Chapter Five

Section 5.1

1. Apply the ratio test :

$$\lim_{n\to\infty}\frac{\left|(x-3)^{n+1}\right|}{\left|(x-3)^{n}\right|} = \lim_{n\to\infty}|x-3| = |x-3|.$$

Hence the series converges absolutely for $|x-3|<1$. The radius of convergence is $\rho=1$. The series diverges for $x=2$ and $x=4$, since the *n-th* term does not approach zero.

3. Applying the ratio test,

$$\lim_{n\to\infty}\frac{|n!\,x^{2n+2}|}{|(n+1)!\,x^{2n}|} = \lim_{n\to\infty}\frac{x^2}{n+1} = 0\,.$$

The series converges absolutely for *all* values of x . Thus the radius of convergence is $\rho=\infty$.

4. Apply the ratio test :

$$\lim_{n\to\infty}\frac{|2^{n+1}x^{n+1}|}{|2^n x^n|} = \lim_{n\to\infty}2|x| = 2|x|.$$

Hence the series converges absolutely for $2|x|$, or $|x|<1/2$. The radius of convergence is $\rho=1/2$. The series diverges for $x=\pm1/2$, since the *n-th* term does not approach zero.

6. Applying the ratio test,

$$\lim_{n\to\infty}\frac{\left|n(x-x_0)^{n+1}\right|}{\left|(n+1)(x-x_0)^{n}\right|} = \lim_{n\to\infty}\frac{n}{n+1}|(x-x_0)| = |(x-x_0)|.$$

Hence the series converges absolutely for $|(x-x_0)|<1$. The radius of convergence is $\rho=1$. At $x=x_0+1$, we obtain the *harmonic series*, which is *divergent*. At the other endpoint, $x=x_0-1$, we obtain

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\,,$$

which is *conditionally* convergent.

7. Apply the ratio test :

$$\lim_{n\to\infty}\frac{\left|3^n(n+1)^2(x+2)^{n+1}\right|}{\left|3^{n+1}n^2(x+2)^n\right|}=\lim_{n\to\infty}\frac{(n+1)^2}{3\,n^2}|(x+2)|=\frac{1}{3}|(x+2)|.$$

Hence the series converges absolutely for $\frac{1}{3}|x+2|<1$, or $|x+2|<3$. The radius of convergence is $\rho=3$. At $x=-5$ and $x=+1$, the series diverges, since the *n-th* term does not approach zero.

8. Applying the ratio test,

$$\lim_{n\to\infty}\frac{|n^n(n+1)!\,x^{n+1}|}{\left|(n+1)^{n+1}n!\,x^n\right|}=\lim_{n\to\infty}\frac{n^n}{(n+1)^n}|x|=\frac{1}{e}|x|,$$

since

$$\lim_{n\to\infty}\frac{n^n}{(n+1)^n}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{-n}=e^{-1}.$$

Hence the series converges absolutely for $|x|<e$. The radius of convergence is $\rho=e$. At $x=\pm e$, the series *diverges*, since the *n-th* term does not approach zero. This follows from the fact that

$$\lim_{n\to\infty}\frac{n!\,e^n}{n^n\sqrt{2\pi n}}=1\,.$$

10. We have $f(x)=e^x$, with $f^{(n)}(x)=e^x$, for $n=1,2,\cdots$. Therefore $f^{(n)}(0)=1$. Hence the Taylor expansion about $x_0=0$ is

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}\,.$$

Applying the ratio test,

$$\lim_{n\to\infty}\frac{|n!x^{n+1}|}{|(n+1)!\,x^n|}=\lim_{n\to\infty}\frac{1}{n+1}|x|=0.$$

The radius of convergence is $\rho=\infty$.

11. We have $f(x)=x$, with $f'(x)=1$ and $f^{(n)}(x)=0$, for $n=2,\cdots$. Clearly, $f(1)=1$ and $f'(1)=1$, with all other derivatives equal to *zero*. Hence the Taylor expansion about $x_0=1$ is

$$x=1+(x-1).$$

Since the series has only a finite number of terms, the converges absolutely for all x.

14. We have $f(x)=1/(1+x)$, $f'(x)=-1/(1+x)^2$, $f''(x)=2/(1+x)^3,\cdots$ with $f^{(n)}(x)=(-1)^n n!/(1+x)^{n+1}$, for $n\geq 1$. It follows that $f^{(n)}(0)=(-1)^n n!$

for $n \geq 0$. Hence the Taylor expansion about $x_0 = 0$ is

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n.$$

Applying the ratio test,

$$\lim_{n\to\infty} \frac{|x^{n+1}|}{|x^n|} = \lim_{n\to\infty} |x| = |x|.$$

The series converges absolutely for $|x| < 1$, but diverges at $x = \pm 1$.

15. We have $f(x) = 1/(1-x)$, $f'(x) = 1/(1-x)^2$, $f''(x) = 2/(1-x)^3, \cdots$ with $f^{(n)}(x) = n!/(1-x)^{n+1}$, for $n \geq 1$. It follows that $f^{(n)}(0) = n!$, for $n \geq 0$. Hence the Taylor expansion about $x_0 = 0$ is

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

Applying the ratio test,

$$\lim_{n\to\infty} \frac{|x^{n+1}|}{|x^n|} = \lim_{n\to\infty} |x| = |x|.$$

The series converges absolutely for $|x| < 1$, but diverges at $x = \pm 1$.

16. We have $f(x) = 1/(1-x)$, $f'(x) = 1/(1-x)^2$, $f''(x) = 2/(1-x)^3, \cdots$ with $f^{(n)}(x) = n!/(1-x)^{n+1}$, for $n \geq 1$. It follows that $f^{(n)}(2) = (-1)^{n+1} n!$ for $n \geq 0$. Hence the Taylor expansion about $x_0 = 2$ is

$$\frac{1}{1-x} = -\sum_{n=0}^{\infty} (-1)^n (x-2)^n.$$

Applying the ratio test,

$$\lim_{n\to\infty} \frac{|(x-2)^{n+1}|}{|(x-2)^n|} = \lim_{n\to\infty} |x-2| = |x-2|.$$

The series converges absolutely for $|x-2| < 1$, but diverges at $x = 1$ and $x = 3$.

17. Applying the ratio test,

$$\lim_{n\to\infty} \frac{|(n+1)x^{n+1}|}{|n\,x^n|} = \lim_{n\to\infty} \frac{n+1}{n}|x| = |x|.$$

The series converges absolutely for $|x| < 1$. Term-by-term differentiation results in

$$y' = \sum_{n=1}^{\infty} n^2 x^{n-1} = 1 + 4x + 9x^2 + 16\,x^3 + \cdots$$

$$y'' = \sum_{n=2}^{\infty} n^2 (n-1)\, x^{n-2} = 4 + 18x + 48x^2 + 100x^3 + \cdots$$

Shifting the indices, we can also write

$$y' = \sum_{n=0}^{\infty} (n+1)^2 x^n \quad \text{and} \quad y'' = \sum_{n=0}^{\infty} (n+2)^2 (n+1)\, x^n.$$

20. Shifting the index in the *second* series, that is, setting $n = k + 1$,

$$\sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{n=1}^{\infty} a_{n-1} x^n \,.$$

Hence

$$\begin{aligned}\sum_{k=0}^{\infty} a_{k+1} x^k + \sum_{k=0}^{\infty} a_k x^{k+1} &= \sum_{k=0}^{\infty} a_{k+1} x^k + \sum_{k=1}^{\infty} a_{k-1} x^k \\ &= a_1 + \sum_{k=1}^{\infty} (a_{k+1} + a_{k-1}) x^{k+1}.\end{aligned}$$

21. Shifting the index by 2, that is, setting $m = n - 2$,

$$\begin{aligned}\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} &= \sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2}\, x^m \\ &= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n \,.\end{aligned}$$

22. Shift the index *down* by 2, that is, set $m = n + 2$. It follows that

$$\begin{aligned}\sum_{n=0}^{\infty} a_n x^{n+2} &= \sum_{m=2}^{\infty} a_{m-2}\, x^m \\ &= \sum_{n=2}^{\infty} a_{n-2}\, x^n \,.\end{aligned}$$

24. Clearly,

$$\left(1-x^2\right)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n.$$

Shifting the index in the *first* series, that is, setting $k = n - 2$,

$$\begin{aligned}\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} &= \sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2}\, x^k \\ &= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\, x^n.\end{aligned}$$

Hence

$$(1-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\, x^n - \sum_{n=2}^{\infty} n(n-1)a_n\, x^n.$$

Note that when $n = 0$ and $n = 1$, the coefficients in the *second* series are *zero*. So that

$$(1-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} - n(n-1)a_n]x^n.$$

26. Clearly,

$$\sum_{n=1}^{\infty} na_n\, x^{n-1} + x\sum_{n=0}^{\infty} a_n\, x^n = \sum_{n=1}^{\infty} na_n\, x^{n-1} + \sum_{n=0}^{\infty} a_n\, x^{n+1}.$$

Shifting the index in the *first* series, that is, setting $k = n - 1$,

$$\sum_{n=1}^{\infty} na_n\, x^{n-1} = \sum_{k=0}^{\infty} (k+1)a_{k+1} x^k.$$

Shifting the index in the *second* series, that is, setting $k = n + 1$,

$$\sum_{n=0}^{\infty} a_n\, x^{n+1} = \sum_{k=1}^{\infty} a_{k-1} x^k.$$

Combining the series, and starting the summation at $n = 1$,

$$\sum_{n=1}^{\infty} na_n\, x^{n-1} + x\sum_{n=0}^{\infty} a_n\, x^n = a_1 + \sum_{n=1}^{\infty} [(n+1)a_{n+1} + a_{n-1}]x^n.$$

27. We note that

$$x\sum_{n=2}^{\infty}n(n-1)a_n\,x^{n-2}+\sum_{n=0}^{\infty}a_n\,x^n=\sum_{n=2}^{\infty}n(n-1)a_n\,x^{n-1}+\sum_{n=0}^{\infty}a_n\,x^n.$$

Shifting the index in the *first* series, that is, setting $k=n-1$,

$$\begin{aligned}\sum_{n=2}^{\infty}n(n-1)a_n\,x^{n-1} &= \sum_{k=1}^{\infty}k(k+1)a_{k+1}x^k\\ &= \sum_{k=0}^{\infty}k(k+1)a_{k+1}x^k,\end{aligned}$$

since the coefficient of the term associated with $k=0$ is *zero*. Combining the series,

$$x\sum_{n=2}^{\infty}n(n-1)a_n\,x^{n-2}+\sum_{n=0}^{\infty}a_n\,x^n=\sum_{n=0}^{\infty}[n(n+1)a_{n+1}+a_n]x^n.$$

Section 5.2

1. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Then

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\, x^n.$$

Substitution into the ODE results in

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\, x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

or

$$\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} - a_n]x^n = 0\,.$$

Equating all the coefficients to *zero*,

$$(n+2)(n+1)a_{n+2} - a_n = 0\,, \qquad n = 0, 1, 2, \cdots.$$

We obtain the recurrence relation

$$a_{n+2} = \frac{a_n}{(n+1)(n+2)}\,, \qquad n = 0, 1, 2, \cdots.$$

The subscripts differ by *two*, so for $k = 1, 2, \cdots$

$$a_{2k} = \frac{a_{2k-2}}{(2k-1)2k} = \frac{a_{2k-4}}{(2k-3)(2k-2)(2k-1)2k} = \cdots = \frac{a_0}{(2k)!}$$

and

$$a_{2k+1} = \frac{a_{2k-1}}{2k(2k+1)} = \frac{a_{2k-3}}{(2k-2)(2k-1)2k(2k+1)} = \cdots = \frac{a_1}{(2k+1)!}\,.$$

Hence

$$y = a_0 \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} + a_1 \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}\,.$$

The linearly independent solutions are

$$y_1 = a_0\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots\right) = a_0 cosh\, x$$

$$y_2 = a_1\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots\right) = a_1 sinh\, x\,.$$

4. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Then

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\, x^n .$$

Substitution into the ODE results in

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\, x^n + k^2 x^2 \sum_{n=0}^{\infty} a_n x^n = 0\,.$$

Rewriting the *second* summation,

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\, x^n + \sum_{n=2}^{\infty} k^2 a_{n-2}\, x^n = 0\,,$$

that is,

$$2a_2 + 3\cdot 2\, a_3 x + \sum_{n=2}^{\infty} \left[(n+2)(n+1)a_{n+2} + k^2 a_{n-2}\right] x^n = 0\,.$$

Setting the coefficients equal to *zero*, we have $a_2 = 0$, $a_3 = 0$, and

$$(n+2)(n+1)a_{n+2} + k^2 a_{n-2} = 0\,, \quad \text{for } n = 2, 3, 4, \cdots .$$

