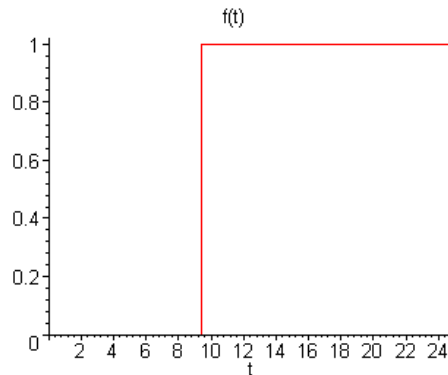
$$\frac{1}{s(s^2 + 1)} = \frac{1}{s} - \frac{s}{s^2 + 1}.$$

Hence

$$Y(s) = \frac{s}{s^2 + 1} + e^{-3\pi s} \left[ \frac{1}{s} - \frac{s}{s^2 + 1} \right].$$

Taking the inverse transform, the solution of the IVP is

$$\begin{aligned} y(t) &= \cos t + [1 - \cos(t - 3\pi)]u_{3\pi}(t) \\ &= \cos t + [1 + \cos t]u_{3\pi}(t). \end{aligned}$$



Due to initial conditions, the solution temporarily oscillates about  $y = 0$ . After the forcing is applied, the response is a steady oscillation about  $y_m = 1$ .

9. Let  $g(t)$  be the *forcing function* on the right-hand-side. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2 Y(s) - s y(0) - y'(0) + Y(s) = \mathcal{L}[g(t)].$$

Applying the initial conditions,

$$s^2 Y(s) + Y(s) - 1 = \mathcal{L}[g(t)].$$

The forcing function can be written as

$$\begin{aligned} g(t) &= \frac{t}{2}[1 - u_6(t)] + 3u_6(t) \\ &= \frac{t}{2} - \frac{1}{2}(t - 6)u_6(t) \end{aligned}$$

with Laplace transform

$$\mathcal{L}[g(t)] = \frac{1}{2s^2} - \frac{e^{-6s}}{2s^2}.$$

Solving for the transform,

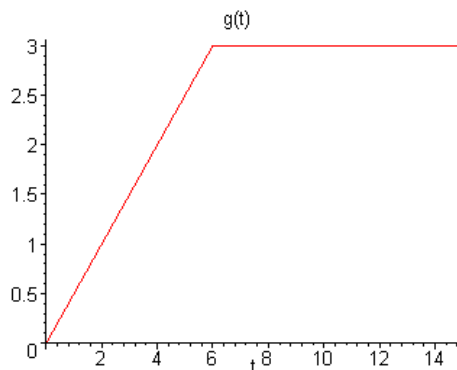
$$Y(s) = \frac{1}{s^2 + 1} + \frac{1}{2s^2(s^2 + 1)} - \frac{e^{-6s}}{2s^2(s^2 + 1)}.$$

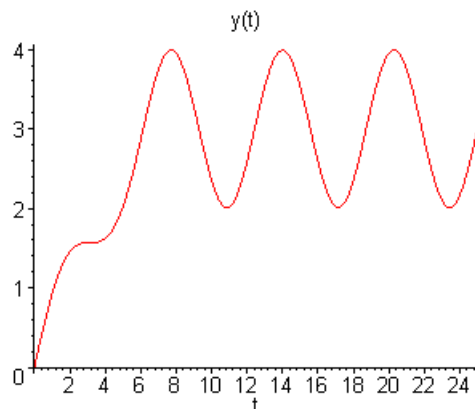
Using partial fractions,

$$\frac{1}{2s^2(s^2 + 1)} = \frac{1}{2} \left[ \frac{1}{s^2} - \frac{1}{s^2 + 1} \right].$$

Taking the inverse transform, and using Theorem 6.3.1, the solution of the IVP is

$$\begin{aligned} y(t) &= \sin t + \frac{1}{2}[t - \sin t] - \frac{1}{2}[(t - 6) - \sin(t - 6)]u_6(t) \\ &= \frac{1}{2}[t + \sin t] - \frac{1}{2}[(t - 6) - \sin(t - 6)]u_6(t). \end{aligned}$$





The solution increases, in response to the *ramp input*, and thereafter oscillates about a mean value of  $y_m = 3$ .

11. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2 Y(s) - s y(0) - y'(0) + 4Y(s) = \frac{e^{-\pi s}}{s} - \frac{e^{-3\pi s}}{s}.$$

Applying the initial conditions,

$$s^2 Y(s) + 4Y(s) = \frac{e^{-\pi s}}{s} - \frac{e^{-3\pi s}}{s}.$$

Solving for the transform,

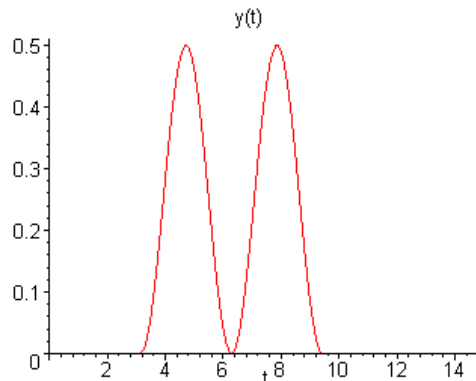
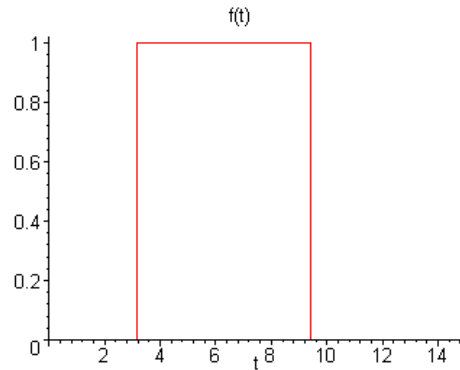
$$Y(s) = \frac{e^{-\pi s}}{s(s^2 + 4)} - \frac{e^{-3\pi s}}{s(s^2 + 4)}.$$

Using partial fractions,

$$\frac{1}{s(s^2 + 4)} = \frac{1}{4} \left[ \frac{1}{s} - \frac{s}{s^2 + 4} \right].$$

Taking the inverse transform, and applying Theorem 6.3.1,

$$\begin{aligned} y(t) &= \frac{1}{4} [1 - \cos(2t - 2\pi)] u_\pi(t) - \frac{1}{4} [1 - \cos(2t - 6\pi)] u_{3\pi}(t) \\ &= \frac{1}{4} [u_\pi(t) - u_{3\pi}(t)] - \frac{1}{4} \cos 2t \cdot [u_\pi(t) - u_{3\pi}(t)]. \end{aligned}$$



Since there is no damping term, the solution responds immediately to the forcing input. There is a temporary oscillation about  $y = 1/4$ .

12. Taking the Laplace transform of the ODE, we obtain

$$s^4 Y(s) - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) - Y(s) = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}.$$

Applying the *initial conditions*,

$$s^4 Y(s) - Y(s) = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}.$$

Solving for the transform of the solution,

$$Y(s) = \frac{e^{-s}}{s(s^4 - 1)} - \frac{e^{-2s}}{s(s^4 - 1)}.$$

Using partial fractions,

$$\frac{1}{s(s^4 - 1)} = \frac{1}{4} \left[ -\frac{4}{s} + \frac{1}{s+1} + \frac{1}{s-1} + \frac{2s}{s^2+1} \right].$$

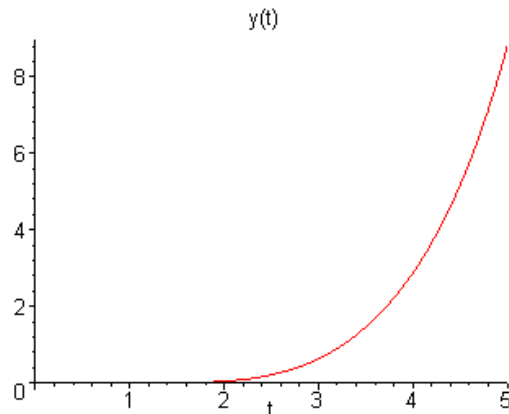
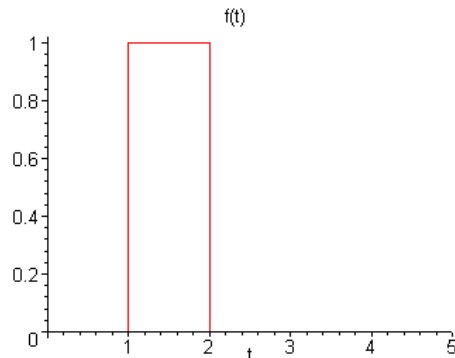
It follows that



$$\mathcal{L}^{-1}\left[\frac{1}{s(s^4-1)}\right] = \frac{1}{4}[-4 + e^{-t} + e^t + 2\cos t].$$

Based on Theorem 6.3.1, the solution of the IVP is

$$y(t) = -[u_1(t) - u_2(t)] + \frac{1}{4}[e^{-(t-1)} + e^{(t-1)} + 2\cos(t-1)]u_1(t) - \frac{1}{4}[e^{-(t-2)} + e^{(t-2)} + 2\cos(t-2)]u_2(t).$$



The solution increases without bound, exponentially.

13. Taking the Laplace transform of the ODE, we obtain

$$s^4 Y(s) - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) + 5[s^2 Y(s) - s y(0) - y'(0)] + 4 Y(s) = \frac{1}{s} - \frac{e^{-\pi s}}{s}.$$

Applying the *initial conditions*,

$$s^4 Y(s) + 5s^2 Y(s) + 4 Y(s) = \frac{1}{s} - \frac{e^{-\pi s}}{s}.$$

Solving for the transform of the solution,

$$Y(s) = \frac{1}{s(s^4 + 5s^2 + 4)} - \frac{e^{-\pi s}}{s(s^4 + 5s^2 + 4)}.$$

Using partial fractions,

$$\frac{1}{s(s^4 + 5s^2 + 4)} = \frac{1}{12} \left[ \frac{3}{s} + \frac{s}{s^2 + 4} - \frac{4s}{s^2 + 1} \right].$$

It follows that

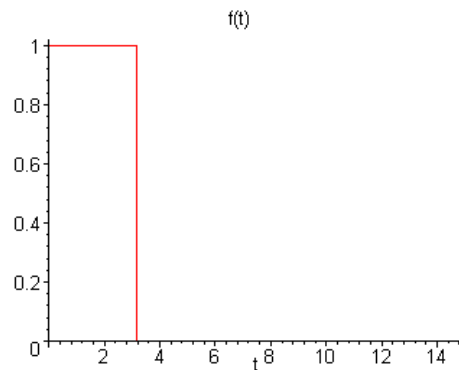
$$\mathcal{L}^{-1} \left[ \frac{1}{s(s^4 + 5s^2 + 4)} \right] = \frac{1}{12} [3 + \cos 2t - 4 \cos t].$$

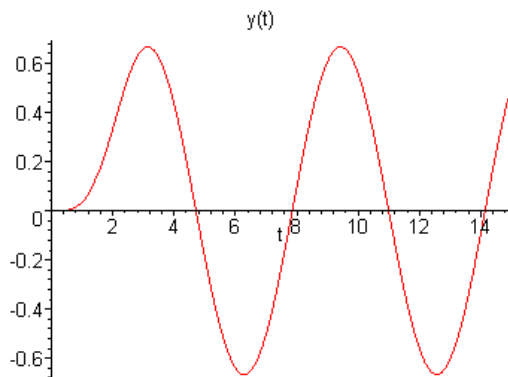
Based on Theorem 6.3.1, the solution of the IVP is

$$\begin{aligned} y(t) &= \frac{1}{4} [1 - u_\pi(t)] + \frac{1}{12} [\cos 2t - 4 \cos t] - \\ &\quad - \frac{1}{12} [\cos 2(t - \pi) - 4 \cos(t - \pi)] u_\pi(t). \end{aligned}$$

That is,

$$\begin{aligned} y(t) &= \frac{1}{4} [1 - u_\pi(t)] + \frac{1}{12} [\cos 2t - 4 \cos t] - \\ &\quad - \frac{1}{12} [\cos 2t + 4 \cos t] u_\pi(t). \end{aligned}$$





After an initial transient, the solution oscillates about  $y_m = 0$ .