The recurrence relation can be written as

$$a_{n+2} = -\frac{k^2 a_{n-2}}{(n+2)(n+1)}\,, \quad n = 2, 3, 4, \cdots .$$

The indices differ by *four*, so $a_4\,, a_8\,, a_{12}\,, \cdots$ are defined by

$$a_4 = -\frac{k^2 a_0}{4\cdot 3}\,,\; a_8 = -\frac{k^2 a_4}{8\cdot 7}\,, a_{12} = -\frac{k^2 a_8}{12\cdot 11}\,, \cdots .$$

Similarly, $a_5\,, a_9\,, a_{13}\,, \cdots$ are defined by

$$a_5 = -\frac{k^2 a_1}{5\cdot 4}\,,\; a_9 = -\frac{k^2 a_5}{9\cdot 8}\,, a_{13} = -\frac{k^2 a_9}{13\cdot 12}\,, \cdots .$$

The remaining coefficients are *zero*. Therefore the general solution is

$$y = a_0\left[1 - \frac{k^2}{4\cdot 3}x^4 + \frac{k^4}{8\cdot 7\cdot 4\cdot 3}x^8 - \frac{k^6}{12\cdot 11\cdot 8\cdot 7\cdot 4\cdot 3}x^{12} + \cdots\right] +$$
$$+ a_1\left[x - \frac{k^2}{5\cdot 4}x^5 + \frac{k^4}{9\cdot 8\cdot 5\cdot 4}x^9 - \frac{k^6}{13\cdot 12\cdot 9\cdot 8\cdot 4\cdot 4}x^{13} + \cdots\right].$$

Note that for the *even* coefficients,

$$a_{4m} = -\frac{k^2 a_{4m-4}}{(4m-1)4m}, \quad m = 1, 2, 3, \cdots$$

and for the *odd* coefficients,

$$a_{4m+1} = -\frac{k^2 a_{4m-3}}{4m(4m+1)}, \quad m = 1, 2, 3, \cdots.$$

Hence the linearly independent solutions are

$$y_1(x) = 1 + \sum_{m=0}^{\infty} \frac{(-1)^{m+1}\left(k^2 x^4\right)^{m+1}}{3 \cdot 4 \cdot 7 \cdot 8 \cdots (4m+3)(4m+4)}$$

$$y_2(x) = x\left[1 + \sum_{m=0}^{\infty} \frac{(-1)^{m+1}\left(k^2 x^4\right)^{m+1}}{4 \cdot 5 \cdot 8 \cdot 9 \cdots (4m+4)(4m+5)}\right].$$

6. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n.$$

Substitution into the ODE results in

$$\left(2 + x^2\right) \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n - x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 4 \sum_{n=0}^{\infty} a_n x^n = 0.$$

Before proceeding, write

$$x^2 \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n = \sum_{n=2}^{\infty} n(n-1) a_n x^n$$

and

$$x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=1}^{\infty} n\, a_n x^n.$$

It follows that

$$4a_0 + 4a_2 + (3a_1 + 12a_3)x + \sum_{n=2}^{\infty} \left[2(n+2)(n+1) a_{n+2} + n(n-1) a_n - n\, a_n + 4a_n\right] x^n = 0.$$

Equating the coefficients to *zero*, we find that $a_2 = -a_0$, $a_3 = -a_1/4$, and

$$a_{n+2} = -\frac{n^2 - 2n + 4}{2(n+2)(n+1)}\, a_n\,, \quad n = 0, 1, 2, \cdots.$$

The indices differ by *two*, so for $k = 0, 1, 2, \cdots$

$$a_{2k+2} = -\frac{(2k)^2 - 4k + 4}{2(2k+2)(2k+1)}\, a_{2k}$$

and

$$a_{2k+3} = -\frac{(2k+1)^2 - 4k + 2}{2(2k+3)(2k+2)}\, a_{2k+1}.$$

Hence the linearly independent solutions are

$$y_1(x) = 1 - x^2 + \frac{x^4}{6} - \frac{x^6}{30} + \cdots$$

$$y_2(x) = x - \frac{x^3}{4} + \frac{7x^5}{160} - \frac{19x^7}{1920} + \cdots.$$

7. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n.$$

Substitution into the ODE results in

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n + x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 2 \sum_{n=0}^{\infty} a_n x^n = 0\,.$$

First write

$$x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=1}^{\infty} n\, a_n x^n.$$

We then obtain

$$2a_2 + 2a_0 + \sum_{n=1}^{\infty} [(n+2)(n+1) a_{n+2} + n\, a_n + 2a_n] x^n = 0\,.$$

It follows that $a_2 = -a_0$ and $a_{n+2} = -a_n/(n+1)$, $n = 0, 1, 2, \cdots$. Note that the indices differ by *two*, so for $k = 1, 2, \cdots$

$$a_{2k} = -\frac{a_{2k-2}}{2k-1} = \frac{a_{2k-4}}{(2k-3)(2k-1)} = \cdots = \frac{(-1)^k a_0}{1 \cdot 3 \cdot 5 \cdots (2k-1)}$$

and

$$a_{2k+1} = -\frac{a_{2k-1}}{2k} = \frac{a_{2k-3}}{(2k-2)2k} = \cdots = \frac{(-1)^k a_1}{2 \cdot 4 \cdot 6 \cdots (2k)}.$$

Hence the linearly independent solutions are

$$y_1(x) = 1 - \frac{x^2}{1} + \frac{x^4}{1 \cdot 3} - \frac{x^6}{1 \cdot 3 \cdot 5} + \cdots = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$$

$$y_2(x) = x - \frac{x^3}{2} + \frac{x^5}{2 \cdot 4} - \frac{x^7}{2 \cdot 4 \cdot 6} + \cdots = x + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{2 \cdot 4 \cdot 6 \cdots (2n)}.$$

9. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n.$$

Substitution into the ODE results in

$$(1 + x^2) \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n - 4x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 6 \sum_{n=0}^{\infty} a_n x^n = 0.$$

Before proceeding, write

$$x^2 \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n = \sum_{n=2}^{\infty} n(n-1) a_n x^n$$

and

$$x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=1}^{\infty} n\, a_n x^n.$$

It follows that

$$6a_0 + 2a_2 + (2a_1 + 6a_3)x + \sum_{n=2}^{\infty} \left[(n+2)(n+1)a_{n+2} + n(n-1)a_n - 4n\,a_n + 6a_n\right]x^n = 0\,.$$

Setting the coefficients equal to *zero*, we obtain $a_2 = -3a_0$, $a_3 = -a_1/3$, and

$$a_{n+2} = -\frac{(n-2)(n-3)}{(n+1)(n+2)}\,a_n\,, \quad n = 0, 1, 2, \cdots.$$

Observe that for $n = 2$ and $n = 3$, we obtain $a_4 = a_5 = 0$. Since the indices differ by *two*, we also have $a_n = 0$ for $n \geq 4$. Therefore the general solution is a polynomial

$$y = a_0 + a_1 x - 3a_0 x^2 - a_1 x^3/3\,.$$

Hence the linearly independent solutions are

$$y_1(x) = 1 - 3x^2 \quad \text{and} \quad y_2(x) = x - x^3/3\,.$$

10. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Then

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\,x^n.$$

Substitution into the ODE results in

$$\left(4 - x^2\right)\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\,x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0\,.$$

First write

$$x^2 \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\,x^n = \sum_{n=2}^{\infty} n(n-1)a_n x^n.$$

It follows that

$$2a_0 + 8a_2 + (2a_1 + 24a_3)x + \sum_{n=2}^{\infty} \left[4(n+2)(n+1)a_{n+2} - n(n-1)a_n + 2a_n\right]x^n = 0\,.$$

We obtain $a_2 = -a_0/4$, $a_3 = -a_1/12$ and

$$4(n+2)a_{n+2} = (n-2)a_n\,, \quad n = 0, 1, 2, \cdots.$$

Note that for $n = 2$, $a_4 = 0$. Since the indices differ by *two*, we also have $a_{2k} = 0$ for $k = 2, 3, \cdots$. On the other hand, for $k = 1, 2, \cdots$,

$$a_{2k+1} = \frac{(2k-3)a_{2k-1}}{4(2k+1)} = \frac{(2k-5)(2k-3)a_{2k-3}}{4^2(2k-1)(2k+1)} = \cdots = \frac{-a_1}{4^k(2k-1)(2k+1)}\,.$$

Therefore the general solution is

$$y = a_0 + a_1 x - a_0 \frac{x^2}{4} - a_1 \sum_{n=1}^{\infty} \frac{x^{2n+1}}{4^n (2n-1)(2n+1)} \,.$$

Hence the linearly independent solutions are $y_1(x) = 1 - x^2/4$ and

$$y_2(x) = x - \frac{x^3}{12} - \frac{x^5}{240} - \frac{x^7}{2240} - \cdots = x - \sum_{n=1}^{\infty} \frac{x^{2n+1}}{4^n (2n-1)(2n+1)} \,.$$

11. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n .$$

Substitution into the ODE results in

$$\left(3 - x^2\right) \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n - 3x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} a_n x^n = 0 \,.$$

Before proceeding, write

$$x^2 \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n = \sum_{n=2}^{\infty} n(n-1) a_n x^n$$

and

$$x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=1}^{\infty} n\, a_n x^n .$$

It follows that

$$6a_2 - a_0 + (-4a_1 + 18a_3)x + \sum_{n=2}^{\infty} \left[3(n+2)(n+1) a_{n+2} - n(n-1) a_n - 3n\, a_n - a_n\right] x^n = 0.$$

We obtain $a_2 = a_0/6$, $2a_3 = a_1/9$, and

$$3(n+2) a_{n+2} = (n+1) a_n \,, \quad n = 0, 1, 2, \cdots .$$

The indices differ by *two*, so for $k = 1, 2, \cdots$

$$a_{2k} = \frac{(2k-1) a_{2k-2}}{3(2k)} = \frac{(2k-3)(2k-1) a_{2k-4}}{3^2 (2k-2)(2k)} = \cdots = \frac{3 \cdot 5 \cdots (2k-1)\, a_0}{3^k \cdot 2 \cdot 4 \cdots (2k)}$$

and

$$a_{2k+1} = \frac{(2k)a_{2k-1}}{3(2k+1)} = \frac{(2k-2)(2k)a_{2k-3}}{3^2(2k-1)(2k+1)} = \cdots = \frac{2\cdot 4\cdot 6\cdots(2k)\,a_1}{3^k\cdot 3\cdot 5\cdots(2k+1)}\,.$$

Hence the linearly independent solutions are

$$y_1(x) = 1 + \frac{x^2}{6} + \frac{x^4}{24} + \frac{5x^6}{432} + \cdots = 1 + \sum_{n=1}^{\infty}\frac{3\cdot 5\cdots(2n-1)\,x^{2n}}{3^n\cdot 2\cdot 4\cdots(2n)}$$

$$y_2(x) = x + \frac{2x^3}{9} + \frac{8x^5}{135} + \frac{16x^7}{945} + \cdots = x + \sum_{n=1}^{\infty}\frac{2\cdot 4\cdot 6\cdots(2n)\,x^{2n+1}}{3^n\cdot 3\cdot 5\cdots(2n+1)}\,.$$

12. Let $y = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$. Then

$$y' = \sum_{n=1}^{\infty} na_nx^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^n.$$

Substitution into the ODE results in

$$(1-x)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^n + x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - \sum_{n=0}^{\infty}a_nx^n = 0\,.$$

Before proceeding, write

$$x\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^n = \sum_{n=1}^{\infty}(n+1)n\,a_{n+1}x^n$$

and

$$x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n = \sum_{n=1}^{\infty}n\,a_nx^n.$$

It follows that

$$2a_2 - a_0 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} - (n+1)n\,a_{n+1} + n\,a_n - a_n\right]x^n = 0.$$

We obtain $a_2 = a_0/2$ and

$$(n+2)(n+1)a_{n+2} - (n+1)n\,a_{n+1} + (n-1)a_n = 0$$

for $n = 0, 1, 2, \cdots$. Writing out the individual equations,

$$\begin{aligned} 3\cdot 2\,a_3 - 2\cdot 1\,a_2 &= 0 \\ 4\cdot 3\,a_4 - 3\cdot 2\,a_3 + a_2 &= 0 \\ 5\cdot 4\,a_5 - 4\cdot 3\,a_4 + 2\,a_3 &= 0 \\ 6\cdot 5\,a_6 - 5\cdot 4\,a_5 + 3\,a_4 &= 0 \\ &\vdots \end{aligned}$$

The coefficients can be calculated successively as $a_3 = a_0/(2\cdot 3)$, $a_4 = a_3/2 - a_2/12$ $= a_0/24$, $a_5 = 3a_4/5 - a_3/10 = a_0/120$, $\cdots$. We can now see that for $n \geq 2$, a_n is proportional to a_0. In fact, for $n \geq 2$, $a_n = a_0/(n!)$. Therefore the general solution is

$$y = a_0 + a_1 x + \frac{a_0 x^2}{2!} + \frac{a_0 x^3}{3!} + \frac{a_0 x^4}{4!} + \cdots.$$

Hence the linearly independent solutions are $y_2(x) = x$ and

$$y_1(x) = 1 + \sum_{n=2}^{\infty} \frac{x^n}{n!}.$$

13. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n.$$

Substitution into the ODE results in

$$2\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n + x\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 3\sum_{n=0}^{\infty} a_n x^n = 0.$$

First write

$$x\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=1}^{\infty} n\, a_n x^n.$$

We then obtain

$$4a_2 + 3a_0 + \sum_{n=1}^{\infty} [2(n+2)(n+1) a_{n+2} + n\, a_n + 3a_n] x^n = 0.$$

It follows that $a_2 = -3a_0/4$ and

$$2(n+2)(n+1) a_{n+2} + (n+3) a_n = 0$$

for $n = 0, 1, 2, \cdots$. The indices differ by *two*, so for $k = 1, 2, \cdots$

$$\begin{aligned} a_{2k} &= -\frac{(2k+1)a_{2k-2}}{2(2k-1)(2k)} = \frac{(2k-1)(2k+1)a_{2k-4}}{2^2(2k-3)(2k-2)(2k-1)(2k)} = \cdots \\ &= \frac{(-1)^k 3 \cdot 5 \cdots (2k+1)}{2^k\,(2k)!}\, a_0 \,. \end{aligned}$$

and

$$\begin{aligned} a_{2k+1} &= -\frac{(2k+2)a_{2k-1}}{2(2k)(2k+1)} = \frac{(2k)(2k+2)a_{2k-3}}{2^2(2k-2)(2k-1)(2k)(2k+1)} = \cdots \\ &= \frac{(-1)^k\, 4 \cdot 6 \cdots (2k)(2k+2)}{2^k\,(2k+1)!}\, a_1 \,. \end{aligned}$$

Hence the linearly independent solutions are

$$y_1(x) = 1 - \frac{3}{4}x^2 + \frac{5}{32}x^4 - \frac{7}{384}x^6 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n 3 \cdot 5 \cdots (2n+1)}{2^n\,(2n)!} x^{2n}$$

$$y_2(x) = x - \frac{1}{3}x^3 + \frac{1}{20}x^5 - \frac{1}{210}x^7 + \cdots = x + \sum_{n=1}^{\infty} \frac{(-1)^n 4 \cdot 6 \cdots (2n+2)}{2^n\,(2n+1)!} x^{2n+1}.$$

15(a). From Prob. 2 , we have

$$y_1(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n\, n!} \quad \text{and} \quad y_2(x) = \sum_{n=0}^{\infty} \frac{2^n\, n!\, x^{2n+1}}{(2n+1)!} \;.$$

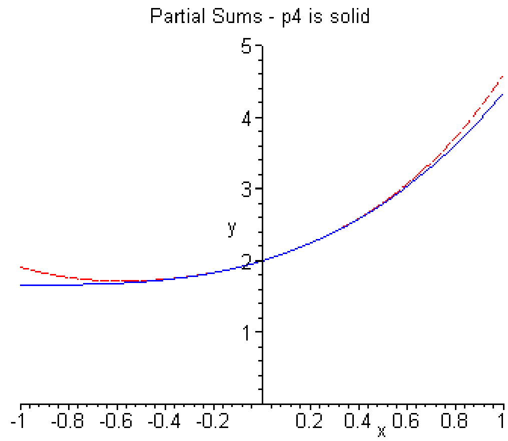
Since $a_0 = y(0)$ and $a_1 = y'(0)$, we have $y(x) = 2\, y_1(x) + y_2(x)$. That is,

$$y(x) = 2 + x + x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{15}x^5 + \frac{1}{24}x^6 + \cdots.$$

The *four*- and *five*-term polynomial approximations are

$$\begin{aligned} p_4 &= 2 + x + x^2 + x^3/3 \\ p_5 &= 2 + x + x^2 + x^3/3 + x^4/4 \,. \end{aligned}$$

(b).



(c). The *four-term* approximation p_4 appears to be reasonably accurate (within 10%) on the interval $|x| < 0.7$.

$17(a)$. From Prob. 7, the linearly independent solutions are

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$$

$$y_2(x) = x + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{2 \cdot 4 \cdot 6 \cdots (2n)}.$$

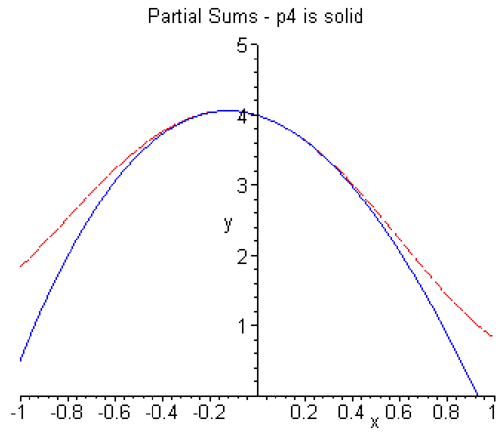
Since $a_0 = y(0)$ and $a_1 = y'(0)$, we have $y(x) = 4\,y_1(x) - y_2(x)$. That is,

$$y(x) = 4 - x - 4x^2 + \frac{1}{2}x^3 + \frac{4}{3}x^4 - \frac{1}{8}x^5 - \frac{4}{15}x^6 + \cdots.$$

The *four-* and *five*-term polynomial approximations are

$$p_4 = 4 - x - 4x^2 + \frac{1}{2}x^3$$
$$p_5 = 4 - x - 4x^2 + \frac{1}{2}x^3 + \frac{4}{3}x^4.$$

(b).



(c). The *four-term* approximation p_4 appears to be reasonably accurate (within 10%) on the interval $|x| < 0.5$.

$18(a)$. From Prob. 12, we have

$$y_1(x) = 1 + \sum_{n=2}^{\infty} \frac{x^n}{n!} \quad \text{and} \quad y_2(x) = x\,.$$

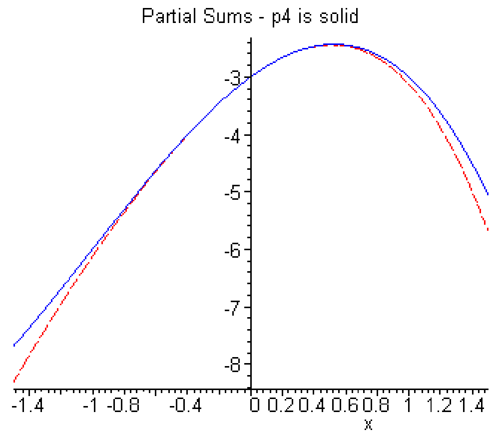
Since $a_0 = y(0)$ and $a_1 = y'(0)$, we have $y(x) = -3\,y_1(x) + 2\,y_2(x)$. That is,

$$y(x) = -3 + 2x - \frac{3}{2}x^2 - \frac{1}{2}x^3 - \frac{1}{8}x^4 - \frac{1}{40}x^5 - \frac{1}{240}x^6 + \cdots.$$

The *four*- and *five*-term polynomial approximations are

$$p_4 = -3 + 2x - \frac{3}{2}x^2 - \frac{1}{2}x^3$$
$$p_5 = -3 + 2x - \frac{3}{2}x^2 - \frac{1}{2}x^3 - \frac{1}{8}x^4\,.$$

(b).



(c). The *four-term* approximation p_4 appears to be reasonably accurate (within 10%) on the interval $|x| < 0.9$.

20. Two linearly independent solutions of *Airy's equation* (about $x_0 = 0$) are

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{x^{3n}}{2 \cdot 3 \cdots (3n-1)(3n)}$$

$$y_2(x) = x + \sum_{n=1}^{\infty} \frac{x^{3n+1}}{3 \cdot 4 \cdots (3n)(3n+1)}.$$

Applying the *ratio test* to the terms of $y_1(x)$,

$$\lim_{n\to\infty} \frac{|2 \cdot 3 \cdots (3n-1)(3n)\, x^{3n+3}|}{|2 \cdot 3 \cdots (3n+2)(3n+3)\, x^{3n}|} = \lim_{n\to\infty} \frac{1}{(3n+1)(3n+2)(3n+3)} |x|^3 = 0.$$

Similarly, applying the *ratio test* to the terms of $y_2(x)$,

$$\lim_{n\to\infty} \frac{|3 \cdot 4 \cdots (3n)(3n+1)\, x^{3n+4}|}{|3 \cdot 4 \cdots (3n+3)(3n+4)\, x^{3n+1}|} = \lim_{n\to\infty} \frac{1}{(3n+2)(3n+3)(3n+4)} |x|^3 = 0.$$

Hence both series converge *absolutely* for all x.

21. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\, x^n.$$

Substitution into the ODE results in

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\, x^n - 2x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \lambda\sum_{n=0}^{\infty} a_n x^n = 0\,.$$

First write

$$x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n = \sum_{n=1}^{\infty} n\, a_n x^n.$$

We then obtain

$$2a_2 + \lambda\, a_0 + \sum_{n=1}^{\infty}[(n+2)(n+1)a_{n+2} - 2n\, a_n + \lambda\, a_n]x^n = 0\,.$$

Setting the coefficients equal to *zero*, it follows that

$$a_{n+2} = \frac{(2n-\lambda)}{(n+1)(n+2)}\, a_n$$

for $n = 0, 1, 2, \cdots$. Note that the indices differ by *two*, so for $k = 1, 2, \cdots$

$$\begin{aligned} a_{2k} &= \frac{(4k-4-\lambda)a_{2k-2}}{(2k-1)2k} = \frac{(4k-8-\lambda)(4k-4-\lambda)a_{2k-4}}{(2k-3)(2k-2)(2k-1)2k} = \cdots \\ &= (-1)^k \frac{\lambda\cdots(\lambda-4k+8)(\lambda-4k+4)}{(2k)!}\, a_0\,. \end{aligned}$$

and

$$\begin{aligned} a_{2k+1} &= \frac{(4k-2-\lambda)a_{2k-1}}{2k(2k+1)} = \frac{(4k-6-\lambda)(4k-2-\lambda)a_{2k-3}}{(2k-2)(2k-1)2k(2k+1)} = \cdots \\ &= (-1)^k \frac{(\lambda-2)\cdots(\lambda-4k+6)(\lambda-4k+2)}{(2k+1)!}\, a_1. \end{aligned}$$

Hence the linearly independent solutions of the *Hermite equation* (about $x_0 = 0$) are

$$y_1(x) = 1 - \frac{\lambda}{2!}x^2 + \frac{\lambda(\lambda-4)}{4!}x^4 - \frac{\lambda(\lambda-4)(\lambda-8)}{6\,!}x^6 + \cdots$$

$$y_2(x) = x - \frac{\lambda-2}{3\,!}x^3 + \frac{(\lambda-2)(\lambda-6)}{5\,!}x^5 - \frac{(\lambda-2)(\lambda-6)(\lambda-10)}{7\,!}x^7 + \cdots.$$

(b). Based on the recurrence relation

$$a_{n+2} = \frac{(2n - \lambda)}{(n+1)(n+2)}\, a_n \,,$$

the series solution will *terminate* as long as λ is a *nonnegative* even integer. If $\lambda = 2m$, then *one or the other* of the solutions in Part (b) will contain at most $m/2 + 1$ terms. In particular, we obtain the polynomial solutions corresponding to $\lambda = 0, 2, 4, 6, 8, 10$:

$\lambda = 0$	$y_1(x) = 1$
$\lambda = 2$	$y_2(x) = x$
$\lambda = 4$	$y_1(x) = 1 - 2x^2$
$\lambda = 6$	$y_2(x) = x - 2x^3/3$
$\lambda = 8$	$y_1(x) = 1 - 4x^2 + 4x^4/3$
$\lambda = 10$	$y_2(x) = x - 4x^3/3 + 4x^5/15$

(c). Observe that if $\lambda = 2n$, and $a_0 = a_1 = 1$, then

$$a_{2k} = (\,-1)^k\, \frac{2n \cdots (2n - 4k + 8)(2n - 4k + 4)}{(2k)!}$$

and

$$a_{2k+1} = (\,-1)^k\, \frac{(2n-2)\cdots(2n - 4k + 6)(2n - 4k + 2)}{(2k+1)!}\,.$$

for $k = 1, 2, \cdots [n/2]$. It follows that the *coefficient* of x^n, in y_1 and y_2, is

$$a_n = \begin{cases} (\,-1)^k\, \frac{4^k\, k!}{(2k)!} & \text{for } n = 2k \\ (\,-1)^k\, \frac{4^k\, k!}{(2k+1)!} & \text{for } n = 2k+1 \end{cases}$$

Then by definition,

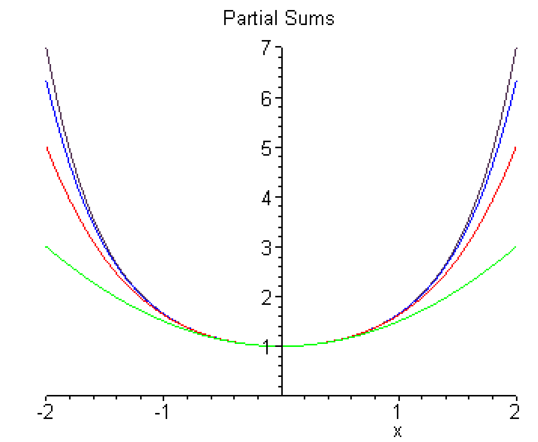
$$H_n(x) = \begin{cases} (\,-1)^k\, 2^n\, \frac{(2k)!}{4^k\, k!} y_1(x) = (\,-1)^k\, \frac{(2k)!}{k!} y_1(x) & \text{for } n = 2k \\ (\,-1)^k\, 2^n\, \frac{(2k+1)!}{4^k\, k!} y_2(x) = (\,-1)^k\, \frac{2\,(2k+1)!}{k!} y_2(x) & \text{for } n = 2k+1 \end{cases}$$