14. The specified function is defined by

$$f(t) = \begin{cases} 0, & 0 \leq t < t_0 \\ \frac{h}{k}(t - t_0), & t_0 \leq t < t_0 + k \\ h, & t \geq t_0 + k \end{cases}$$

which can conveniently be expressed as

$$f(t) = \frac{h}{k}(t - t_0) u_{t_0}(t) - \frac{h}{k}(t - t_0 - k) u_{t_0+k}(t).$$

15. The function is defined by

$$g(t) = \begin{cases} 0, & 0 \leq t < t_0 \\ \frac{h}{k}(t - t_0), & t_0 \leq t < t_0 + k \\ -\frac{h}{k}(t - t_0 - 2k), & t_0 + k \leq t < t_0 + 2k \\ 0, & t \geq t_0 + 2k \end{cases}$$

which can also be written as

$$g(t) = \frac{h}{k}(t - t_0) u_{t_0}(t) - \frac{2h}{k}(t - t_0 - k) u_{t_0+k}(t) + \frac{h}{k}(t - t_0 - 2k) u_{t_0+2k}(t).$$

16(d). From Part (c), the solution is

$$u(t) = 4k u_{3/2}(t) h \left( t - \frac{3}{2} \right) - 4k u_{5/2}(t) h \left( t - \frac{5}{2} \right),$$

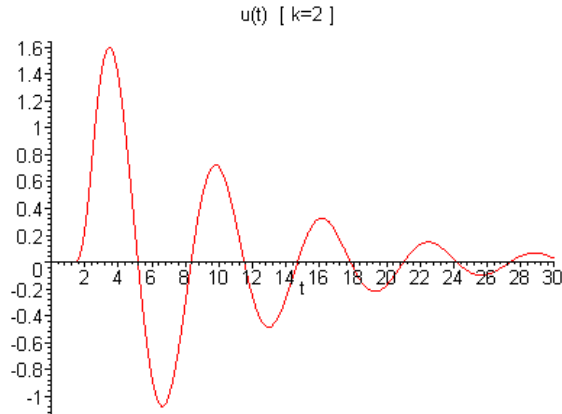
where

$$h(t) = \frac{1}{4} - \frac{\sqrt{7}}{84} e^{-t/8} \sin \left( \frac{3\sqrt{7}t}{8} \right) - \frac{1}{4} e^{-t/8} \cos \left( \frac{3\sqrt{7}t}{8} \right).$$

Due to the *damping term*, the solution will decay to *zero*. The maximum will occur

shortly after the forcing ceases. By plotting the various solutions, it appears that the solution will reach a value of  $y = 2$ , as long as  $k > 2.51$ .

(e).



Based on the graph, and numerical calculation,  $|u(t)| < 0.1$  for  $t > 25.6773$ .

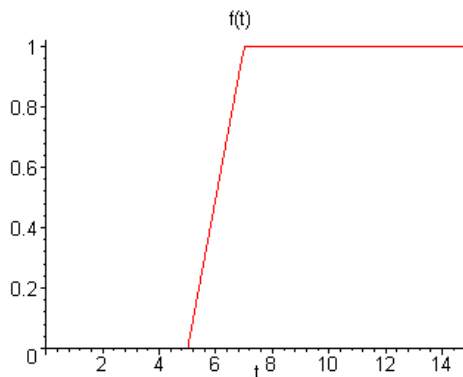
17. We consider the initial value problem

$$y'' + 4y = \frac{1}{k}[(t - 5) u_5(t) - (t - 5 - k) u_{5+k}(t)],$$

with  $y(0) = y'(0) = 0$ .

(a). The specified function is defined by

$$f(t) = \begin{cases} 0, & 0 \leq t < 5 \\ \frac{1}{k}(t - 5), & 5 \leq t < 5 + k \\ 1, & t \geq 5 + k \end{cases}$$



(b). Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2 Y(s) - s y(0) - y'(0) + 4Y(s) = \frac{e^{-5s}}{ks^2} - \frac{e^{-(5+k)s}}{ks^2}.$$

Applying the initial conditions,

$$s^2 Y(s) + 4Y(s) = \frac{e^{-5s}}{ks^2} - \frac{e^{-(5+k)s}}{ks^2}.$$

Solving for the transform,

$$Y(s) = \frac{e^{-5s}}{ks^2(s^2 + 4)} - \frac{e^{-(5+k)s}}{ks^2(s^2 + 4)}.$$

Using partial fractions,

$$\frac{1}{s^2(s^2 + 4)} = \frac{1}{4} \left[ \frac{1}{s^2} - \frac{1}{s^2 + 4} \right].$$

It follows that

$$\mathcal{L}^{-1} \left[ \frac{1}{s^2(s^2 + 4)} \right] = \frac{1}{4}t - \frac{1}{8}\sin 2t.$$

Using Theorem 6.3.1, the solution of the IVP is

$$y(t) = \frac{1}{k} [h(t - 5) u_5(t) - h(t - 5 - k) u_{5+k}(t)],$$

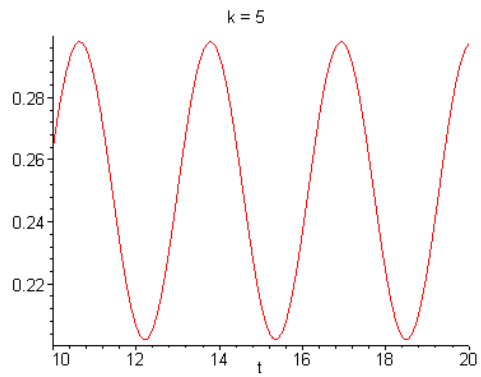
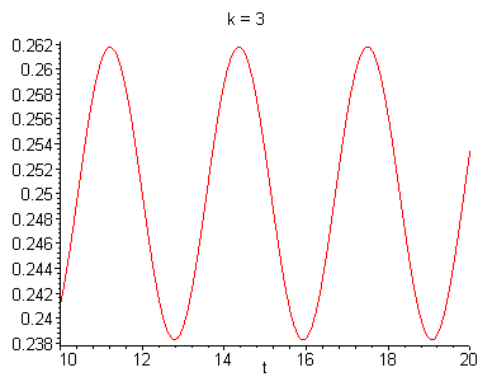
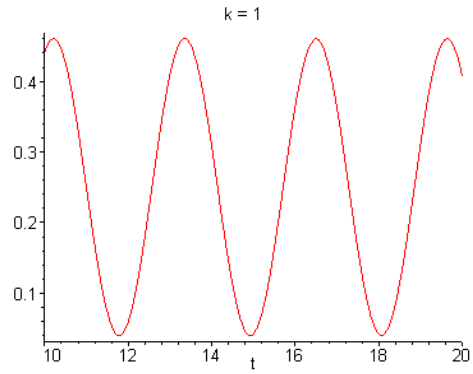
in which  $h(t) = \frac{1}{4}t - \frac{1}{8}\sin 2t$ .

(c). Note that for  $t > 5 + k$ , the solution is given by

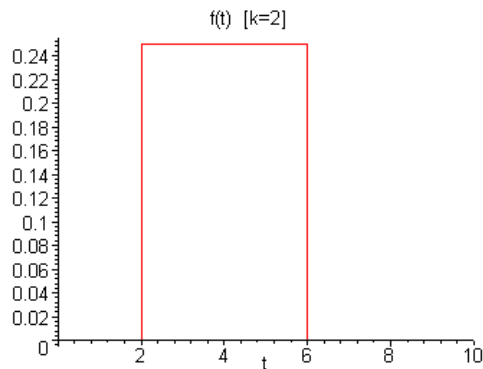
$$\begin{aligned} y(t) &= \frac{1}{4} - \frac{1}{8k} \sin(2t - 10) + \frac{1}{8k} \sin(2t - 10 - 2k) \\ &= \frac{1}{4} - \frac{\sin k}{4k} \cos(2t - 10 - k). \end{aligned}$$

So for  $t > 5 + k$ , the solution oscillates about  $y_m = 1/4$ , with an amplitude of

$$A = \frac{|\sin(k)|}{4k}.$$



18(a).



(b). The forcing function can be expressed as

$$f_k(t) = \frac{1}{2k} [u_{4-k}(t) - u_{4+k}(t)].$$

Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2 Y(s) - s y(0) - y'(0) + \frac{1}{3} [s Y(s) - y(0)] + 4 Y(s) = \frac{e^{-(4-k)s}}{2ks} - \frac{e^{-(4+k)s}}{2ks}.$$

Applying the initial conditions,

$$s^2 Y(s) + \frac{1}{3} s Y(s) + 4 Y(s) = \frac{e^{-(4-k)s}}{2ks} - \frac{e^{-(4+k)s}}{2ks}.$$

Solving for the transform,

$$Y(s) = \frac{3 e^{-(4-k)s}}{2ks(3s^2 + s + 12)} - \frac{3 e^{-(4+k)s}}{2ks(3s^2 + s + 12)}.$$

Using partial fractions,

$$\begin{aligned} \frac{1}{s(3s^2 + s + 12)} &= \frac{1}{12} \left[ \frac{1}{s} - \frac{1 + 3s}{3s^2 + s + 12} \right] \\ &= \frac{1}{12} \left[ \frac{1}{s} - \frac{1}{6} \frac{1 + 6(s + \frac{1}{6})}{(s + \frac{1}{6})^2 + \frac{143}{36}} \right]. \end{aligned}$$

Let

$$H(s) = \frac{1}{8k} \left[ \frac{1}{s} - \frac{\frac{1}{6}}{(s + \frac{1}{6})^2 + \frac{143}{36}} - \frac{s + \frac{1}{6}}{(s + \frac{1}{6})^2 + \frac{143}{36}} \right].$$

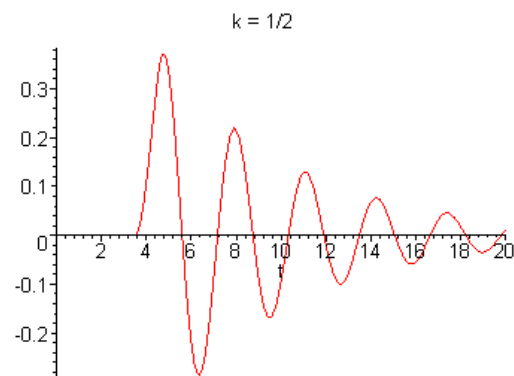
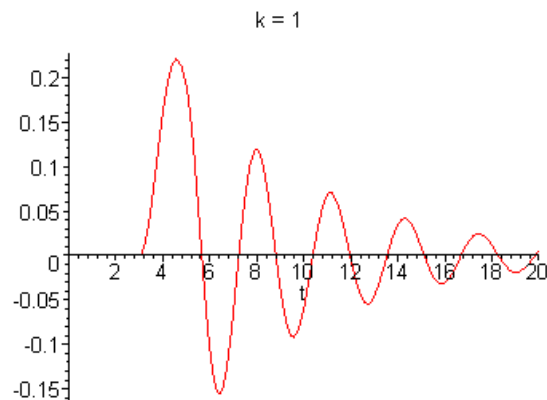
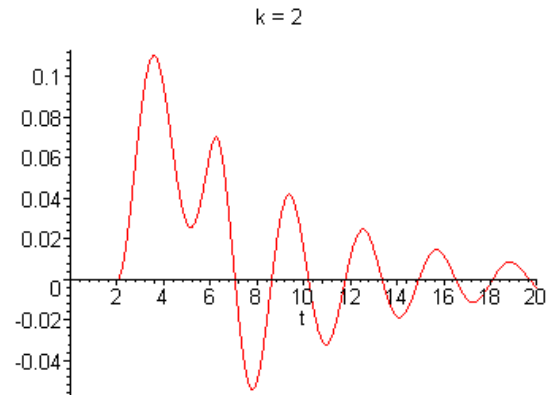
It follows that

$$h(t) = \mathcal{L}^{-1}[H(s)] = \frac{1}{8k} - \frac{e^{-t/6}}{8k} \left[ \frac{1}{\sqrt{143}} \sin\left(\frac{\sqrt{143}t}{6}\right) + \cos\left(\frac{\sqrt{143}t}{6}\right) \right].$$

Based on Theorem 6.3.1, the solution of the IVP is

$$y(t) = h(t - 4 + k) u_{4-k}(t) - h(t - 4 - k) u_{4+k}(t).$$

(c).

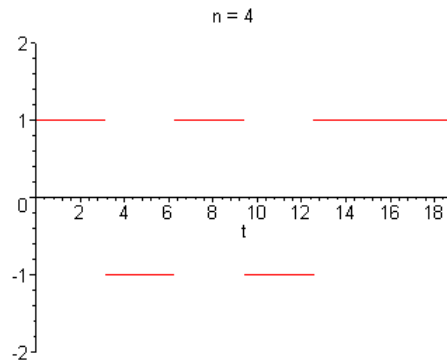


As the parameter  $k$  decreases, the solution remains *null* for a longer period of time.



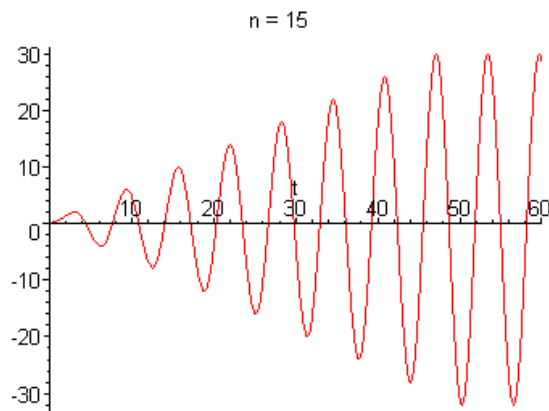
Since the *magnitude* of the impulsive force *increases*, the initial *overshoot* of the response also increases. The *duration* of the impulse decreases. All solutions eventually decay to  $y = 0$ .

19(a).

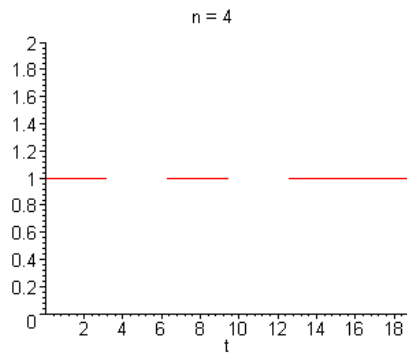
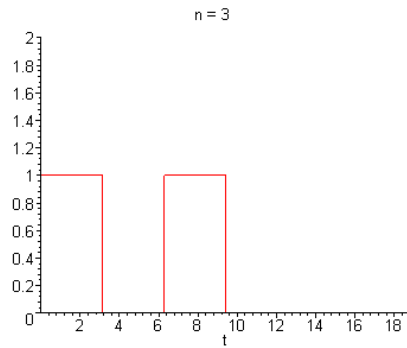


(c). From Part (b),

$$u(t) = 1 - \cos t + 2 \sum_{k=1}^n (-1)^k [1 - \cos(t - k\pi)] u_{k\pi}(t).$$



21(a).



(b). Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2 U(s) - s u(0) - u'(0) + U(s) = \frac{1}{s} + \sum_{k=1}^n \frac{(-1)^k e^{-k\pi s}}{s}.$$

Applying the initial conditions,

$$s^2 U(s) + U(s) = \frac{1}{s} + \sum_{k=1}^n \frac{(-1)^k e^{-k\pi s}}{s}.$$

Solving for the transform,

$$U(s) = \frac{1}{s(s^2 + 1)} + \sum_{k=1}^n \frac{(-1)^k e^{-k\pi s}}{s(s^2 + 1)}.$$

Using partial fractions,

$$\frac{1}{s(s^2 + 1)} = \frac{1}{s} - \frac{s}{s^2 + 1}.$$

Let

$$h(t) = \mathcal{L}^{-1}\left[\frac{1}{s(s^2 + 1)}\right] = 1 - \cos t.$$

Applying Theorem 6.3.1, term-by-term, the solution of the IVP is

$$u(t) = h(t) + \sum_{k=1}^n (-1)^k h(t - k\pi) u_{k\pi}(t).$$

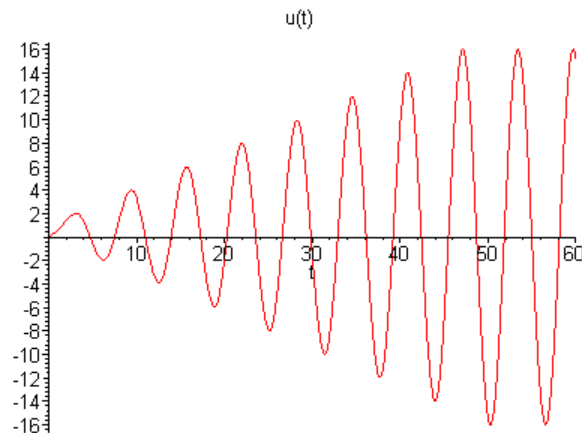
Note that

$$\begin{aligned} h(t - k\pi) &= u_0(t - k\pi) - \cos(t - k\pi) \\ &= u_{k\pi}(t) - (-1)^k \cos t. \end{aligned}$$

Hence

$$u(t) = 1 - \cos t + \sum_{k=1}^n (-1)^k u_{k\pi}(t) - (\cos t) \sum_{k=1}^n u_{k\pi}(t).$$

(c).



The ODE has no *damping term*. Each interval of forcing adds to the energy of the system.

Hence the amplitude will increase. For  $n = 15$ ,  $g(t) = 0$  when  $t > 15\pi$ . Therefore the oscillation will eventually become *steady*, with an amplitude depending on the values of  $u(15\pi)$  and  $u'(15\pi)$ .

(d). As  $n$  increases, the interval of forcing also increases. Hence the amplitude of the transient will increase with  $n$ . Eventually, the forcing function will be *constant*. In fact, for *large* values of  $t$ ,

$$g(t) = \begin{cases} 1, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

Further, for  $t > n\pi$ ,

$$u(t) = 1 - \cos t - n \cos t - \frac{1 - (-1)^n}{2}.$$

Hence the steady state solution will oscillate about 0 or 1, depending on  $n$ , with an amplitude of  $A = n + 1$ .

In the limit, as  $n \rightarrow \infty$ , the forcing function will be a periodic function, with period  $2\pi$ . From Prob. 27, in Section 6.3,

$$\mathcal{L}[g(t)] = \frac{1}{s(1 + e^{-s})}.$$

As  $n$  increases, the duration and magnitude of the transient will increase without bound.

22(a). Taking the initial conditions into consideration, the transform of the ODE is

$$s^2 U(s) + 0.1 s U(s) + U(s) = \frac{1}{s} + \sum_{k=1}^n \frac{(-1)^k e^{-k\pi s}}{s}.$$

Solving for the transform,

$$U(s) = \frac{1}{s(s^2 + 0.1s + 1)} + \sum_{k=1}^n \frac{(-1)^k e^{-k\pi s}}{s(s^2 + 0.1s + 1)}.$$

Using partial fractions,

$$\frac{1}{s(s^2 + 0.1s + 1)} = \frac{1}{s} - \frac{s + 0.1}{s^2 + 0.1s + 1}.$$

Since the denominator in the second term is irreducible, write

$$\frac{s + 0.1}{s^2 + 0.1s + 1} = \frac{(s + 0.05) + 0.05}{(s + 0.05)^2 + (399/400)}.$$

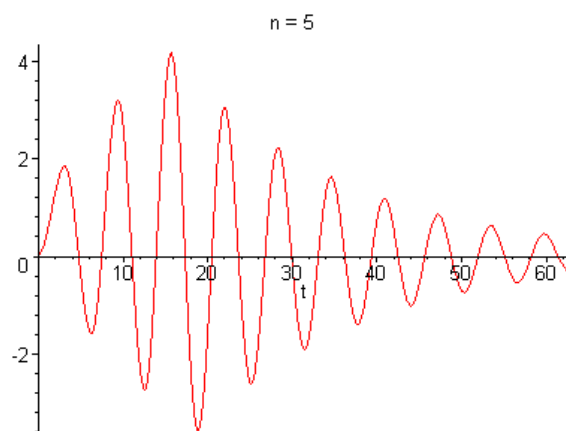
Let

$$\begin{aligned} h(t) &= \mathcal{L}^{-1} \left[ \frac{1}{s} - \frac{(s + 0.05)}{(s + 0.05)^2 + (399/400)} - \frac{0.05}{(s + 0.05)^2 + (399/400)} \right] \\ &= 1 - e^{-t/20} \left[ \cos \left( \frac{\sqrt{399}}{20} t \right) + \frac{1}{\sqrt{399}} \sin \left( \frac{\sqrt{399}}{20} t \right) \right]. \end{aligned}$$

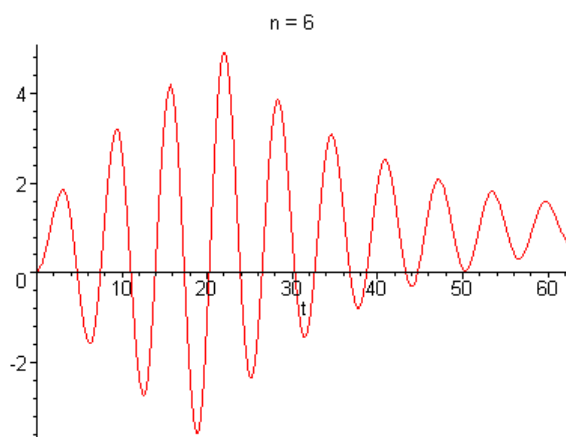
Applying Theorem 6.3.1, term-by-term, the solution of the IVP is

$$u(t) = h(t) + \sum_{k=1}^n (-1)^k h(t - k\pi) u_{k\pi}(t).$$

For *odd* values of  $n$ , the solution approaches  $y = 0$ .