Therefore the first six *Hermite polynomials* are

$$\begin{aligned} H_0(x) &= 1 \\ H_1(x) &= 2x \\ H_2(x) &= 4x^2 - 2 \\ H_3(x) &= 8x^8 - 12x \\ H_4(x) &= 16x^4 - 48x^2 + 12 \\ H_5(x) &= 32x^5 - 160x^3 + 120x \end{aligned}$$

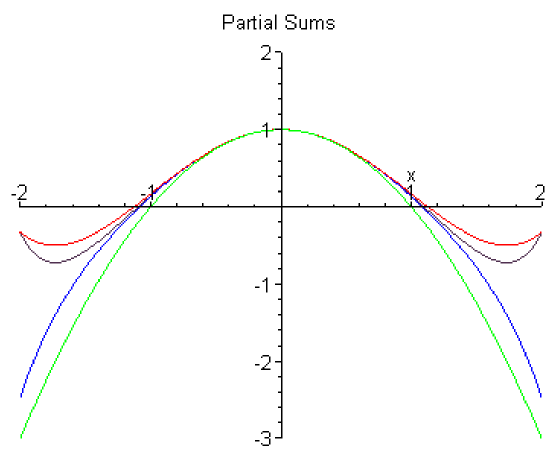
23. The series solution is given by

$$y(x) = 1 + \frac{1}{2}x^2 + \frac{1}{2^2\,2!}x^4 + \frac{1}{2^3 3!}x^6 + \frac{1}{2^4 4!}x^8 + \cdots .$$



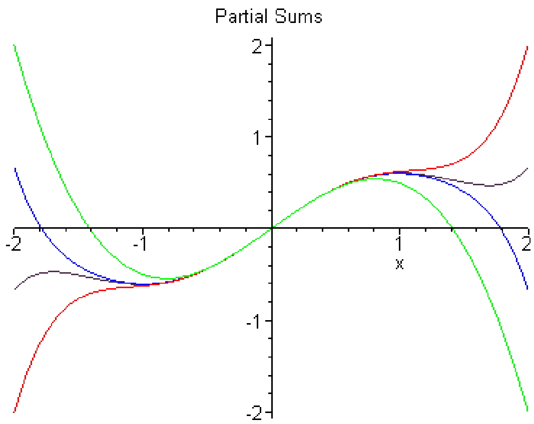
24. The series solution is given by

$$y(x) = 1 - x^2 + \frac{x^4}{6} - \frac{x^6}{30} + \frac{x^8}{120} + \cdots .$$



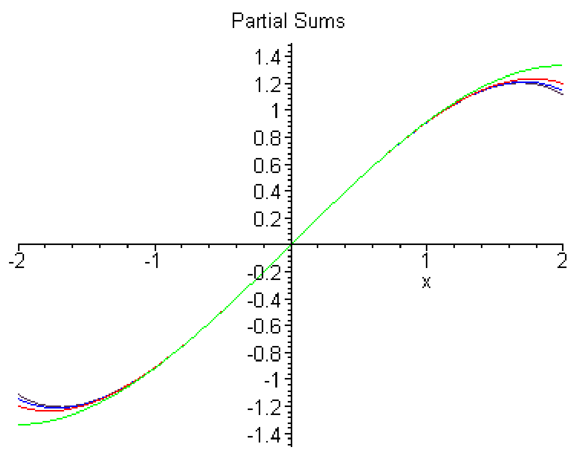
25. The series solution is given by

$$y(x) = x - \frac{x^3}{2} + \frac{x^5}{2\cdot 4} - \frac{x^7}{2\cdot 4\cdot 6} + \frac{x^9}{2\cdot 4\cdot 6\cdot 8} - \cdots .$$



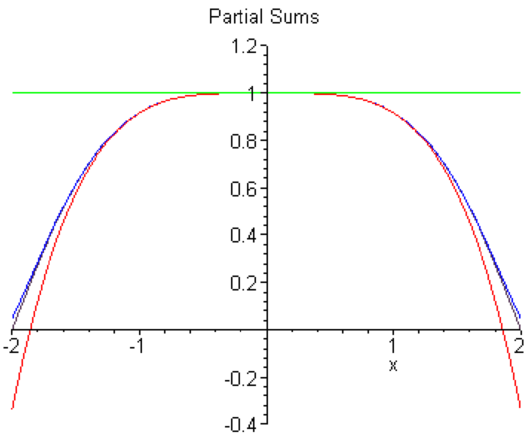
26. The series solution is given by

$$y(x) = x - \frac{x^3}{12} - \frac{x^5}{240} - \frac{x^7}{2240} - \frac{x^9}{16128} - \cdots.$$



27. The series solution is given by

$$y(x) = 1 - \frac{x^4}{12} + \frac{x^8}{672} - \frac{x^{12}}{88704} + \cdots.$$



28. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n .$$

Substitution into the ODE results in

$$(1-x)\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n + x\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - 2\sum_{n=0}^{\infty} a_n x^n = 0 \, .$$

After appropriately shifting the indices, it follows that

$$2a_2 - 2a_0 + \sum_{n=1}^{\infty} [(n+2)(n+1)a_{n+2} - (n+1)n\, a_{n+1} + n\, a_n - 2\, a_n] x^n = 0.$$

We find that $a_2 = a_0$ and

$$(n+2)(n+1)a_{n+2} - (n+1)n\, a_{n+1} + (n-2)a_n = 0$$

for $n = 1, 2, \cdots$. Writing out the individual equations,

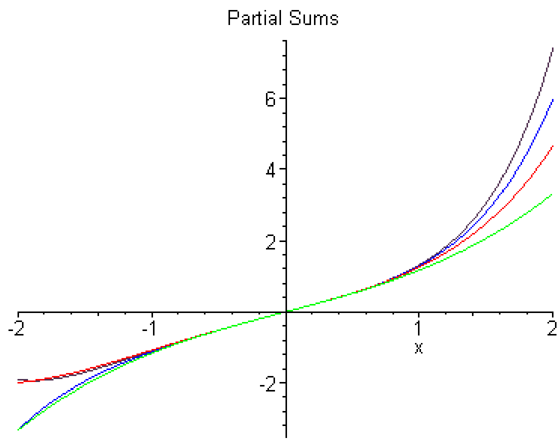
$$\begin{aligned} 3\cdot 2\, a_3 - 2\cdot 1\, a_2 - a_1 &= 0 \\ 4\cdot 3\, a_4 - 3\cdot 2\, a_3 &= 0 \\ 5\cdot 4\, a_5 - 4\cdot 3\, a_4 + a_3 &= 0 \\ 6\cdot 5\, a_6 - 5\cdot 4\, a_5 + 2\, a_4 &= 0 \\ &\vdots \end{aligned}$$

Since $a_0 = 0$ and $a_1 = 1$, the remaining coefficients satisfy the equations

$$\begin{aligned} 3 \cdot 2\, a_3 - 1 &= 0 \\ 4 \cdot 3\, a_4 - 3 \cdot 2\, a_3 &= 0 \\ 5 \cdot 4\, a_5 - 4 \cdot 3\, a_4 + a_3 &= 0 \\ 6 \cdot 5\, a_6 - 5 \cdot 4\, a_5 + 2\, a_4 &= 0 \\ &\vdots \end{aligned}$$

That is, $a_3 = 1/6\,, a_4 = 1/12\,, a_5 = 1/24\,, a_6 = 1/45\,, \cdots$. Hence the series solution of the initial value problem is

$$y(x) = x + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{24}x^5 + \frac{1}{45}x^6 + \frac{13}{1008}x^7 + \cdots.$$



Section 5.3

2. Let $y = \phi(x)$ be a solution of the initial value problem. First note that

$$y'' = -(sin\,x)y' - (cos\,x)y\,.$$

Differentiating twice,

$$y''' = -(sin\,x)y'' - 2(cos\,x)y' + (sin\,x)y$$
$$y^{iv} = -(sin\,x)y''' - 3(cos\,x)y'' + 3(sin\,x)y' + (cos\,x)y\,.$$

Given that $\phi(0) = 0$ and $\phi'(0) = 1$, the *first* equation gives $\phi''(0) = 0$ and the last two equations give $\phi'''(0) = -2$ and $\phi^{iv}(0) = 0$.

3. Let $y = \phi(x)$ be a solution of the initial value problem. First write

$$y'' = -\frac{1+x}{x^2}\,y' - \frac{3\,ln\,x}{x^2}\,y\,.$$

Differentiating twice,

$$y''' = \frac{-1}{x^3}\left[\left(x + x^2\right)y'' + (3x\,ln\,x - x - 2)y' + (3 - 6\,ln\,x)y\right].$$

$$y^{iv} = \frac{-1}{x^4}\Big[\left(x^2 + x^3\right)y''' + \left(3x^2 ln\,x - 2x^2 - 4x\right)y'' + \\ + (6 + 8x - 12x\,ln\,x)y' + (18\,ln\,x - 15)y\Big].$$

Given that $\phi(1) = 2$ and $\phi'(1) = 0$, the *first* equation gives $\phi''(1) = 0$ and the last two equations give $\phi'''(0) = -6$ and $\phi^{iv}(0) = 42$.

4. Let $y = \phi(x)$ be a solution of the initial value problem. First note that

$$y'' = -x^2\,y' - (sin\,x)y\,.$$

Differentiating twice,

$$y''' = -x^2\,y'' - (2x + sin\,x)y' - (cos\,x)y$$
$$y^{iv} = -x^2\,y''' - (4x + sin\,x)y'' - (2 + 2cos\,x)y' + (sin\,x)y\,.$$

Given that $\phi(0) = a_0$ and $\phi'(0) = a_1$, the *first* equation gives $\phi''(0) = 0$ and the last two equations give $\phi'''(0) = -a_0$ and $\phi^{iv}(0) = -4a_1$.

5. Clearly, $p(x) = 4$ and $q(x) = 6x$ are analytic for all x. Hence the series solutions converge *everywhere*.

7. The *zeroes* of $P(x) = 1 + x^3$ are the *three* cube roots of -1. They all lie on the unit circle in the complex plane. So for $x_0 = 0$, $\rho_{min} = 1$. For $x_0 = 2$, the *nearest*

root is $e^{i\pi/3} = \left(1 + i\sqrt{3}\right)/2$, hence $\rho_{min} = \sqrt{3}$.

8. The only root of $P(x) = x$ is *zero*. Hence $\rho_{min} = 1$.

9(b). $p(x) = -x$ and $q(x) = -1$ are analytic for all x .
(c). $p(x) = -x$ and $q(x) = -1$ are analytic for all x .
(d). $p(x) = 0$ and $q(x) = kx^2$ are analytic for all x .
(e). The only root of $P(x) = 1 - x$ is 1 . Hence $\rho_{min} = 1$.
(g). $p(x) = x$ and $q(x) = 2$ are analytic for all x .
(i). The *zeroes* of $P(x) = 1 + x^2$ are $\pm i$. Hence $\rho_{min} = 1$.
(j). The *zeroes* of $P(x) = 4 - x^2$ are ± 2 . Hence $\rho_{min} = 2$.
(k). The *zeroes* of $P(x) = 3 - x^2$ are $\pm\sqrt{3}$. Hence $\rho_{min} = \sqrt{3}$.
(l). The only root of $P(x) = 1 - x$ is 1 . Hence $\rho_{min} = 1$.
(m). $p(x) = x/2$ and $q(x) = 3/2$ are analytic for all x .
(n). $p(x) = (1 + x)/2$ and $q(x) = 3/2$ are analytic for all x .