For *even* values of  $n$ , the solution approaches  $y = 1$ .



(b). The solution is a sum of *damped sinusoids*, each of frequency  $\omega = \sqrt{399}/20 \approx 1$ . Each term has an 'initial' amplitude of approximately 1. For any given  $n$ , the solution contains  $n + 1$  such terms. Although the amplitude will *increase* with  $n$ , the amplitude will also be bounded by  $n + 1$ .

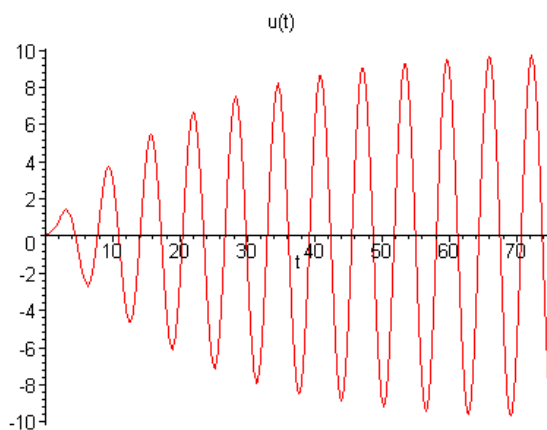
(c). Suppose that the forcing function is replaced by  $g(t) = \sin t$ . Based on the methods in Chapter 3, the general solution of the differential equation is

$$u(t) = e^{-t/20} \left[ c_1 \cos\left(\frac{\sqrt{399}}{20} t\right) + c_2 \sin\left(\frac{\sqrt{399}}{20} t\right) \right] + u_p(t).$$

Note that  $u_p(t) = A \cos t + B \sin t$ . Using the method of *undetermined coefficients*,  $A = -10$  and  $B = 0$ . Based on the initial conditions, the solution of the IVP is

$$u(t) = 10 e^{-t/20} \left[ \cos \left( \frac{\sqrt{399}}{20} t \right) + \frac{1}{\sqrt{399}} \sin \left( \frac{\sqrt{399}}{20} t \right) \right] - 10 \cos t.$$

Observe that both solutions have the same frequency,  $\omega = \sqrt{399}/20 \approx 1$ .



23(a). Taking the initial conditions into consideration, the transform of the ODE is

$$s^2 U(s) + U(s) = \frac{1}{s} + 2 \sum_{k=1}^n \frac{(-1)^k e^{-(11k/4)s}}{s}.$$

Solving for the transform,

$$U(s) = \frac{1}{s(s^2 + 1)} + 2 \sum_{k=1}^n \frac{(-1)^k e^{-(11k/4)s}}{s(s^2 + 1)}.$$

Using partial fractions,

$$\frac{1}{s(s^2 + 1)} = \frac{1}{s} - \frac{s}{s^2 + 1}.$$

Let

$$h(t) = \mathcal{L}^{-1} \left[ \frac{1}{s(s^2 + 1)} \right] = 1 - \cos t.$$

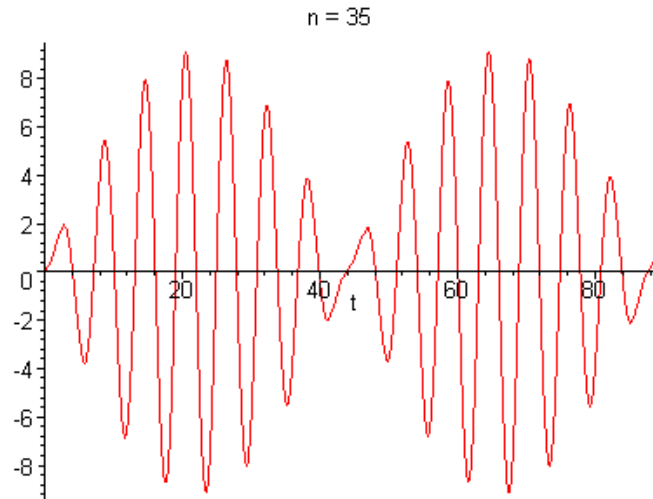
Applying Theorem 6.3.1, term-by-term, the solution of the IVP is

$$u(t) = h(t) + 2 \sum_{k=1}^n (-1)^k h \left( t - \frac{11k}{4} \right) u_{11k/4}(t).$$

That is,

$$u(t) = 1 - \cos t + 2 \sum_{k=1}^n (-1)^k \left[ 1 - \cos \left( t - \frac{11k}{4} \right) \right] u_{11k/4}(t).$$

(b).



(c). Based on the plot, the '*slow period*' appears to be 88. The '*fast period*' appears to be about 6. These values correspond to a '*slow frequency*' of  $\omega_s = 0.0714$  and a '*fast frequency*'  $\omega_f = 1.0472$ .

(d). The natural frequency of the system is  $\omega_0 = 1$ . The forcing function is initially periodic, with period  $T = 11/2 = 5.5$ . Hence the corresponding forcing frequency is  $\omega = 1.1424$ . Using the results in Section 3.9, the '*slow frequency*' is given by

$$\omega_s = \frac{|\omega - \omega_0|}{2} = 0.0712$$

and the '*fast frequency*' is given by

$$\omega_f = \frac{|\omega + \omega_0|}{2} = 1.0712.$$

Based on these values, the '*slow period*' is predicted as 88.247 and the '*fast period*' is given as 5.8656.

## Section 6.5

2. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2 Y(s) - s y(0) - y'(0) + 4Y(s) = e^{-\pi s} - e^{-2\pi s}.$$

Applying the initial conditions,

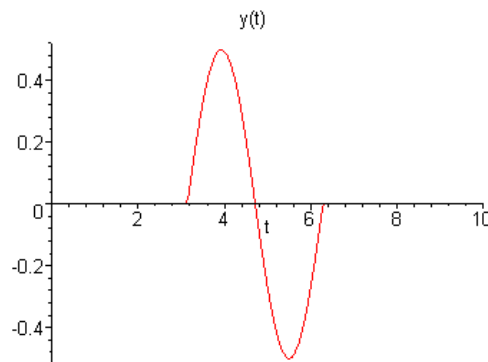
$$s^2 Y(s) + 4Y(s) = e^{-\pi s} - e^{-2\pi s}.$$

Solving for the transform,

$$Y(s) = \frac{e^{-\pi s} - e^{-2\pi s}}{s^2 + 4} = \frac{e^{-\pi s}}{s^2 + 4} - \frac{e^{-2\pi s}}{s^2 + 4}.$$

Applying Theorem 6.3.1, the solution of the IVP is

$$\begin{aligned} y(t) &= \frac{1}{2} \sin(2t - 2\pi) u_{\pi}(t) - \frac{1}{2} \sin(2t - 4\pi) u_{2\pi}(t) \\ &= \frac{1}{2} \sin(2t) [u_{\pi}(t) - u_{2\pi}(t)]. \end{aligned}$$



4. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2 Y(s) - s y(0) - y'(0) - Y(s) = -20 e^{-3s}.$$

Applying the initial conditions,

$$s^2 Y(s) - Y(s) - s = -20 e^{-3s}.$$

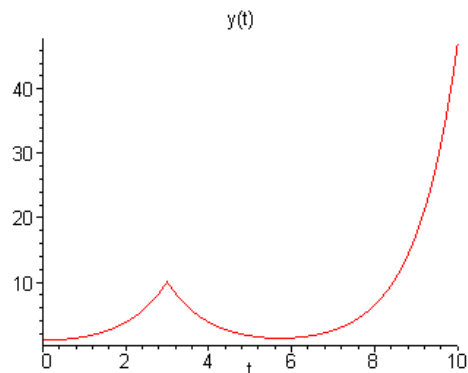
Solving for the transform,

$$Y(s) = \frac{s}{s^2 - 1} - \frac{20 e^{-3s}}{s^2 - 1}.$$

Using a *table of transforms*, and Theorem 6.3.1, the solution of the IVP is

$$y(t) = \cosh t - 20 \sinh(t - 3) u_3(t).$$





6. Taking the initial conditions into consideration, the transform of the ODE is

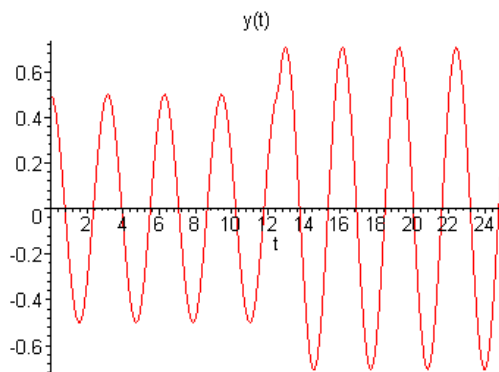
$$s^2 Y(s) + 4Y(s) - s/2 = e^{-4\pi s}.$$

Solving for the transform,

$$Y(s) = \frac{s/2}{s^2 + 4} + \frac{e^{-4\pi s}}{s^2 + 4}.$$

Using a *table of transforms*, and Theorem 6.3.1, the solution of the IVP is

$$\begin{aligned} y(t) &= \frac{1}{2} \cos 2t + \frac{1}{2} \sin(2t - 8\pi) u_{4\pi}(t) \\ &= \frac{1}{2} \cos 2t + \frac{1}{2} \sin(2t) u_{4\pi}(t). \end{aligned}$$



8. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2 Y(s) - s y(0) - y'(0) + 4Y(s) = 2 e^{-(\pi/4)s}.$$

Applying the initial conditions,

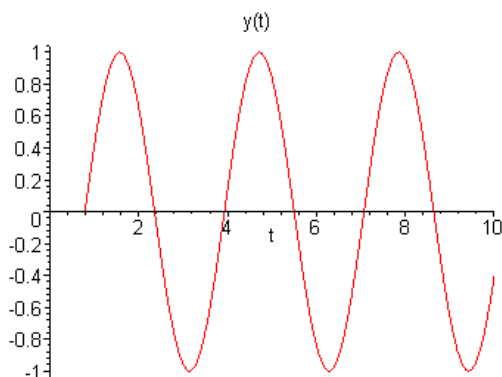
$$s^2 Y(s) + 4Y(s) = 2 e^{-(\pi/4)s}.$$

Solving for the transform,

$$Y(s) = \frac{2e^{-(\pi/4)s}}{s^2 + 4}.$$

Applying Theorem 6.3.1, the solution of the IVP is

$$y(t) = \sin\left(2t - \frac{\pi}{2}\right)u_{\pi/4}(t) = -\cos(2t)u_{\pi/4}(t).$$