12. The Taylor series expansion of e^x, about $x_0 = 0$, is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} .$$

Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Substituting into the ODE,

$$\left[\sum_{n=0}^{\infty} \frac{x^n}{n!}\right]\left[\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\, x^n\right] + x\sum_{n=0}^{\infty} a_n x^n = 0 .$$

First note that

$$x\sum_{n=0}^{\infty} a_n x^n = \sum_{n=1}^{\infty} a_{n-1} x^n = a_0 x + a_1 x^2 + a_2 x^3 + \cdots + a_{n-1} x^n + \cdots .$$

The coefficient of x^n in the *product* of the two series is

$$c_n = 2a_2 \frac{1}{n!} + 6a_3 \frac{1}{(n-1)!} + 12a_4 \frac{1}{(n-2)!} + \cdots + (n+1)n\, a_{n+1} + (n+2)(n+1)a_{n+2} .$$

Expanding the individual series, it follows that

$$2a_2 + (2a_2 + 6a_3)x + (a_2 + 6a_3 + 12a_4)x^2 + (a_2 + 6a_3 + 12a_4 + 20a_5)x^3 + \cdots + \\ + a_0 x + a_1 x^2 + a_2 x^3 + \cdots = 0 .$$

Setting the coefficients equal to *zero*, we obtain the system $2a_2 = 0$, $2a_2 + 6a_3 + a_0 = 0$, $a_2 + 6a_3 + 12a_4 + a_1 = 0$, $a_2 + 6a_3 + 12a_4 + 20a_5 + a_2 = 0$, $\cdots$. Hence the general solution is

$$y(x) = a_0 + a_1 x - a_0 \frac{x^3}{6} + (a_0 - a_1)\frac{x^4}{12} + (2a_1 - a_0)\frac{x^5}{40} + \left(\tfrac{4}{3}a_0 - 2a_1\right)\frac{x^6}{120} + \cdots.$$

We find that two linearly independent solutions are

$$y_1(x) = 1 - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{40} + \cdots$$

$$y_2(x) = x - \frac{x^4}{12} + \frac{x^5}{20} - \frac{x^6}{60} + \cdots$$

Since $p(x) = 0$ and $q(x) = xe^{-x}$ converge everywhere, $\rho = \infty$.

13. The Taylor series expansion of $cos\, x$, about $x_0 = 0$, is

$$cos\, x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} .$$

Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Substituting into the ODE,

$$\left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}\right]\left[\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\, x^n\right] + \sum_{n=1}^{\infty} n a_n x^n - 2\sum_{n=0}^{\infty} a_n x^n = 0 .$$

The coefficient of x^n in the *product* of the two series is

$$c_n = 2a_2 b_n + 6a_3 b_{n-1} + 12a_4 b_{n-2} + \cdots + (n+1)n\, a_{n+1} b_1 + (n+2)(n+1)a_{n+2} b_0 ,$$

in which $cos\, x = b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n + \cdots$. It follows that

$$2a_2 - 2a_0 + \sum_{n=1}^{\infty} c_n x^n + \sum_{n=1}^{\infty}(n-2)a_n x^n = 0 .$$

Expanding the product of the series, it follows that

$$2a_2 - 2a_0 + 6a_3 x + (-a_2 + 12a_4)x^2 + (-3a_3 + 20a_5)x^3 + \cdots - \\ - a_1 x + a_3 x^3 + 2a_4 x^4 + \cdots = 0 .$$

Setting the coefficients equal to *zero*, $a_2 - a_0 = 0$, $6a_3 - a_1 = 0$, $-a_2 + 12a_4 = 0$, $-3a_3 + 20a_5 + a_3 = 0$, $\cdots$. Hence the general solution is

$$y(x) = a_0 + a_1 x + a_0 x^2 + a_1 \frac{x^3}{6} + a_0 \frac{x^4}{12} + a_1 \frac{x^5}{60} + a_0 \frac{x^6}{120} + a_1 \frac{x^7}{560} + \cdots.$$

We find that two linearly independent solutions are

$$y_1(x) = 1 + x^2 + \frac{x^4}{12} + \frac{x^6}{120} + \cdots$$

$$y_2(x) = x + \frac{x^3}{6} + \frac{x^5}{60} + \frac{x^7}{560} + \cdots$$

The *nearest* zero of $P(x) = cos\, x$ is at $x = \pm\pi/2$. Hence $\rho_{min} = \pi/2$.

14. The Taylor series expansion of $ln(1+x)$, about $x_0 = 0$, is

$$ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}.$$

Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Substituting into the ODE,

$$\left[\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}\right] \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n + \\ + \left[\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}\right] \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - x \sum_{n=0}^{\infty} a_n x^n = 0.$$

The *first* product is the series

$$2a_2 + (-2a_2 + 6a_3)x + (a_2 - 6a_3 + 12a_4)x^2 + (-a_2 + 6a_3 - 12a_4 + 20a_5)x^3 + \cdots.$$

The *second* product is the series

$$a_1 x + (2a_2 - a_1/2)x^2 + (3a_3 - a_2 + a_1/3)x^3 + (4a_4 - 3a_3/2 + 2a_2/3 - a_1/4)x^3 + \cdots.$$

Combining the series and equating the coefficients to *zero*, we obtain

$$\begin{aligned} 2a_2 &= 0 \\ -2a_2 + 6a_3 + a_1 - a_0 &= 0 \\ 12a_4 - 6a_3 + 3a_2 - 3a_1/2 &= 0 \\ 20a_5 - 12a_4 + 9a_3 - 3a_2 + a_1/3 &= 0 \\ &\vdots \end{aligned}$$

Hence the general solution is

$$y(x) = a_0 + a_1 x + (a_0 - a_1)\frac{x^3}{6} + (2a_0 + a_1)\frac{x^4}{24} + a_1 \frac{7x^5}{120} + \left(\frac{5}{3}a_1 - a_0\right)\frac{x^6}{120} + \cdots.$$

We find that two linearly independent solutions are

$$y_1(x) = 1 + \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^6}{120} + \cdots$$

$$y_2(x) = x - \frac{x^3}{6} + \frac{x^4}{24} + \frac{7x^5}{120} + \cdots$$

The coefficient $p(x) = e^x ln(1+x)$ is analytic at $x_0 = 0$, but its power series has a radius of convergence $\rho = 1$.

15. If $y_1 = x$ and $y_2 = x^2$ are solutions, then substituting y_2 into the ODE results in

$$2\,P(x) + 2x\,Q(x) + x^2 R(x) = 0\,.$$

Setting $x = 0$, we find that $P(0) = 0$. Similarly, substituting y_1 into the ODE results in $Q(0) = 0$. Therefore $P(x)/Q(x)$ and $R(x)/P(x)$ may not be analytic. If they were, Theorem 3.2.1 would guarantee that y_1 and y_2 were the *only* two solutions. But note that an *arbitrary* value of $y(0)$ cannot be a linear combination of $y_1(0)$ and $y_2(0)$. Hence $x_0 = 0$ must be a singular point.

16. Let $y = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$. Substituting into the ODE,

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}\,x^n - \sum_{n=0}^{\infty} a_nx^n = 0\,.$$

That is,

$$\sum_{n=0}^{\infty}\left[(n+1)a_{n+1} - a_n\right]x^n = 0\,.$$

Setting the coefficients equal to *zero*, we obtain

$$a_{n+1} = \frac{a_n}{n+1}$$

for $n = 0, 1, 2, \cdots$. It is easy to see that $a_n = a_0/(n\,!)$. Therefore the general solution is

$$\begin{aligned} y(x) &= a_0\left[1 + x + \frac{x^2}{2\,!} + \frac{x^3}{3\,!} + \cdots\right] \\ &= a_0e^x. \end{aligned}$$

The coefficient $a_0 = y(0)$, which can be arbitrary.

17. Let $y = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$. Substituting into the ODE,

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}\,x^n - x\sum_{n=0}^{\infty} a_nx^n = 0\,.$$

That is,

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}\,x^n - \sum_{n=1}^{\infty} a_{n-1}x^n = 0\,.$$

Combining the series, we have

$$a_1 + \sum_{n=1}^{\infty} [(n+1)a_{n+1} - a_{n-1}]\, x^n = 0\,.$$

Setting the coefficient equal to *zero*, $a_1 = 0$ and $a_{n+1} = a_{n-1}/(n+1)$ for $n = 1, 2, \cdots$. Note that the indices differ by *two*, so for $k = 1, 2, \cdots$

$$a_{2k} = \frac{a_{2k-2}}{(2k)} = \frac{a_{2k-4}}{(2k-2)(2k)} = \cdots = \frac{a_0}{2 \cdot 4 \cdots (2k)}$$

and

$$a_{2k+1} = 0\,.$$

Hence the general solution is

$$\begin{aligned} y(x) &= a_0 \left[1 + \frac{x^2}{2} + \frac{x^4}{2^2\, 2\,!} + \frac{x^6}{2^3\, 3\,!} + \cdots + \frac{x^{2n}}{2^n\, n\,!} + \cdots \right] \\ &= a_0 exp\big(x^2/2\big). \end{aligned}$$

The coefficient $a_0 = y(0)$, which can be arbitrary.

19. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Substituting into the ODE,

$$(1-x)\sum_{n=0}^{\infty}(n+1)a_{n+1}\, x^n - \sum_{n=0}^{\infty} a_n x^n = 0\,.$$

That is,

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}\, x^n - \sum_{n=1}^{\infty} n\, a_n\, x^n - \sum_{n=0}^{\infty} a_n x^n = 0\,.$$

Combining the series, we have

$$a_1 - a_0 + \sum_{n=1}^{\infty} [(n+1)a_{n+1} - n\, a_n - a_n]\, x^n = 0\,.$$

Setting the coefficients equal to *zero*, $a_1 = a_0$ and $a_{n+1} = a_n$ for $n = 0, 1, 2, \cdots$. Hence the general solution is

$$\begin{aligned} y(x) &= a_0 \left[1 + x + x^2 + x^3 + \cdots + x^n + \cdots \right] \\ &= a_0 \frac{1}{1-x}\,. \end{aligned}$$

The coefficient $a_0 = y(0)$, which can be arbitrary.

21. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Substituting into the ODE,

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}\,x^n + x\sum_{n=0}^{\infty} a_n x^n = 1 + x\,.$$

That is,

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}\,x^n + \sum_{n=1}^{\infty} a_{n-1} x^n = 1 + x\,.$$

Combining the series, and the nonhomogeneous terms, we have

$$(a_1 - 1) + (2a_2 + a_0 - 1)x + \sum_{n=2}^{\infty}\left[(n+1)a_{n+1} + a_{n-1}\right]x^n = 0\,.$$

Setting the coefficients equal to *zero*, we obtain $a_1 = 1$, $2a_2 + a_0 - 1 = 0$, and

$$a_n = -\frac{a_{n-2}}{n}\,, \quad n = 3, 4, \cdots.$$

The indices differ by *two*, so for $k = 2, 3, \cdots$

$$a_{2k} = -\frac{a_{2k-2}}{(2k)} = \frac{a_{2k-4}}{(2k-2)(2k)} = \cdots = \frac{(-1)^{k-1}\,a_2}{4\cdot 6\cdots(2k)} = \frac{(-1)^k(a_0 - 1)}{2\cdot 4\cdot 6\cdots(2k)}\,,$$

and for $k = 1, 2, \cdots$

$$a_{2k+1} = = -\frac{a_{2k-1}}{(2k+1)} = \frac{a_{2k-3}}{(2k-1)(2k+1)} = \cdots = \frac{(-1)^k}{3\cdot 5\cdots(2k+1)}\,.$$

Hence the general solution is

$$y(x) = a_0 + x + \frac{1-a_0}{2}x^2 - \frac{x^3}{3} + a_0\frac{x^4}{2^2\,2\,!} + \frac{x^5}{3\cdot 5} - a_0\frac{x^6}{2^3\,3\,!} - \cdots$$

Collecting the terms containing a_0,

$$\begin{aligned} y(x) \;&= a_0\left[1 - \frac{x^2}{2} + \frac{x^4}{2^2\,2\,!} - \frac{x^6}{2^3\,3\,!} + \cdots\right] + \\ &+ \left[x + \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{2^2\,2\,!} + \frac{x^5}{3\cdot 5} + \frac{x^6}{2^3\,3\,!} - \frac{x^7}{3\cdot 5\cdot 7} + \cdots\right]. \end{aligned}$$

Upon inspection, we find that

$$y(x) = a_0 exp(-x^2/2) + \left[x + \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{2^2\,2\,!} + \frac{x^5}{3\cdot 5} + \frac{x^6}{2^3\,3\,!} - \frac{x^7}{3\cdot 5\cdot 7} + \cdots\right].$$

Note that the given ODE is *first order linear*, with integrating factor $\mu(t) = e^{x^2/2}$. The general solution is given by

$$y(x) = e^{-x^2/2}\int_0^x e^{u^2/2}du + (y(0)-1)e^{-x^2/2} + 1\,.$$

23. If $\alpha = 0$, then $y_1(x) = 1$. If $\alpha = 2n$, then $a_{2m} = 0$ for $m \geq n+1$. As a result,

$$y_1(x) = 1 + \sum_{m=1}^{n} (-1)^m \frac{2^m n(n-1)\cdots(n-m+1)(2n+1)(2n+3)\cdots(2n+2m-1)}{(2m)!}\, x^{2m}.$$

$\alpha = 0$	1
$\alpha = 2$	$1 - 3x^2$
$\alpha = 4$	$1 - 10x^2 + \frac{35}{3}x^4$

If $\alpha = 2n+1$, then $a_{2m+1} = 0$ for $m \geq n+1$. As a result,

$$y_2(x) = x + \sum_{m=1}^{n} (-1)^m \frac{2^m n(n-1)\cdots(n-m+1)(2n+3)(2n+5)\cdots(2n+2m+1)}{(2m+1)!}\, x^{2m+1}.$$

$\alpha = 1$	x
$\alpha = 3$	$x - \frac{5}{3}x^3$
$\alpha = 5$	$x - \frac{14}{3}x^3 + \frac{21}{5}x^5$

24(a). Based on Prob. 23,

$\alpha = 0$	1	$y_1(1) = 1$
$\alpha = 2$	$1 - 3x^2$	$y_1(1) = -2$
$\alpha = 4$	$1 - 10x^2 + \frac{35}{3}x^4$	$y_1(1) = \frac{8}{3}$

Normalizing the polynomials, we obtain

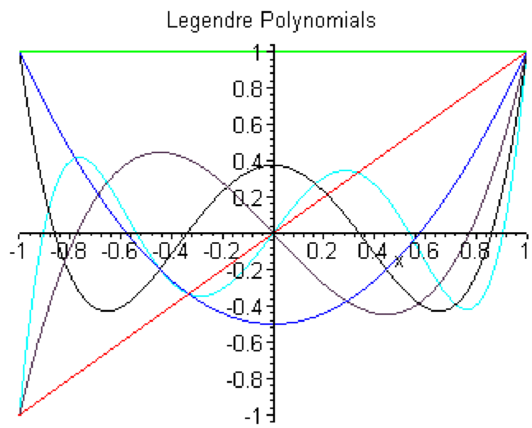
$$P_0(x) = 1$$
$$P_2(x) = -\frac{1}{2} + \frac{3}{2}x^2$$
$$P_4(x) = \frac{3}{8} - \frac{15}{4}x^2 + \frac{35}{8}x^4$$

$\alpha = 1$	x	$y_2(1) = 1$
$\alpha = 3$	$x - \frac{5}{3}x^3$	$y_2(1) = -\frac{2}{3}$
$\alpha = 5$	$x - \frac{14}{3}x^3 + \frac{21}{5}x^5$	$y_2(1) = \frac{8}{15}$

Similarly,

$$P_1(x) = x$$
$$P_3(x) = -\frac{3}{2}x + \frac{5}{2}x^3$$
$$P_5(x) = \frac{15}{8}x - \frac{35}{4}x^3 + \frac{63}{8}x^5$$

(b).



(c). $P_0(x)$ has no roots. $P_1(x)$ has one root at $x = 0$. The *zeros* of $P_2(x)$ are at $x = \pm 1/\sqrt{3}$. The *zeros* of $P_3(x)$ are $x = 0, \pm\sqrt{3/5}$. The roots of $P_4(x)$ are given by $x^2 = \left(15 + 2\sqrt{30}\right)/35$, $\left(15 - 2\sqrt{30}\right)/35$. The roots of $P_5(x)$ are given by $x = 0$ and $x^2 = \left(35 + 2\sqrt{70}\right)/63$, $\left(35 - 2\sqrt{70}\right)/63$.

25. Observe that

$$\begin{aligned} P_n(-1) &= \frac{(-1)^n}{2^n} \sum_{k=0}^{[n/2]} \frac{(-1)^k (2n-2k)!}{k!(n-k)!(n-2k)!} \\ &= (-1)^n P_n(1). \end{aligned}$$

But $P_n(1) = 1$ for *all* nonnegative integers n.

27. We have

$$(x^2 - 1)^n = \sum_{k=0}^{n} \frac{(-1)^{n-k}\, n!}{k!(n-k)!}\, x^{2k},$$

which is a polynomial of *degree* $2n$. Differentiating *n times*,

$$\frac{d^n}{dx^n}(x^2-1)^n = \sum_{k=\mu}^{n} \frac{(-1)^{n-k}\, n!}{k!(n-k)!}(2k)(2k-1)\cdots(2k-n+1)x^{2k-n},$$

in which the *lower index* is $\mu = [n/2] + 1$. Note that if $n = 2m+1$, then $\mu = m + 1$.

Now shift the index, by setting

$$k = n - j\,.$$

Hence

$$\begin{aligned}\frac{d^n}{dx^n}\left(x^2-1\right)^n &= \sum_{j=0}^{[n/2]}\frac{(-1)^j\,n\,!}{(n-j)!j\,!}(2n-2j)(2n-2j-1)\cdots(n-2j+1)x^{n-2j}\\ &= n!\sum_{j=0}^{[n/2]}\frac{(-1)^j(2n-2j)!}{(n-j)!j\,!(n-2j)!}\,x^{n-2j}.\end{aligned}$$

Based on Prob. 25,

$$\frac{d^n}{dx^n}\left(x^2-1\right)^n = n!\,2^n P_n(x).$$

29. Since the $n+1$ polynomials $P_0, P_1, \cdots, P_n$ are *linearly independent*, and the *degree* of P_k is k , any polynomial, f , of degree n can be expressed as a linear combination

$$f(x) = \sum_{k=0}^{n} a_k P_k(x)\,.$$

Multiplying both sides by P_m and integrating,

$$\int_{-1}^{1} f(x)P_m(x)dx = \sum_{k=0}^{n} a_k \int_{-1}^{1} P_k(x)P_m(x)dx\,.$$

Based on Prob. 28,

$$\int_{-1}^{1} P_k(x)P_m(x)dx = \frac{2}{2m+1}\delta_{km}.$$

Hence

$$\int_{-1}^{1} f(x)P_m(x)dx = \frac{2}{2m+1}\,a_m\,.$$

Section 5.4

2. We see that $P(x) = 0$ when $x = 0$ and 1. Since the three coefficients have no factors
in common, both of these points are singular points. Near $x = 0$,

$$\lim_{x \to 0} x\, p(x) = \lim_{x \to 0} x \frac{2x}{x^2 (1-x)^2} = 2\,.$$

$$\lim_{x \to 0} x^2 q(x) = \lim_{x \to 0} x^2 \frac{4}{x^2 (1-x)^2} = 4\,.$$

The singular point $x = 0$ is *regular*. Considering $x = 1$,

$$\lim_{x \to 1} (x-1) p(x) = \lim_{x \to 1} (x-1) \frac{2x}{x^2 (1-x)^2}\,.$$

The latter limit *does not exist*. Hence $x = 1$ is an *irregular* singular point.

3. $P(x) = 0$ when $x = 0$ and 1. Since the three coefficients have no common factors, both of these points are singular points. Near $x = 0$,

$$\lim_{x \to 0} x\, p(x) = \lim_{x \to 0} x \frac{x-2}{x^2 (1-x)}\,.$$

The limit *does not exist*, and so $x = 0$ is an *irregular* singular point. Considering $x = 1$,

$$\lim_{x \to 1} (x-1) p(x) = \lim_{x \to 1} (x-1) \frac{x-2}{x^2 (1-x)} = 1\,.$$

$$\lim_{x \to 1} (x-1)^2 q(x) = \lim_{x \to 1} (x-1)^2 \frac{-3x}{x^2 (1-x)} = 0\,.$$

Hence $x = 1$ is a *regular* singular point.

4. $P(x) = 0$ when $x = 0$ and ± 1. Since the three coefficients have no common factors, both of these points are singular points. Near $x = 0$,

$$\lim_{x \to 0} x\, p(x) = \lim_{x \to 0} x \frac{2}{x^3 (1-x^2)}\,.$$

The limit *does not exist*, and so $x = 0$ is an *irregular* singular point. Near $x = -1$,

$$\lim_{x \to -1} (x+1) p(x) = \lim_{x \to -1} (x+1) \frac{2}{x^3 (1-x^2)} = -1\,.$$

$$\lim_{x\to -1}(x+1)^2 q(x) = \lim_{x\to -1}(x+1)^2 \frac{2}{x^3(1-x^2)} = 0\,.$$

Hence $x = -1$ is a *regular* singular point. At $x = 1$,

$$\lim_{x\to 1}(x-1)p(x) = \lim_{x\to 1}(x-1)\frac{2}{x^3(1-x^2)} = -1\,.$$

$$\lim_{x\to 1}(x-1)^2 q(x) = \lim_{x\to 1}(x-1)^2 \frac{2}{x^3(1-x^2)} = 0\,.$$

Hence $x = 1$ is a *regular* singular point.

6. The only singular point is at $x = 0$. We find that

$$\lim_{x\to 0} x\, p(x) = \lim_{x\to 0} x \frac{x}{x^2} = 1\,.$$

$$\lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} x^2 \, \frac{x^2 - \nu^2}{x^2} = -\nu^2.$$

Hence $x = 0$ is a *regular* singular point.

7. The only singular point is at $x = -3$. We find that

$$\lim_{x\to -3}(x+3)p(x) = \lim_{x\to -3}(x+3)\frac{-2x}{x+3} = 6\,.$$

$$\lim_{x\to -3}(x+3)^2 q(x) = \lim_{x\to -3}(x+3)^2 \, \frac{1-x^2}{x+3} = 0.$$

Hence $x = -3$ is a *regular* singular point.

8. Dividing the ODE by $x(1-x^2)^3$, we find that

$$p(x) = \frac{1}{x(1-x^2)} \quad \text{and} \quad q(x) = \frac{2}{x(1+x)^2(1-x)^3}\,.$$

The singular points are at $x = 0$ and ± 1. For $x = 0$,

$$\lim_{x\to 0} x\, p(x) = \lim_{x\to 0} x \frac{1}{x(1-x^2)} = 1\,.$$

$$\lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} x^2 \, \frac{2}{x(1+x)^2(1-x)^3} = 0\,.$$

Hence $x = 0$ is a *regular* singular point. For $x = -1$,

$$\lim_{x\to-1}(x+1)p(x)=\lim_{x\to-1}(x+1)\frac{1}{x(1-x^2)} = -\frac{1}{2}.$$

$$\lim_{x\to-1}(x+1)^2 q(x)=\lim_{x\to-1}(x+1)^2\frac{2}{x(1+x)^2(1-x)^3} = -\frac{1}{4}.$$

Hence $x = -1$ is a *regular* singular point. For $x = 1$,

$$\lim_{x\to1}(x-1)p(x)=\lim_{x\to1}(x-1)\frac{1}{x(1-x^2)} = -\frac{1}{2}.$$

$$\lim_{x\to1}(x-1)^2 q(x)=\lim_{x\to1}(x-1)^2\frac{2}{x(1+x)^2(1-x)^3}.$$

The latter limit *does not exist*. Hence $x = 1$ is an *irregular* singular point.

9. Dividing the ODE by $(x+2)^2(x-1)$, we find that

$$p(x)=\frac{3}{(x+2)^2} \quad\text{and}\quad q(x)=\frac{-2}{(x+2)(x-1)}.$$

The singular points are at $x = -2$ and 1. For $x = -2$,

$$\lim_{x\to-2}(x+2)p(x)=\lim_{x\to-2}(x+2)\frac{3}{(x+2)^2}.$$

The limit *does not exist*. Hence $x = -2$ is an *irregular* singular point. For $x = 1$,

$$\lim_{x\to1}(x-1)p(x)=\lim_{x\to1}(x-1)\frac{3}{(x+2)^2}=0.$$

$$\lim_{x\to1}(x-1)^2 q(x)=\lim_{x\to1}(x-1)^2\frac{-2}{(x+2)(x-1)}=0.$$

Hence $x = 1$ is a *regular* singular point.

10. $P(x) = 0$ when $x = 0$ and 3. Since the three coefficients have no common factors, both of these points are singular points. Near $x = 0$,

$$\lim_{x\to0}x\,p(x)=\lim_{x\to0}x\frac{x+1}{x(3-x)}=\frac{1}{3}.$$

$$\lim_{x\to0}x^2 q(x)=\lim_{x\to0}x^2\frac{-2}{x(3-x)}=0.$$

Hence $x = 0$ is a *regular* singular point. For $x = 3$,

$$\lim_{x\to 3}(x-3)p(x) = \lim_{x\to 3}(x-3)\frac{x+1}{x(3-x)} = -\frac{4}{3}.$$

$$\lim_{x\to 3}(x-3)^2 q(x) = \lim_{x\to 3}(x-3)^2\frac{-2}{x(3-x)} = 0.$$

Hence $x=3$ is a *regular* singular point.

11. Dividing the ODE by (x^2+x-2), we find that

$$p(x) = \frac{x+1}{(x+2)(x-1)} \quad \text{and} \quad q(x) = \frac{2}{(x+2)(x-1)}.$$

The singular points are at $x = -2$ and 1. For $x = -2$,

$$\lim_{x\to -2}(x+2)p(x) = \lim_{x\to -2}\frac{x+1}{x-1} = \frac{1}{3}.$$

$$\lim_{x\to -2}(x+2)^2 q(x) = \lim_{x\to -2}\frac{2(x+2)}{x-1} = 0.$$

Hence $x = -2$ is a *regular* singular point. For $x=1$,

$$\lim_{x\to 1}(x-1)p(x) = \lim_{x\to 1}\frac{x+1}{x+2} = \frac{2}{3}.$$

$$\lim_{x\to 1}(x-1)^2 q(x) = \lim_{x\to 1}\frac{2(x-1)}{(x+2)} = 0.$$

Hence $x=1$ is a *regular* singular point.