9. Taking the initial conditions into consideration, the transform of the ODE is

$$s^2 Y(s) + Y(s) = \frac{e^{-(\pi/2)s}}{s} + 3e^{-(3\pi/2)s} - \frac{e^{-2\pi s}}{s}.$$

Solving for the transform,

$$Y(s) = \frac{e^{-(\pi/2)s}}{s(s^2 + 1)} + \frac{3e^{-(3\pi/2)s}}{s^2 + 1} - \frac{e^{-2\pi s}}{s(s^2 + 1)}.$$

Using partial fractions,

$$\frac{1}{s(s^2 + 1)} = \frac{1}{s} - \frac{s}{s^2 + 1}.$$

Hence

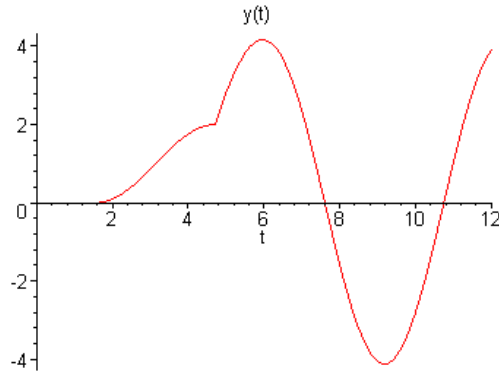
$$Y(s) = \frac{e^{-(\pi/2)s}}{s} - \frac{s e^{-(\pi/2)s}}{s^2 + 1} + \frac{3e^{-(3\pi/2)s}}{s^2 + 1} - \frac{e^{-2\pi s}}{s} + \frac{s e^{-2\pi s}}{s^2 + 1}.$$

Based on Theorem 6.3.1, the solution of the IVP is

$$y(t) = u_{\pi/2}(t) - \cos\left(t - \frac{\pi}{2}\right)u_{\pi/2}(t) + 3\sin\left(t - \frac{3\pi}{2}\right)u_{3\pi/2}(t) - u_{2\pi}(t) + \cos(t - 2\pi)u_{2\pi}(t).$$

That is,

$$y(t) = [1 - \sin(t)]u_{\pi/2}(t) + 3\cos(t)u_{3\pi/2}(t) - [1 - \cos(t)]u_{2\pi}(t).$$



10. Taking the transform of both sides of the ODE,

$$\begin{aligned} 2s^2Y(s) + sY(s) + 4Y(s) &= \int_0^{\infty} e^{-st} \delta\left(t - \frac{\pi}{6}\right) \sin t \, dt \\ &= \frac{1}{2} e^{-(\pi/6)s}. \end{aligned}$$

Solving for the transform,

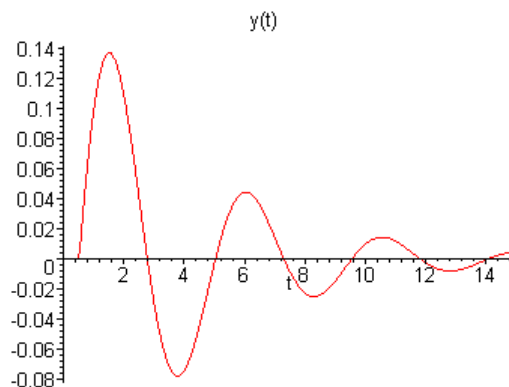
$$Y(s) = \frac{e^{-(\pi/6)s}}{2(2s^2 + s + 4)}.$$

First write

$$\frac{1}{2(2s^2 + s + 4)} = \frac{\frac{1}{4}}{\left(s + \frac{1}{4}\right)^2 + \frac{31}{16}}.$$

It follows that

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \frac{1}{\sqrt{31}} e^{-(t-\pi/6)/4} \cdot \sin \frac{\sqrt{31}}{4} \left(t - \frac{\pi}{6}\right) u_{\pi/6}(t).$$



11. Taking the initial conditions into consideration, the transform of the ODE is

$$s^2 Y(s) + 2s Y(s) + 2 Y(s) = \frac{s}{s^2 + 1} + e^{-(\pi/2)s}.$$

Solving for the transform,

$$Y(s) = \frac{s}{(s^2 + 1)(s^2 + 2s + 2)} + \frac{e^{-(\pi/2)s}}{s^2 + 2s + 2}.$$

Using partial fractions,

$$\frac{s}{(s^2 + 1)(s^2 + 2s + 2)} = \frac{1}{5} \left[ \frac{s}{s^2 + 1} + \frac{2}{s^2 + 1} - \frac{s + 4}{s^2 + 2s + 2} \right].$$

We can also write

$$\frac{s + 4}{s^2 + 2s + 2} = \frac{(s + 1) + 3}{(s + 1)^2 + 1}.$$

Let

$$Y_1(s) = \frac{s}{(s^2 + 1)(s^2 + 2s + 2)}.$$

Then

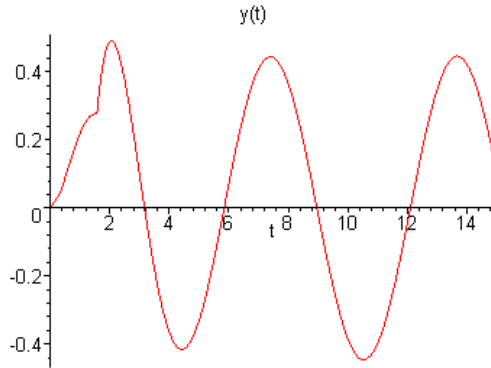
$$\mathcal{L}^{-1}[Y_1(s)] = \frac{1}{5} \cos t + \frac{2}{5} \sin t - \frac{1}{5} e^{-t} [\cos t + 3 \sin t].$$

Applying Theorem 6.3.1,

$$\mathcal{L}^{-1} \left[ \frac{e^{-(\pi/2)s}}{s^2 + 2s + 2} \right] = e^{-(t-\frac{\pi}{2})} \sin \left( t - \frac{\pi}{2} \right) u_{\pi/2}(t).$$

Hence the solution of the IVP is

$$y(t) = \frac{1}{5} \cos t + \frac{2}{5} \sin t - \frac{1}{5} e^{-t} [\cos t + 3 \sin t] - e^{-(t-\frac{\pi}{2})} \cos(t) u_{\pi/2}(t).$$



12. Taking the initial conditions into consideration, the transform of the ODE is

$$s^4 Y(s) - Y(s) = e^{-s}.$$

Solving for the transform,

$$Y(s) = \frac{e^{-s}}{s^4 - 1}.$$

Using partial fractions,

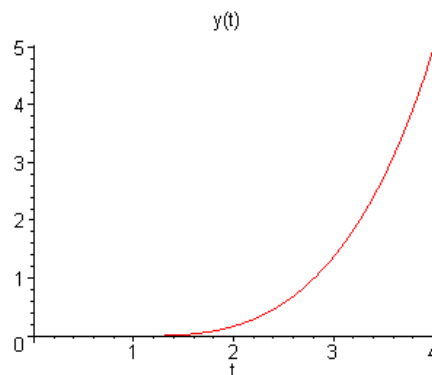
$$\frac{1}{s^4 - 1} = \frac{1}{2} \left[ \frac{1}{s^2 - 1} - \frac{1}{s^2 + 1} \right].$$

It follows that

$$\mathcal{L}^{-1} \left[ \frac{1}{s^4 - 1} \right] = \frac{1}{2} \sinh t - \frac{1}{2} \sin t.$$

Applying Theorem 6.3.1, the solution of the IVP is

$$y(t) = \frac{1}{2} [\sinh(t - 1) - \sin(t - 1)] u_1(t).$$



14(a). The Laplace transform of the ODE is

$$s^2 Y(s) + \frac{1}{2}s Y(s) + Y(s) = e^{-s}.$$

Solving for the transform of the solution,

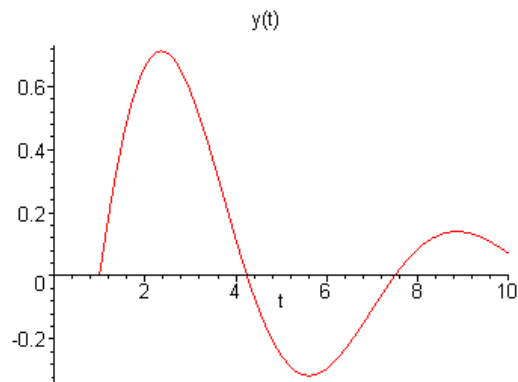
$$Y(s) = \frac{e^{-s}}{s^2 + s/2 + 1}.$$

First write

$$\frac{1}{s^2 + s/2 + 1} = \frac{1}{\left(s + \frac{1}{4}\right)^2 + \frac{15}{16}}.$$

Taking the inverse transform and applying both *shifting theorems*,

$$y(t) = \frac{4}{\sqrt{15}} e^{-(t-1)/4} \sin \frac{\sqrt{15}}{4} (t-1) u_1(t).$$



(b). As shown on the graph, the maximum is attained at some  $t_1 > 2$ . Note that for  $t > 2$ ,

$$y(t) = \frac{4}{\sqrt{15}} e^{-(t-1)/4} \sin \frac{\sqrt{15}}{4} (t-1).$$

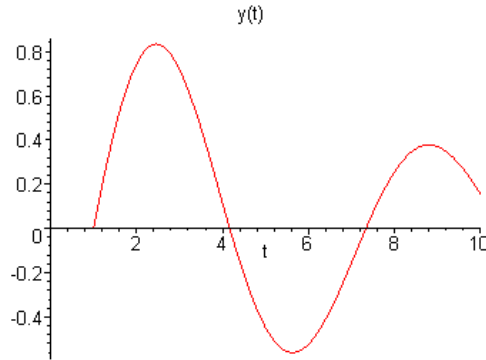
Setting  $y'(t) = 0$ , we find that  $t_1 \approx 2.3613$ . The maximum value is calculated as  $y(2.3613) \approx 0.71153$ .

(c). Setting  $\gamma = 1/4$ , the transform of the solution is

$$Y(s) = \frac{e^{-s}}{s^2 + s/4 + 1}.$$

Following the same steps, it follows that

$$y(t) = \frac{8}{3\sqrt{7}} e^{-(t-1)/8} \sin \frac{3\sqrt{7}}{8} (t-1) u_1(t).$$



Once again, the maximum is attained at some  $t_1 > 2$ . Setting  $y'(t) = 0$ , we find that  $t_1 \approx 2.4569$ , with  $y(t_1) \approx 0.8335$ .

(d). Now suppose that  $0 < \gamma < 1$ . Then the transform of the solution is

$$Y(s) = \frac{e^{-s}}{s^2 + \gamma s + 1}.$$

First write

$$\frac{1}{s^2 + \gamma s + 1} = \frac{1}{(s + \gamma/2)^2 + (1 - \gamma^2/4)}.$$

It follows that

$$h(t) = \mathcal{L}^{-1} \left[ \frac{1}{s^2 + \gamma s + 1} \right] = \frac{2}{\sqrt{4 - \gamma^2}} e^{-\gamma t/2} \sin \left( \sqrt{1 - \gamma^2/4} \cdot t \right).$$

Hence the solution is

$$y(t) = h(t-1) u_1(t).$$

The solution is nonzero only if  $t > 1$ , in which case  $y(t) = h(t-1)$ . Setting  $y'(t) = 0$ , we obtain

$$\tan \left[ \sqrt{1 - \gamma^2/4} \cdot (t-1) \right] = \frac{1}{\gamma} \sqrt{4 - \gamma^2},$$

that is,

$$\frac{\tan \left[ \sqrt{1 - \gamma^2/4} \cdot (t-1) \right]}{\sqrt{1 - \gamma^2/4}} = \frac{2}{\gamma}.$$

As  $\gamma \rightarrow 0$ , we obtain the *formal* equation  $\tan(t - 1) = \infty$ . Hence  $t_1 \rightarrow 1 + \frac{\pi}{2}$ . Setting  $t = \pi/2$  in  $h(t)$ , and letting  $\gamma \rightarrow 0$ , we find that  $y_1 \rightarrow 1$ . These conclusions agree with the case  $\gamma = 0$ , for which it is easy to show that the solution is

$$y(t) = \sin(t - 1) u_1(t).$$

15(a). See Prob. 14. It follows that the solution of the IVP is

$$y(t) = \frac{4k}{\sqrt{15}} e^{-(t-1)/4} \sin \frac{\sqrt{15}}{4} (t - 1) u_1(t).$$

This function is a *multiple* of the answer in Prob. 14(a). Hence the peak value occurs at  $t_1 \approx 2.3613$ . The maximum value is calculated as  $y(2.3613) \approx 0.71153 k$ . We find that the appropriate value of  $k$  is  $k_1 = 2/0.71153 \approx 2.8108$ .

(b). Based on Prob. 14(c), the solution is

$$y(t) = \frac{8k}{3\sqrt{7}} e^{-(t-1)/8} \sin \frac{3\sqrt{7}}{8} (t - 1) u_1(t).$$

Since this function is a *multiple* of the solution in Prob. 14(c), we have  $t_1 \approx 2.4569$ , with  $y(t_1) \approx 0.8335 k$ . The solution attains a value of  $y = 2$ , for  $k_1 = 2/0.8335$ , that is,  $k_1 \approx 2.3995$ .

(c). Similar to Prob. 14(d), for  $0 < \gamma < 1$ , the solution is

$$y(t) = h(t - 1) u_1(t),$$

in which

$$h(t) = \frac{2k}{\sqrt{4 - \gamma^2}} e^{-\gamma t/2} \sin\left(\sqrt{1 - \gamma^2/4} \cdot t\right).$$

It follows that  $t_1 - 1 \rightarrow \pi/2$ . Setting  $t = \pi/2$  in  $h(t)$ , and letting  $\gamma \rightarrow 0$ , we find that  $y_1 \rightarrow k$ . Requiring that the *peak value* remains at  $y = 2$ , the limiting value of  $k$  is  $k_1 = 2$ . These conclusions agree with the case  $\gamma = 0$ , for which it is easy to show that the solution is

$$y(t) = k \sin(t - 1) u_1(t).$$

16(a). Taking the initial conditions into consideration, the transformation of the ODE is

$$s^2 Y(s) + Y(s) = \frac{1}{2k} \left[ \frac{e^{-(4-k)s}}{s} - \frac{e^{-(4+k)s}}{s} \right].$$

Solving for the transform of the solution,



$$Y(s) = \frac{1}{2k} \left[ \frac{e^{-(4-k)s}}{s(s^2 + 1)} - \frac{e^{-(4+k)s}}{s(s^2 + 1)} \right].$$

Using partial fractions,

$$\frac{1}{s(s^2 + 1)} = \frac{1}{s} - \frac{s}{s^2 + 1}.$$

Now let

$$h(t) = \mathcal{L}^{-1} \left[ \frac{1}{s(s^2 + 1)} \right] = 1 - \cos t.$$

Applying Theorem 6.3.1, the solution is

$$\phi(t, k) = \frac{1}{2k} [h(t - 4 + k) u_{4-k}(t) - h(t - 4 - k) u_{4+k}(t)].$$

That is,

$$\begin{aligned} \phi(t, k) &= \frac{1}{2k} [u_{4-k}(t) - u_{4+k}(t)] - \\ &\quad - \frac{1}{2k} [\cos(t - 4 + k) u_{4-k}(t) - \cos(t - 4 - k) u_{4+k}(t)]. \end{aligned}$$

(b). Consider various values of  $t$ . For any fixed  $t < 4$ ,  $\phi(t, k) = 0$ , as long as  $4 - k > t$ . If  $t \geq 4$ , then for  $4 + k < t$ ,

$$\phi(t, k) = -\frac{1}{2k} [\cos(t - 4 + k) - \cos(t - 4 - k)].$$

It follows that

$$\begin{aligned} \lim_{k \rightarrow 0} \phi(t, k) &= \lim_{k \rightarrow 0} -\frac{\cos(t - 4 + k) - \cos(t - 4 - k)}{2k} \\ &= \sin(t - 4). \end{aligned}$$

Hence

$$\lim_{k \rightarrow 0} \phi(t, k) = \sin(t - 4) u_4(t).$$

(c). The Laplace transform of the differential equation

$$y'' + y = \delta(t - 4),$$

with  $y(0) = y'(0) = 0$ , is

$$s^2 Y(s) + Y(s) = e^{-4s}.$$

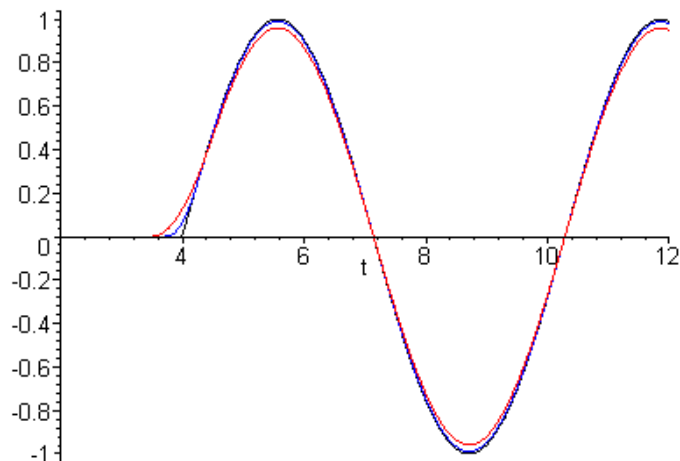
Solving for the transform of the solution,

$$Y(s) = \frac{e^{-4s}}{s^2 + 1}.$$

It follows that the solution is

$$\phi_0(t) = \sin(t - 4) u_4(t).$$

(d).



18(b). The transform of the ODE (given the specified initial conditions) is

$$s^2 Y(s) + Y(s) = \sum_{k=1}^{20} (-1)^{k+1} e^{-k\pi s}.$$

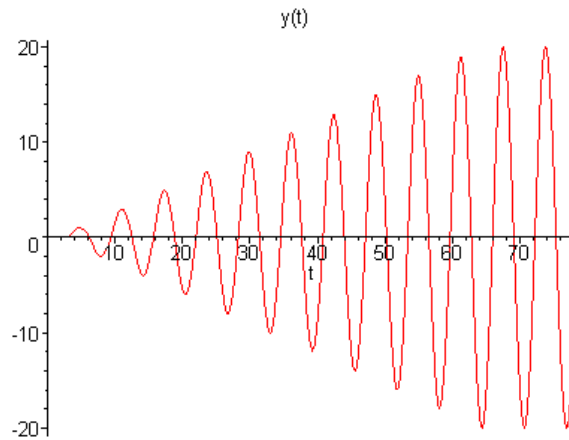
Solving for the transform of the solution,

$$Y(s) = \frac{1}{s^2 + 1} \sum_{k=1}^{20} (-1)^{k+1} e^{-k\pi s}.$$

Applying Theorem 6.3.1, term-by-term,

$$\begin{aligned} y(t) &= \sum_{k=1}^{20} (-1)^{k+1} \sin(t - k\pi) u_{k\pi}(t) \\ &= -\sin(t) \cdot \sum_{k=1}^{20} u_{k\pi}(t). \end{aligned}$$

(c).



19(b). Taking the initial conditions into consideration, the transform of the ODE is

$$s^2 Y(s) + Y(s) = \sum_{k=1}^{20} e^{-(k\pi/2)s}.$$

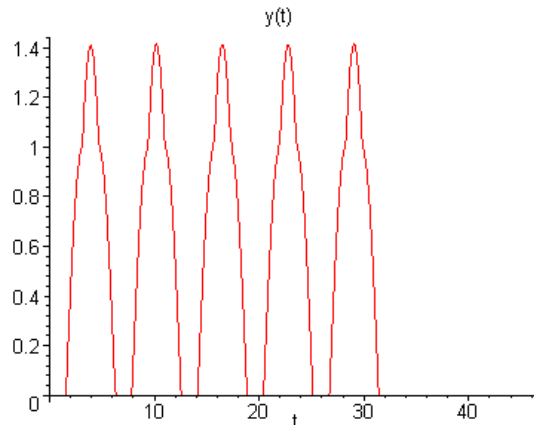
Solving for the transform of the solution,

$$Y(s) = \frac{1}{s^2 + 1} \sum_{k=1}^{20} e^{-(k\pi/2)s}.$$

Applying Theorem 6.3.1, term-by-term,

$$y(t) = \sum_{k=1}^{20} \sin\left(t - \frac{k\pi}{2}\right) u_{k\pi/2}(t).$$

(c).



20(b). The transform of the ODE (given the specified initial conditions) is

$$s^2 Y(s) + Y(s) = \sum_{k=1}^{20} (-1)^{k+1} e^{-(k\pi/2)s}.$$

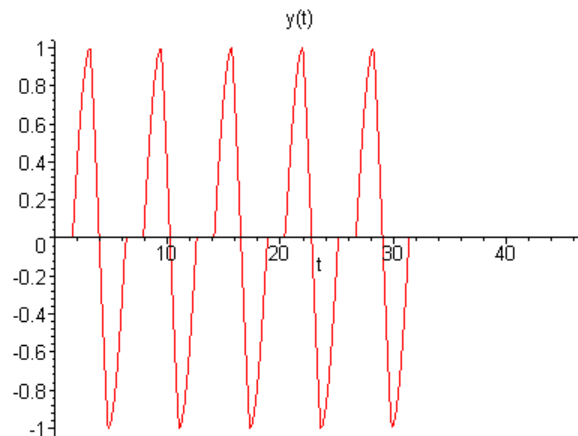
Solving for the transform of the solution,

$$Y(s) = \sum_{k=1}^{20} (-1)^{k+1} \frac{e^{-(k\pi/2)s}}{s^2 + 1}.$$

Applying Theorem 6.3.1 , term-by-term,

$$y(t) = \sum_{k=1}^{20} (-1)^{k+1} \sin\left(t - \frac{k\pi}{2}\right) u_{k\pi/2}(t).$$

(c).