13. Note that $p(x) = ln|x|$ and $q(x) = 3x$. Evidently, $p(x)$ is *not* analytic at $x_0 = 0$. Furthermore, the function $x\,p(x) = x\,ln|x|$ does *not* have a Taylor series about $x_0 = 0$. Hence $x=0$ is an *irregular* singular point.

14. $P(x) = 0$ when $x=0$. Since the three coefficients have no common factors, $x=0$ is a singular point. The Taylor series of $e^x - 1$, about $x=0$, is

$$e^x - 1 = x + x^2/2 + x^3/6 + \cdots.$$

Hence the function $x\,p(x) = 2(e^x-1)/x$ is analytic at $x=0$. Similarly, the Taylor series of $e^{-x}cos\,x$, about $x=0$, is

$$e^{-x}cos\,x = 1 - x + x^3/3 - x^4/6 + \cdots.$$

The function $x^2q(x) = e^{-x}cos\,x$ is also analytic at $x=0$. Hence $x=0$ is a *regular* singular point.

15. $P(x) = 0$ when $x = 0$. Since the three coefficients have no common factors, $x = 0$ is a singular point. The Taylor series of $sin\, x$, about $x = 0$, is

$$sin\, x = x - x^3/3! + x^5/5! - \cdots .$$

Hence the function $x\, p(x) = -3sin\, x/x$ is analytic at $x = 0$. On the other hand, $q(x)$ is a rational function, with

$$\lim_{x \to 0} x^2 q(x) = \lim_{x \to 0} x^2 \frac{1 + x^2}{x^2} = 1 .$$

Hence $x = 0$ is a *regular* singular point.

16. $P(x) = 0$ when $x = 0$. Since the three coefficients have no common factors, $x = 0$ is a singular point. We find that

$$\lim_{x \to 0} x\, p(x) = \lim_{x \to 0} x \frac{1}{x} = 1 .$$

Although the function $R(x) = cot\, x$ does not have a Taylor series about $x = 0$, note that $x^2 q(x) = x\, cot\, x = 1 - x^2/3 - x^4/45 - 2x^6/945 - \cdots$. Hence $x = 0$ is a *regular* singular point. Furthermore, $q(x) = cot\, x/x^2$ is undefined at $x = \pm n\pi$. Therefore the points $x = \pm n\pi$ are *also* singular points. First note that

$$\lim_{x \to \pm n\pi} (x \mp n\pi) p(x) = \lim_{x \to \pm n\pi} (x \mp n\pi) \frac{1}{x} = 0 .$$

Furthermore, since $cot\, x$ has period π,

$$\begin{aligned} q(x) &= cot\, x/x = cot(x \mp n\pi)/x \\ &= cot(x \mp n\pi) \frac{1}{(x \mp n\pi) \pm n\pi} . \end{aligned}$$

Therefore

$$(x \mp n\pi)^2 q(x) = (x \mp n\pi) cot(x \mp n\pi) \left[\frac{(x \mp n\pi)}{(x \mp n\pi) \pm n\pi} \right] .$$

From above,

$$(x \mp n\pi) cot(x \mp n\pi) = 1 - (x \mp n\pi)^2/3 - (x \mp n\pi)^4/45 - \cdots .$$

Note that the function in *brackets* is analytic *near* $x = \pm n\pi$. It follows that the function $(x \mp n\pi)^2 q(x)$ is also analytic near $x = \pm n\pi$. Hence all the singular points are *regular*.

18. The singular points are located at $x = \pm n\pi$, $n = 0, 1, \cdots$. Dividing the ODE by $x\, sin\, x$, we find that $x\, p(x) = 3\, csc\, x$ and $x^2 q(x) = x^2 csc\, x$. Evidently, $x\, p(x)$ is not even defined at $x = 0$. Hence $x = 0$ is an *irregular* singular point. On the other hand, the Taylor series of $x\, csc\, x$, about $x = 0$, is

$$x\,csc\,x = 1 + x^2/6 + 7x^4 360 + \cdots.$$

Noting that $csc(x \mp n\pi) = (-1)^n csc\,x$,

$$\begin{aligned}(x \mp n\pi)p(x) &= 3(-1)^n (x \mp n\pi)csc(x \mp n\pi)/x \\ &= 3(-1)^n (x \mp n\pi)csc(x \mp n\pi)\left[\frac{1}{(x \mp n\pi) \pm n\pi}\right].\end{aligned}$$

It is apparent that $(x \mp n\pi)p(x)$ is analytic at $x = \pm n\pi$. Similarly,

$$\begin{aligned}(x \mp n\pi)^2 q(x) &= (x \mp n\pi)^2 csc\,x \\ &= (-1)^n (x \mp n\pi)^2 csc(x \mp n\pi),\end{aligned}$$

which is also analytic at $x = \pm n\pi$. Hence all other singular points are *regular*.

20. $x = 0$ is the only singular point. Dividing the ODE by $2x^2$, we have $p(x) = 3/(2x)$ and $q(x) = -x^{-2}(1+x)/2$. It follows that

$$\lim_{x \to 0} x\,p(x) = \lim_{x \to 0} x\frac{3}{2x} = \frac{3}{2},$$

$$\lim_{x \to 0} x^2 q(x) = \lim_{x \to 0} x^2 \frac{-(1+x)}{2x^2} = -\frac{1}{2}.$$

Hence $x = 0$ is a *regular* singular point. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Substitution into the ODE results in

$$2x^2 \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\,x^n + 3x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - (1+x)\sum_{n=0}^{\infty} a_n x^n = 0.$$

That is,

$$2\sum_{n=2}^{\infty} n(n-1)a_n\,x^n + 3\sum_{n=1}^{\infty} n\,a_n x^n - \sum_{n=0}^{\infty} a_n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0.$$

It follows that

$$-a_0 + (2a_1 - a_0)x + \sum_{n=2}^{\infty}[2n(n-1)a_n + 3n\,a_n - a_n - a_{n-1}]x^n = 0.$$

Equating the coefficients to *zero*, we find that $a_0 = 0$, $2a_1 - a_0 = 0$, and

$$(2n-1)(n+1)a_n = a_{n-1}, \quad n = 2, 3, \cdots.$$

We conclude that *all* the a_n are *equal to zero*. Hence $y(x) = 0$ is the only solution that can be obtained.

22. Based on Prob. 21, the change of variable, $x = 1/\xi$, transforms the ODE into the

form

$$\xi^4 \frac{d^2y}{d\xi^2} + 2\xi^3 \frac{dy}{d\xi} + y = 0\,.$$

Evidently, $\xi = 0$ is a singular point. Now $p(\xi) = 2/\xi$ and $q(\xi) = 1/\xi^4$. Since the value of $\lim_{\xi \to 0} \xi^2 q(\xi)$ does not exist, $\xi = 0$, that is, $x = \infty$, is an *irregular* singular point.

24. Under the transformation $x = 1/\xi$, the ODE becomes

$$\xi^4 \left(1 - \frac{1}{\xi^2}\right) \frac{d^2y}{d\xi^2} + \left[2\xi^3 \left(1 - \frac{1}{\xi^2}\right) + 2\xi^2 \frac{1}{\xi}\right] \frac{dy}{d\xi} + \alpha(\alpha+1)y = 0\,,$$

that is,

$$\left(\xi^4 - \xi^2\right) \frac{d^2y}{d\xi^2} + 2\xi^3 \frac{dy}{d\xi} + \alpha(\alpha+1)y = 0\,.$$

Therefore $\xi = 0$ is a singular point. Note that

$$p(\xi) = \frac{2\xi}{\xi^2 - 1} \text{ and } q(\xi) = \frac{\alpha(\alpha+1)}{\xi^2(\xi^2-1)}\,.$$

It follows that

$$\lim_{\xi \to 0} \xi\, p(\xi) = \lim_{\xi \to 0} \xi \frac{2\xi}{\xi^2 - 1} = 0\,,$$

$$\lim_{\xi \to 0} \xi^2 q(\xi) = \lim_{\xi \to 0} \xi^2 \frac{\alpha(\alpha+1)}{\xi^2(\xi^2-1)} = -\alpha(\alpha+1)\,.$$

Hence $\xi = 0$ $(x = \infty)$ is a *regular* singular point.

26. Under the transformation $x = 1/\xi$, the ODE becomes

$$\xi^4 \frac{d^2y}{d\xi^2} + \left[2\xi^3 + 2\xi^2 \frac{1}{\xi}\right] \frac{dy}{d\xi} + \lambda\, y = 0\,,$$

that is,

$$\xi^4 \frac{d^2y}{d\xi^2} + 2\left(\xi^3 + \xi\right) \frac{dy}{d\xi} + \lambda\, y = 0\,.$$

Therefore $\xi = 0$ is a singular point. Note that

$$p(\xi) = \frac{2(\xi^2+1)}{\xi^3} \text{ and } q(\xi) = \frac{\lambda}{\xi^4}\,.$$

It immediately follows that the limit $\lim_{\xi \to 0} \xi\, p(\xi)$ *does not exist*. Hence $\xi = 0$ $(x = \infty)$

is an *irregular* singular point.

27. Under the transformation $x = 1/\xi$, the ODE becomes

$$\xi^4 \frac{d^2y}{d\xi^2} + 2\xi^3 \frac{dy}{d\xi} - \frac{1}{\xi} y = 0 \,.$$

Therefore $\xi = 0$ is a singular point. Note that

$$p(\xi) = \frac{2}{\xi} \text{ and } q(\xi) = \frac{-1}{\xi^5}.$$

We find that

$$\lim_{\xi \to 0} \xi \, p(\xi) = \lim_{\xi \to 0} \xi \frac{2}{\xi} = 2 \,,$$

but

$$\lim_{\xi \to 0} \xi^2 q(\xi) = \lim_{\xi \to 0} \xi^2 \, \frac{(-1)}{\xi^5} \,.$$

The latter limit *does not exist*. Hence $\xi = 0 \ (x = \infty)$ is an *irregular* singular point.

Section 5.5

1. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where

$$\begin{aligned} F(r) &= r(r-1) + 4r + 2 \\ &= r^2 + 3r + 2\,. \end{aligned}$$

The roots are $r = -2\,,\ -1$. Hence the general solution, for $x \neq 0$, is

$$y = c_1 x^{-2} + c_2 x^{-1}.$$

3. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where

$$\begin{aligned} F(r) &= r(r-1) - 3r + 4 \\ &= r^2 - 4r + 4\,. \end{aligned}$$

The root is $r = 2$, with multiplicity *two*. Hence the general solution, for $x \neq 0$, is

$$y = (c_1 + c_2\, ln|x|)\, x^2.$$

5. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where

$$\begin{aligned} F(r) &= r(r-1) - r + 1 \\ &= r^2 - 2r + 1\,. \end{aligned}$$

The root is $r = 1$, with multiplicity *two*. Hence the general solution, for $x \neq 0$, is

$$y = (c_1 + c_2\, ln|x|)\, x\,.$$

6. Substitution of $y = (x-1)^r$ results in the quadratic equation $F(r) = 0$, where

$$F(r) = r^2 + 7r + 12\,.$$

The roots are $r = -3\,,\ -4$. Hence the general solution, for $x \neq 1$, is

$$y = c_1 (x-1)^{-3} + c_2 (x-1)^{-4}.$$

7. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where

$$F(r) = r^2 + 5r - 1\,.$$

The roots are $r = -\left(5 \pm \sqrt{29}\right)/2$. Hence the general solution, for $x \neq 0$, is

$$y = c_1 |x|^{-\left(5+\sqrt{29}\right)/2} + c_2 |x|^{-\left(5-\sqrt{29}\right)/2}.$$

8. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where

$$F(r) = r^2 - 3r + 3\,.$$

The roots are complex, with $r = \left(3 \pm i\sqrt{3}\right)/2$. Hence the general solution, for $x \neq 0$, is

$$y = c_1\,|x|^{3/2} cos\left(\frac{\sqrt{3}}{2}\,ln|x|\right) + c_2|x|^{3/2} sin\left(\frac{\sqrt{3}}{2}\,ln|x|\right).$$

10. Substitution of $y = (x-2)^r$ results in the quadratic equation $F(r) = 0$, where

$$F(r) = r^2 + 4r + 8\,.$$

The roots are complex, with $r = -2 \pm 2i$. Hence the general solution, for $x \neq 2$, is

$$y = c_1\,(x-2)^{-2} cos(2\,ln|x-2|) + c_2(x-2)^{-2} sin(2\,ln|x-2|).$$

11. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where

$$F(r) = r^2 + r + 4\,.$$

The roots are complex, with $r = -\left(1 \pm i\sqrt{15}\right)/2$. Hence the general solution, for $x \neq 0$, is

$$y = c_1\,|x|^{-1/2} cos\left(\frac{\sqrt{15}}{2}\,ln|x|\right) + c_2|x|^{-1/2} sin\left(\frac{\sqrt{15}}{2}\,ln|x|\right).$$

12. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where

$$F(r) = r^2 - 5r + 4\,.$$

The roots are $r = 1\,, 4$. Hence the general solution, for $x \neq 0$, is

$$y = c_1\,x + c_2\,x^4.$$

14. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where

$$F(r) = 4r^2 + 4r + 17\,.$$

The roots are complex, with $r = -1/2 \pm 2i$. Hence the general solution, for $x > 0$, is

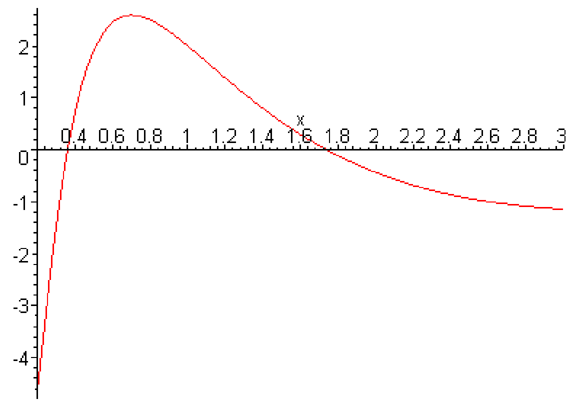
$$y = c_1\,x^{-1/2} cos(2\,ln\,x) + c_2\,x^{-1/2} sin(2\,ln\,x).$$

Invoking the initial conditions, we obtain the system of equations

$$c_1 = 2$$
$$-\frac{1}{2}c_1 + 2c_2 = -3$$

Hence the solution of the initial value problem is

$$y(x) = 2\,x^{-1/2}cos(2\,ln\,x) - x^{-1/2}sin(2\,ln\,x).$$



As $x \to 0^+$, the solution decreases without bound.

15. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where

$$F(r) = r^2 - 4r + 4\,.$$

The root is $r = 2$, with multiplicity *two*. Hence the general solution, for $x < 0$, is

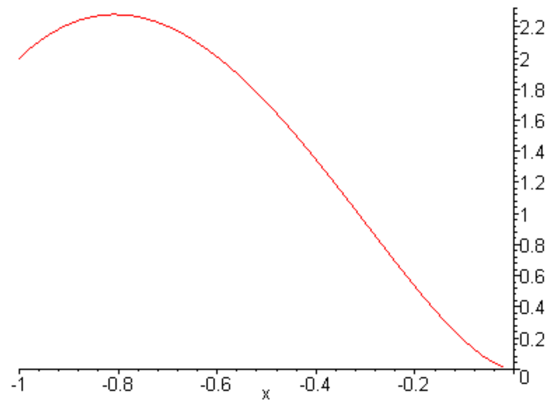
$$y = (c_1 + c_2\,ln\,|x|)\,x^2.$$

Invoking the initial conditions, we obtain the system of equations

$$c_1 = 2$$
$$-2c_1 - c_2 = 3$$

Hence the solution of the initial value problem is

$$y(x) = (2 - 7\,ln\,|x|)\,x^2.$$



We find that $y(x) \to 0$ as $x \to 0^-$.

18. Substitution of $y = x^r$ results in the quadratic equation $r^2 - r + \beta = 0$. The roots are

$$r = \frac{1 \pm \sqrt{1 - 4\beta}}{2}.$$

If $\beta > 1/4$, the roots are complex, with $r_{1,2} = \left(1 \pm i\sqrt{4\beta - 1}\right)/2$. Hence the general solution, for $x \neq 0$, is

$$y = c_1 |x|^{1/2} cos\left(\frac{1}{2}\sqrt{4\beta - 1}\ ln|x|\right) + c_2 |x|^{1/2} sin\left(\frac{1}{2}\sqrt{4\beta - 1}\ ln|x|\right).$$

Since the trigonometric factors are *bounded*, $y(x) \to 0$ as $x \to 0$. If $\beta = 1/4$, the roots are *equal*, and

$$y = c_1 |x|^{1/2} + c_2 |x|^{1/2}\ ln\,|x|\,.$$

Since $\lim_{x \to 0} \sqrt{|x|}\ ln\,|x| = 0$, $y(x) \to 0$ as $x \to 0$. If $\beta < 1/4$, the roots are real, with $r_{1,2} = \left(1 \pm \sqrt{1 - 4\beta}\right)/2$. Hence the general solution, for $x \neq 0$, is

$$y = c_1 |x|^{1/2 + \sqrt{1-4\beta}/2} + c_2 |x|^{1/2 - \sqrt{1-4\beta}/2}.$$

Evidently, solutions approach *zero* as long as $1/2 - \sqrt{1 - 4\beta}/2 > 0$. That is,

$$0 < \beta < 1/4\,.$$

Hence *all* solutions approach *zero*, for $\beta > 0$.

19. Substitution of $y = x^r$ results in the quadratic equation $r^2 - r - 2 = 0$. The roots are $r = -1\,, 2$. Hence the general solution, for $x \neq 0$, is

$$y = c_1 x^{-1} + c_2\, x^2.$$

Invoking the initial conditions, we obtain the system of equations

$$c_1 + c_2 = 1$$
$$-c_1 + 2c_2 = \gamma$$

Hence the solution of the initial value problem is

$$y(x) = \frac{2-\gamma}{3}x^{-1} + \frac{1+\gamma}{3}x^2.$$

The solution is *bounded*, as $x \to 0$, if $\gamma = 2$.

20. Substitution of $y = x^r$ results in the quadratic equation $r^2 + (\alpha - 1)r + 5/2 = 0$. Formally, the roots are given by

$$\begin{aligned} r &= \frac{1-\alpha \pm \sqrt{\alpha^2 - 2\alpha - 9}}{2} \\ &= \frac{1-\alpha \pm \sqrt{\left(\alpha - 1 - \sqrt{10}\right)\left(\alpha - 1 + \sqrt{10}\right)}}{2}. \end{aligned}$$

(i) The roots $r_{1,2}$ will be *complex*, if $|1-\alpha| < \sqrt{10}$. For solutions to approach *zero*, as $x \to \infty$, we need $-\sqrt{10} < 1 - \alpha < 0$.
(ii) The roots will be *equal*, if $|1-\alpha| = \sqrt{10}$. In this case, all solutions approach *zero* as long as $1 - \alpha = -\sqrt{10}$.
(iii) The roots will be real and *distinct*, if $|1-\alpha| > \sqrt{10}$. It follows that

$$r_{max} = \frac{1-\alpha + \sqrt{\alpha^2 - 2\alpha - 9}}{2}.$$

For solutions to approach *zero*, we need $1 - \alpha + \sqrt{\alpha^2 - 2\alpha - 9} < 0$. That is, $1 - \alpha < -\sqrt{10}$.

Hence all solutions approach *zero*, as $x \to \infty$, as long as $\alpha > 1$.

23(a). Given that $x = e^z$, $y(x) = y(e^z) = w(z)$. By the chain rule,

$$\frac{dy}{dx} = \frac{d}{dx}w(z) = \frac{dw}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dw}{dz}.$$

Similarly,

$$\begin{aligned} \frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{1}{x}\frac{dw}{dz}\right] &= -\frac{1}{x^2}\frac{dw}{dz} + \frac{1}{x}\frac{d^2w}{dz^2}\frac{dz}{dx} \\ &= -\frac{1}{x^2}\frac{dw}{dz} + \frac{1}{x^2}\frac{d^2w}{dz^2}. \end{aligned}$$

(b). Direct substitution results in

$$x^2\left[\frac{1}{x^2}\frac{d^2w}{dz^2}-\frac{1}{x^2}\frac{dw}{dz}\right]+\alpha\,x\left[\frac{1}{x}\frac{dw}{dz}\right]+\beta\,w=0\,,$$

that is,

$$\frac{d^2w}{dz^2}+(\alpha-1)\frac{dw}{dz}+\beta\,w=0\,.$$

The associated *characteristic equation* is $r^2+(\alpha-1)r+\beta=0$. Since $z=ln\,x$, it follows that $y(x)=w(ln\,x)$.

(c). If the roots $r_{1,2}$ are real and *distinct*, then

$$\begin{aligned} y &= c_1e^{r_1z}+c_2e^{r_2z}\\ &= c_1x^{r_1}+c_2x^{r_2}. \end{aligned}$$

(d). If the roots $r_{1,2}$ are real and *equal*, then

$$\begin{aligned} y &= c_1e^{r_1z}+c_2z\,e^{r_1z}\\ &= c_1x^{r_1}+c_2x^{r_1}ln\,x\,. \end{aligned}$$

(e). If the roots are *complex conjugates*, then $r=\lambda\pm i\mu$, and

$$\begin{aligned} y &= e^{\lambda z}(c_1cos\,\mu z+c_2\,sin\,\mu z)\\ &= x^{\lambda}[c_1cos(\mu\,ln\,x)+c_2\,sin(\mu\,ln\,x)]. \end{aligned}$$

24. Based on Prob. 23, the change of variable $x=e^z$ transforms the ODE into

$$\frac{d^2w}{dz^2}-\frac{dw}{dz}-2w=0\,.$$

The associated *characteristic equation* is $r^2-r-2=0$, with roots $r=-1\,,2\,$. Hence $w(z)=c_1e^{-z}+c_2e^{2z}$, and $y(x)=c_1x^{-1}+c_2\,x^2$.

26. The change of variable $x=e^z$ transforms the ODE into

$$\frac{d^2w}{dz^2}+6\frac{dw}{dz}+5w=e^z.$$

The associated *characteristic equation* is $r^2+6\,r+5=0$, with roots $r=-5\,,\,-1\,$. Hence $w_c(z)=c_1e^{-z}+c_2e^{-5z}$. Since the right hand side is *not* a solution of the homogeneous equation, we can use the *method of undetermined coefficients* to show that a particular solution is $W=e^z/12$. Therefore the general solution is given by $w(z)=c_1e^{-z}+c_2e^{-5z}+e^z/12$, that is, $y(x)=c_1x^{-1}+c_2\,x^{-5}+x/12\,$.

27. The change of variable $x=e^z$ transforms the given ODE into

$$\frac{d^2w}{dz^2} - 3\frac{dw}{dz} + 2w = 3e^{2z} + 2z\,.$$

The associated *characteristic equation* is $r^2 - 3\,r + 2 = 0$, with roots $r = 1\,,2$. Hence $w_c(z) = c_1 e^z + c_2 e^{2z}$. Using the *method of undetermined coefficients*, let $W = Ae^{2z} + Bze^{2z} + Cz + D$. It follows that the general solution is given by $w(z) = c_1 e^z + c_2 e^{2z} + 3ze^{2z} + z + 3/2$, that is,

$$y(x) = c_1 x + c_2\,x^2 + 3\,x^2 ln\,x + ln\,x + 3/2\,.$$

28. The change of variable $x = e^z$ transforms the given ODE into

$$\frac{d^2w}{dz^2} + 4w = sin\,z\,.$$

The solution of the homogeneous equation is $w_c(z) = c_1 cos\,2z + c_2 sin\,2z$. The right hand side is *not* a solution of the homogeneous equation. We can use the *method of undetermined coefficients* to show that a particular solution is $W = \frac{1}{3} sin\,z$. Hence the general solution is given by $w(z) = c_1 cos\,2z + c_2 sin\,2z + \frac{1}{3} sin\,z$, that is, $y(x) = c_1 cos(2\,ln\,x) + c_2 sin(2\,ln\,x) + \frac{1}{3} sin(ln\,x)$.

29. After dividing the equation by 3, the change of variable $x = e^z$ transforms the ODE into

$$\frac{d^2w}{dz^2} + 3\frac{dw}{dz} + 3w = 0\,.$$

The associated *characteristic equation* is $r^2 + 3\,r + 3 = 0$, with complex roots $r = -\left(3 \pm i\sqrt{3}\right)/2$. Hence the general solution is

$$w(z) = e^{-3z/2}\left[c_1 cos\left(\sqrt{3}\;z/2\right) + c_2 sin\left(\sqrt{3}\;z/2\right)\right],$$

and therefore

$$y(x) = x^{-3/2}\left[c_1 cos\left(\frac{\sqrt{3}}{2}\,ln\,x\right) + c_2 sin\left(\frac{\sqrt{3}}{2}\,ln\,x\right)\right].$$

30. Let $x < 0$. Setting $y = (-x)^r$, successive differentiation gives $y' = -r(-x)^{r-1}$ and $y'' = r(r-1)(-x)^{r-2}$. It follows that

$$L[(-x)^r] = r(r-1)x^2(-x)^{r-2} - \alpha\,r\,x(-x)^{r-1} + \beta(-x)^r.$$

Since $x^2 = (-x)^2$, we find that

$$\begin{aligned} L[(-x)^r] &= r(r-1)(-x)^r + \alpha\, r(-x)^r + \beta(-x)^r \\ &= (-x)^r[r(r-1) + \alpha\, r + \beta]. \end{aligned}$$

Given that r_1 and r_2 are roots of $F(r) = r(r-1) + \alpha\, r + \beta$, we have $L[(-x)^{r_i}] = 0$. Therefore $y_1 = (-x)^{r_1}$ and $y_2 = (-x)^{r_2}$ are *linearly independent* solutions of the differential equation, $L[y] = 0$, for $x < 0$, as long as $r_1 \neq r_2$.