22(b). Taking the initial conditions into consideration, the transform of the ODE is

$$s^2 Y(s) + Y(s) = \sum_{k=1}^{40} (-1)^{k+1} e^{-(11k/4)s}.$$

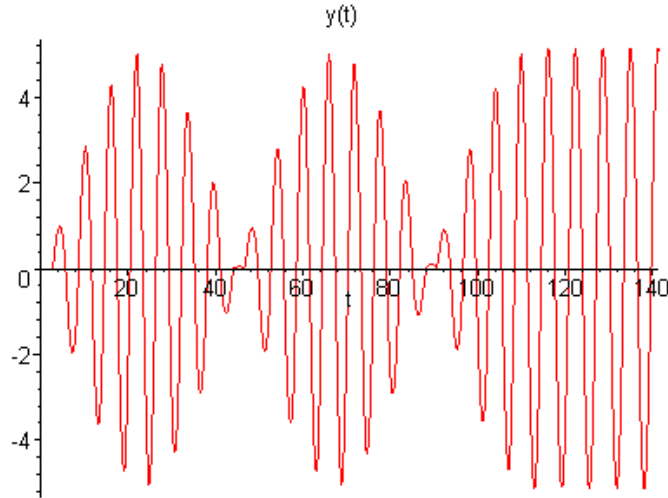
Solving for the transform of the solution,

$$Y(s) = \sum_{k=1}^{40} (-1)^{k+1} \frac{e^{-(11k/4)s}}{s^2 + 1}.$$

Applying Theorem 6.3.1 , term-by-term,

$$y(t) = \sum_{k=1}^{40} (-1)^{k+1} \sin\left(t - \frac{11k}{4}\right) u_{11k/4}(t).$$

(c).



23(b). The transform of the ODE (given the specified initial conditions) is

$$s^2 Y(s) + 0.1s Y(s) + Y(s) = \sum_{k=1}^{20} (-1)^{k+1} e^{-k\pi s}.$$

Solving for the transform of the solution,

$$Y(s) = \sum_{k=1}^{20} \frac{e^{-k\pi s}}{s^2 + 0.1s + 1}.$$

First write

$$\frac{1}{s^2 + 0.1s + 1} = \frac{1}{\left(s + \frac{1}{20}\right)^2 + \frac{399}{400}}.$$

It follows that

$$\mathcal{L}^{-1}\left[\frac{1}{s^2 + 0.1s + 1}\right] = \frac{20}{\sqrt{399}} e^{-t/20} \sin\left(\frac{\sqrt{399}}{20} t\right).$$

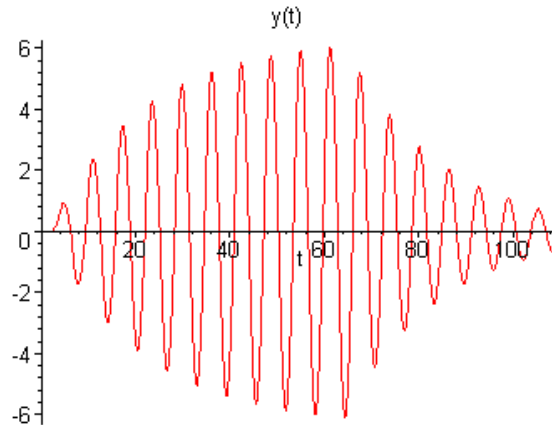
Applying Theorem 6.3.1, term-by-term,

$$y(t) = \sum_{k=1}^{20} (-1)^{k+1} h(t - k\pi) u_{k\pi}(t),$$

in which

$$h(t) = \frac{20}{\sqrt{399}} e^{-t/20} \sin\left(\frac{\sqrt{399}}{20} t\right).$$

(c).



24(b). Taking the initial conditions into consideration, the transform of the ODE is

$$s^2 Y(s) + 0.1s Y(s) + Y(s) = \sum_{k=1}^{15} e^{-(2k-1)\pi s}.$$

Solving for the transform of the solution,

$$Y(s) = \sum_{k=1}^{15} \frac{e^{-(2k-1)\pi s}}{s^2 + 0.1s + 1}.$$

As shown in Prob. 23,

$$\mathcal{L}^{-1}\left[\frac{1}{s^2 + 0.1s + 1}\right] = \frac{20}{\sqrt{399}} e^{-t/20} \sin\left(\frac{\sqrt{399}}{20} t\right).$$

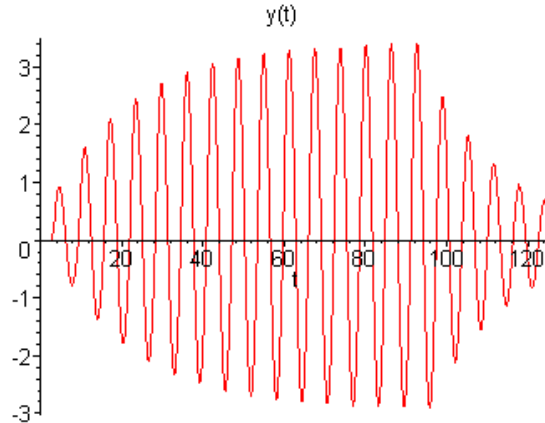
Applying Theorem 6.3.1, term-by-term,

$$y(t) = \sum_{k=1}^{15} h[t - (2k - 1)\pi] u_{(2k-1)\pi}(t),$$

in which

$$h(t) = \frac{20}{\sqrt{399}} e^{-t/20} \sin\left(\frac{\sqrt{399}}{20} t\right).$$

(c).



25(a). A fundamental set of solutions is  $y_1(t) = e^{-t} \cos t$  and  $y_2(t) = e^{-t} \sin t$ . Based on Prob. 22, in Section 3.7, a particular solution is given by

$$y_p(t) = \int_0^t \frac{y_1(s)y_2(t) - y_1(t)y_2(s)}{W(y_1, y_2)(s)} f(s) ds.$$

In the given problem,

$$\begin{aligned} y_p(t) &= \int_0^t \frac{e^{-s-t} [\cos(s) \sin(t) - \sin(s) \cos(t)]}{\exp(-2s)} f(s) ds. \\ &= \int_0^t e^{-(t-s)} \sin(t-s) f(s) ds. \end{aligned}$$

Given the specified initial conditions,

$$y(t) = \int_0^t e^{-(t-s)} \sin(t-s) f(s) ds.$$

(b). Let  $f(t) = \delta(t - \pi)$ . It is easy to see that if  $t < \pi$ ,  $y(t) = 0$ . If  $t > \pi$ ,

$$\int_0^t e^{-(t-s)} \sin(t-s) \delta(s - \pi) ds = e^{-(t-\pi)} \sin(t - \pi).$$

Setting  $t = \pi + \varepsilon$ , and letting  $\varepsilon \rightarrow 0$ , we find that  $y(\pi) = 0$ . Hence

$$y(t) = e^{-(t-\pi)} \sin(t - \pi) u_{\pi}(t).$$

(c). The Laplace transform of the solution is

$$\begin{aligned} Y(s) &= \frac{e^{-\pi s}}{s^2 + 2s + 2} \\ &= \frac{e^{-\pi s}}{(s + 1)^2 + 1}. \end{aligned}$$

Hence the solutions agree.



## Section 6.6

1(a). The *convolution integral* is defined as

$$f * g(t) = \int_0^t f(t - \tau)g(\tau)d\tau.$$

Consider the change of variable  $u = t - \tau$ . It follows that

$$\begin{aligned} \int_0^t f(t - \tau)g(\tau)d\tau &= \int_t^0 f(u)g(t - u)(-du) \\ &= \int_0^t g(t - u)f(u)du \\ &= g * f(t). \end{aligned}$$

(b). Based on the distributive property of the *real numbers*, the convolution is also distributive.

(c). By definition,

$$\begin{aligned} f * (g * h)(t) &= \int_0^t f(t - \tau)[g * h(\tau)]d\tau \\ &= \int_0^t f(t - \tau) \left[ \int_0^\tau g(\tau - \eta)h(\eta)d\eta \right] d\tau \\ &= \int_0^t \int_0^\tau f(t - \tau)g(\tau - \eta)h(\eta) d\eta d\tau. \end{aligned}$$

The region of integration, in the double integral is the area between the straight lines  $\eta = 0$ ,  $\eta = \tau$  and  $\tau = t$ . Interchanging the order of integration,

$$\begin{aligned} \int_0^t \int_0^\tau f(t - \tau)g(\tau - \eta)h(\eta) d\eta d\tau &= \int_0^t \int_\eta^t f(t - \tau)g(\tau - \eta)h(\eta) d\tau d\eta \\ &= \int_0^t \left[ \int_\eta^t f(t - \tau)g(\tau - \eta)d\tau \right] h(\eta) d\eta. \end{aligned}$$

Now let  $\tau - \eta = u$ . Then

$$\begin{aligned} \int_\eta^t f(t - \tau)g(\tau - \eta)d\tau &= \int_0^{t-\eta} f(t - \eta - u)g(u)du \\ &= f * g(t - \eta). \end{aligned}$$

Hence

$$\int_0^t f(t - \tau)[g * h(\tau)]d\tau = \int_0^t [f * g(t - \tau)]h(\tau) d\tau.$$

2. Let  $f(t) = e^t$ . Then

$$\begin{aligned} f * 1(t) &= \int_0^t e^{t-\tau} \cdot 1 \, d\tau \\ &= e^t \int_0^t e^{-\tau} \, d\tau \\ &= e^t - 1. \end{aligned}$$

3. It follows directly that

$$\begin{aligned} f * f(t) &= \int_0^t \sin(t-\tau) \sin(\tau) \, d\tau \\ &= \frac{1}{2} \int_0^t [\cos(t-2\tau) - \cos(t)] \, d\tau \\ &= \frac{1}{2} [\sin(t) - t \cos(t)]. \end{aligned}$$

The *range* of the resulting function is  $\mathbb{R}$ .

5. We have  $\mathcal{L}[e^{-t}] = 1/(s+1)$  and  $\mathcal{L}[\sin t] = 1/(s^2+1)$ . Based on Theorem 6.6.1,

$$\begin{aligned} \mathcal{L}\left[\int_0^t e^{-(t-\tau)} \sin(\tau) \, d\tau\right] &= \frac{1}{s+1} \cdot \frac{1}{s^2+1} \\ &= \frac{1}{(s+1)(s^2+1)}. \end{aligned}$$

6. Let  $g(t) = t$  and  $h(t) = e^t$ . Then  $f(t) = g * h(t)$ . Applying Theorem 6.6.1,

$$\begin{aligned} \mathcal{L}\left[\int_0^t g(t-\tau)h(\tau) \, d\tau\right] &= \frac{1}{s^2} \cdot \frac{1}{s-1} \\ &= \frac{1}{s^2(s-1)}. \end{aligned}$$

7. We have  $f(t) = g * h(t)$ , in which  $g(t) = \sin t$  and  $h(t) = \cos t$ . The transform of the convolution integral is

$$\begin{aligned} \mathcal{L}\left[\int_0^t g(t-\tau)h(\tau) \, d\tau\right] &= \frac{1}{s^2+1} \cdot \frac{s}{s^2+1} \\ &= \frac{s}{(s^2+1)^2}. \end{aligned}$$

9. It is easy to see that

$$\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] = e^{-t} \quad \text{and} \quad \mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right] = \cos 2t.$$

Applying Theorem 6.6.1,

$$\mathcal{L}^{-1}\left[\frac{s}{(s+1)(s^2+4)}\right] = \int_0^t e^{-(t-\tau)} \cos 2\tau \, d\tau.$$

10. We first note that

$$\mathcal{L}^{-1}\left[\frac{1}{(s+1)^2}\right] = t e^{-t} \quad \text{and} \quad \mathcal{L}^{-1}\left[\frac{1}{s^2+4}\right] = \frac{1}{2} \sin 2t.$$

Based on the *convolution theorem*,

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{1}{(s+1)^2(s^2+4)}\right] &= \frac{1}{2} \int_0^t (t-\tau) e^{-(t-\tau)} \sin 2\tau \, d\tau \\ &= \frac{1}{2} \int_0^t \tau e^{-\tau} \sin(2t-2\tau) \, d\tau. \end{aligned}$$

11. Let  $g(t) = \mathcal{L}^{-1}[G(s)]$ . Since  $\mathcal{L}^{-1}[1/(s^2+1)] = \sin t$ , the inverse transform of the product is

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{G(s)}{s^2+1}\right] &= \int_0^t g(t-\tau) \sin \tau \, d\tau \\ &= \int_0^t \sin(t-\tau) g(\tau) \, d\tau. \end{aligned}$$

12. Taking the initial conditions into consideration, the transform of the ODE is

$$s^2 Y(s) - 1 + \omega^2 Y(s) = G(s).$$

Solving for the transform of the solution,

$$Y(s) = \frac{1}{s^2 + \omega^2} + \frac{G(s)}{s^2 + \omega^2}.$$

As shown in a related situation, Prob. 11,

$$\mathcal{L}^{-1}\left[\frac{G(s)}{s^2 + \omega^2}\right] = \frac{1}{\omega} \int_0^t \sin \omega(t-\tau) g(\tau) \, d\tau.$$

Hence the solution of the IVP is

$$y(t) = \frac{1}{\omega} \sin \omega t + \frac{1}{\omega} \int_0^t \sin \omega(t - \tau) g(\tau) d\tau.$$

14. The transform of the ODE (given the specified initial conditions) is

$$4s^2 Y(s) + 4s Y(s) + 17 Y(s) = G(s).$$

Solving for the transform of the solution,

$$Y(s) = \frac{G(s)}{4s^2 + 4s + 17}.$$

First write

$$\frac{1}{4s^2 + 4s + 17} = \frac{\frac{1}{4}}{\left(s + \frac{1}{2}\right)^2 + 4}.$$

Based on the elementary properties of the Laplace transform,

$$\mathcal{L}^{-1}\left[\frac{1}{4s^2 + 4s + 17}\right] = \frac{1}{8} e^{-t/2} \sin 2t.$$

Applying the *convolution theorem*, the solution of the IVP is

$$y(t) = \frac{1}{8} \int_0^t e^{-(t-\tau)/2} \sin 2(t - \tau) g(\tau) d\tau.$$

16. Taking the initial conditions into consideration, the transform of the ODE is

$$s^2 Y(s) - 2s + 3 + 4[s Y(s) - 2] + 4 Y(s) = G(s).$$

Solving for the transform of the solution,

$$Y(s) = \frac{2s + 5}{(s + 2)^2} + \frac{G(s)}{(s + 2)^2}.$$

We can write

$$\frac{2s + 5}{(s + 2)^2} = \frac{2}{s + 2} + \frac{1}{(s + 2)^2}.$$

It follows that

$$\mathcal{L}^{-1}\left[\frac{2}{s + 2}\right] = 2e^{-2t} \quad \text{and} \quad \mathcal{L}^{-1}\left[\frac{1}{(s + 2)^2}\right] = t e^{-2t}.$$

Based on the *convolution theorem*, the solution of the IVP is

$$y(t) = 2e^{-2t} + te^{-2t} + \int_0^t (t - \tau)e^{-2(t-\tau)}g(\tau) d\tau.$$

18. The transform of the ODE (given the specified initial conditions) is

$$s^4 Y(s) - Y(s) = G(s).$$

Solving for the transform of the solution,

$$Y(s) = \frac{G(s)}{s^4 - 1}.$$

First write

$$\frac{1}{s^4 - 1} = \frac{1}{2} \left[ \frac{1}{s^2 - 1} - \frac{1}{s^2 + 1} \right].$$

It follows that

$$\mathcal{L}^{-1} \left[ \frac{1}{s^4 - 1} \right] = \frac{1}{2} [\sinh t - \sin t].$$

Based on the *convolution theorem*, the solution of the IVP is

$$y(t) = \frac{1}{2} \int_0^t [\sinh(t - \tau) - \sin(t - \tau)]g(\tau) d\tau.$$

19. Taking the initial conditions into consideration, the transform of the ODE is

$$s^4 Y(s) - s^3 + 5s^2 Y(s) - 5s + 4Y(s) = G(s).$$

Solving for the transform of the solution,

$$Y(s) = \frac{s^3 + 5s}{(s^2 + 1)(s^2 + 4)} + \frac{G(s)}{(s^2 + 1)(s^2 + 4)}.$$

Using partial fractions, we find that

$$\frac{s^3 + 5s}{(s^2 + 1)(s^2 + 4)} = \frac{1}{3} \left[ \frac{4s}{s^2 + 1} - \frac{s}{s^2 + 4} \right],$$

and

$$\frac{1}{(s^2 + 1)(s^2 + 4)} = \frac{1}{3} \left[ \frac{1}{s^2 + 1} - \frac{1}{s^2 + 4} \right].$$

It follows that

$$\mathcal{L}^{-1}\left[\frac{s(s^2 + 5)}{(s^2 + 1)(s^2 + 4)}\right] = \frac{4}{3} \cos t - \frac{1}{3} \cos 2t,$$

and

$$\mathcal{L}^{-1}\left[\frac{1}{(s^2 + 1)(s^2 + 4)}\right] = \frac{1}{3} \sin t - \frac{1}{6} \sin 2t.$$

Based on the *convolution theorem*, the solution of the IVP is

$$y(t) = \frac{4}{3} \cos t - \frac{1}{3} \cos 2t + \frac{1}{6} \int_0^t [2 \sin(t - \tau) - \sin 2(t - \tau)]g(\tau) d\tau.$$

21(a). Let  $\phi(t) = u''(t)$ . Substitution into the *integral equation* results in

$$u''(t) + \int_0^t (t - \xi) u''(\xi) d\xi = \sin 2t.$$

Integrating by parts,

$$\begin{aligned} \int_0^t (t - \xi) u''(\xi) d\xi &= (t - \xi) u'(\xi) \Big|_{\xi=0}^{\xi=t} + \int_0^t u'(\xi) d\xi \\ &= -t u'(0) + u(t) - u(0). \end{aligned}$$

Hence

$$u''(t) + u(t) - t u'(0) - u(0) = \sin 2t.$$

(b). Substituting the given *initial conditions* for the function  $u(t)$ ,

$$u''(t) + u(t) = \sin 2t.$$

Hence the solution of the IVP is equivalent to solving the integral equation in Part (a).

(c). Taking the Laplace transform of the integral equation, with  $\Phi(s) = \mathcal{L}[\phi(t)]$ ,

$$\Phi(s) + \frac{1}{s^2} \cdot \Phi(s) = \frac{2}{s^2 + 4}.$$

Note that the *convolution theorem* was applied. Solving for the transform  $\Phi(s)$ ,

$$\Phi(s) = \frac{2s^2}{(s^2 + 1)(s^2 + 4)}.$$

Using partial fractions, we can write

$$\frac{2s^2}{(s^2 + 1)(s^2 + 4)} = \frac{2}{3} \left[ \frac{4}{s^2 + 4} - \frac{1}{s^2 + 1} \right].$$

Therefore the solution of the *integral equation* is

$$\phi(t) = \frac{4}{3} \sin 2t - \frac{2}{3} \sin t.$$

(d). Taking the Laplace transform of the ODE, with  $U(s) = \mathcal{L}[u(t)]$ ,

$$s^2 U(s) + U(s) = \frac{2}{s^2 + 4}.$$

Solving for the transform of the solution,

$$U(s) = \frac{2}{(s^2 + 1)(s^2 + 4)}.$$

Using partial fractions, we can write

$$\frac{2}{(s^2 + 1)(s^2 + 4)} = \frac{1}{3} \left[ \frac{2}{s^2 + 1} - \frac{2}{s^2 + 4} \right].$$

It follows that the solution of the IVP is

$$u(t) = \frac{2}{3} \sin t - \frac{1}{3} \sin 2t.$$

We find that  $u''(t) = -\frac{2}{3} \sin t + \frac{4}{3} \sin 2t$ .

22(a). First note that

$$\int_0^b \frac{f(y)}{\sqrt{b-y}} dy = \left( \frac{1}{\sqrt{y}} * f \right)(b).$$

Take the Laplace transformation of both sides of the equation. Using the *convolution theorem*, with  $F(s) = \mathcal{L}[f(y)]$ ,

$$\frac{T_0}{s} = \frac{1}{\sqrt{2g}} F(s) \cdot \mathcal{L} \left[ \frac{1}{\sqrt{y}} \right].$$

It was shown in Prob. 27(c), Section 6.1, that

$$\mathcal{L} \left[ \frac{1}{\sqrt{y}} \right] = \sqrt{\frac{\pi}{s}}.$$

Hence

$$\frac{T_0}{s} = \frac{1}{\sqrt{2g}} F(s) \cdot \sqrt{\frac{\pi}{s}},$$

with

$$F(s) = \sqrt{\frac{2g}{\pi}} \cdot \frac{T_0}{\sqrt{s}}.$$

Taking the inverse transform, we obtain

$$f(y) = \frac{T_0}{\pi} \sqrt{\frac{2g}{y}}.$$

(b). Combining equations (i) and (iv),

$$\frac{2gT_0^2}{\pi^2 y} = 1 + \left(\frac{dx}{dy}\right)^2.$$

Solving for the derivative  $dx/dy$ ,

$$\frac{dx}{dy} = \sqrt{\frac{2\alpha - y}{y}},$$

in which  $\alpha = gT_0^2/\pi^2$ .

(c). Consider the *change of variable*  $y = 2\alpha \sin^2(\theta/2)$ . Using the chain rule,

$$\frac{dy}{d\theta} = 2\alpha \sin(\theta/2)\cos(\theta/2) \cdot \frac{d\theta}{d\theta}$$

and

$$\frac{dx}{d\theta} = \frac{1}{2\alpha \sin(\theta/2)\cos(\theta/2)} \cdot \frac{dx}{d\theta}.$$

It follows that

$$\begin{aligned} \frac{dx}{d\theta} &= 2\alpha \sin(\theta/2)\cos(\theta/2) \sqrt{\frac{\cos^2(\theta/2)}{\sin^2(\theta/2)}} \\ &= 2\alpha \cos^2(\theta/2) \\ &= \alpha + \alpha \cos \theta. \end{aligned}$$

Direct integration results in

$$x(\theta) = \alpha \theta + \alpha \sin \theta + C.$$

Since the curve passes through the *origin*, we require  $y(0) = x(0) = 0$ . Hence  $C = 0$ , and  $x(\theta) = \alpha \theta + \alpha \sin \theta$ . We also have



$$\begin{aligned}y(\theta) &= 2\alpha \sin^2(\theta/2) \\ &= \alpha - \alpha \cos \theta.\end{aligned}$$