

Chapter Four

Section 4.1

1. The differential equation is in standard form. Its coefficients, as well as the function $g(t) = t$, are continuous *everywhere*. Hence solutions are valid on the entire real line.
3. Writing the equation in standard form, the coefficients are *rational* functions with singularities at $t = 0$ and $t = 1$. Hence the solutions are valid on the intervals $(-\infty, 0)$, $(0, 1)$, and $(1, \infty)$.
4. The coefficients are continuous everywhere, but the function $g(t) = \ln t$ is defined and continuous only on the interval $(0, \infty)$. Hence solutions are defined for positive reals.
5. Writing the equation in standard form, the coefficients are *rational* functions with a singularity at $x_0 = 1$. Furthermore, $p_4(x) = \tan x/(x - 1)$ is *undefined*, and hence not continuous, at $x_k = \pm(2k + 1)\pi/2$, $k = 0, 1, 2, \dots$. Hence solutions are defined on any *interval* that *does not* contain x_0 or x_k .
6. Writing the equation in standard form, the coefficients are *rational* functions with singularities at $x = \pm 2$. Hence the solutions are valid on the intervals $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$.
7. Evaluating the Wronskian of the three functions, $W(f_1, f_2, f_3) = -14$. Hence the functions are linearly *independent*.
9. Evaluating the Wronskian of the four functions, $W(f_1, f_2, f_3, f_4) = 0$. Hence the functions are linearly *dependent*. To find a linear relation among the functions, we need to find constants c_1, c_2, c_3, c_4 , not all zero, such that

$$c_1 f_1(t) + c_2 f_2(t) + c_3 f_3(t) + c_4 f_4(t) = 0.$$

Collecting the common terms, we obtain

$$(c_2 + 2c_3 + c_4)t^2 + (2c_1 - c_3 + c_4)t + (-3c_1 + c_2 + c_4) = 0,$$

which results in *three* equations in *four* unknowns. Arbitrarily setting $c_4 = -1$, we can solve the equations $c_2 + 2c_3 = 1$, $2c_1 - c_3 = 1$, $-3c_1 + c_2 = 1$, to find that $c_1 = 2/7$, $c_2 = 13/7$, $c_3 = -3/7$. Hence

$$2f_1(t) + 13f_2(t) - 3f_3(t) - 7f_4(t) = 0.$$

10. Evaluating the Wronskian of the three functions, $W(f_1, f_2, f_3) = 156$. Hence the functions are linearly *independent*.

11. Substitution verifies that the functions are solutions of the ODE. Furthermore, we have

$$W(1, \cos t, \sin t) = 1.$$

12. Substitution verifies that the functions are solutions of the ODE. Furthermore, we have $W(1, t, \cos t, \sin t) = 1$.

14. Substitution verifies that the functions are solutions of the ODE. Furthermore, we have $W(1, t, e^{-t}, t e^{-t}) = e^{-2t}$.

15. Substitution verifies that the functions are solutions of the ODE. Furthermore, we have $W(1, x, x^3) = 6x$.

16. Substitution verifies that the functions are solutions of the ODE. Furthermore, we have $W(x, x^2, 1/x) = 6/x$.

18. The operation of taking a derivative is linear, and hence

$$(c_1 y_1 + c_2 y_2)^{(k)} = c_1 y_1^{(k)} + c_2 y_2^{(k)}.$$

It follows that

$$L[c_1 y_1 + c_2 y_2] = c_1 y_1^{(n)} + c_2 y_2^{(n)} + p_1 [c_1 y_1^{(n-1)} + c_2 y_2^{(n-1)}] + \cdots + p_n [c_1 y_1 + c_2 y_2].$$

Rearranging the terms, we obtain $L[c_1 y_1 + c_2 y_2] = c_1 L[y_1] + c_2 L[y_2]$. Since y_1 and y_2 are solutions, $L[c_1 y_1 + c_2 y_2] = 0$. The rest follows by induction.

19(a). Note that $d^k(t^n)/dt^k = n(n-1)\cdots(n-k+1)t^{n-k}$, for $k = 1, 2, \dots, n$. Hence

$$L[t^n] = a_0 n! + a_1 [n(n-1)\cdots 2]t + \cdots a_{n-1} n t^{n-1} + a_n t^n.$$

(b). We have $d^k(e^{rt})/dt^k = r^k e^{rt}$, for $k = 0, 1, 2, \dots$. Hence

$$\begin{aligned} L[e^{rt}] &= a_0 r^n e^{rt} + a_1 r^{n-1} e^{rt} + \cdots a_{n-1} r e^{rt} + a_n e^{rt} \\ &= [a_0 r^n + a_1 r^{n-1} + \cdots a_{n-1} r + a_n] e^{rt}. \end{aligned}$$

(c). Set $y = e^{rt}$, and substitute into the ODE. It follows that $r^4 - 5r^2 + 4 = 0$, with $r = \pm 1, \pm 2$. Furthermore, $W(e^t, e^{-t}, e^{2t}, e^{-2t}) = 72$.

20(a). Let $f(t)$ and $g(t)$ be arbitrary functions. Then $W(f, g) = fg' - f'g$. Hence $W'(f, g) = f'g' + fg'' - f''g - f'g' = fg'' - f''g$. That is,

$$W'(f, g) = \begin{vmatrix} f & g \\ f'' & g'' \end{vmatrix}.$$

Now expand the 3-by-3 determinant as

$$W(y_1, y_2, y_3) = y_1 \begin{vmatrix} y_2' & y_3' \\ y_2'' & y_3'' \end{vmatrix} - y_2 \begin{vmatrix} y_1' & y_3' \\ y_1'' & y_3'' \end{vmatrix} + y_3 \begin{vmatrix} y_1' & y_2' \\ y_1'' & y_2'' \end{vmatrix}.$$

Differentiating, we obtain

$$\begin{aligned} W'(y_1, y_2, y_3) &= y_1' \begin{vmatrix} y_2' & y_3' \\ y_2'' & y_3'' \end{vmatrix} - y_2' \begin{vmatrix} y_1' & y_3' \\ y_1'' & y_3'' \end{vmatrix} + y_3' \begin{vmatrix} y_1' & y_2' \\ y_1'' & y_2'' \end{vmatrix} + \\ &+ y_1 \begin{vmatrix} y_2' & y_3' \\ y_2''' & y_3''' \end{vmatrix} - y_2 \begin{vmatrix} y_1' & y_3' \\ y_1''' & y_3''' \end{vmatrix} + y_3 \begin{vmatrix} y_1' & y_2' \\ y_1''' & y_2''' \end{vmatrix}. \end{aligned}$$

The *second* line follows from the observation above. Now we find that

$$W'(y_1, y_2, y_3) = \begin{vmatrix} y_1' & y_2' & y_3' \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} + \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix}.$$

Hence the assertion is true, since the first determinant is equal to *zero*.

(b). Based on the properties of determinants,

$$p_2(t)p_3(t)W' = \begin{vmatrix} p_3 y_1 & p_3 y_2 & p_3 y_3 \\ p_2 y_1' & p_2 y_2' & p_2 y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix}$$

Adding the *first two* rows to the *third* row does not change the value of the determinant. Since the functions are assumed to be solutions of the given ODE, addition of the rows results in

$$p_2(t)p_3(t)W' = \begin{vmatrix} p_3 y_1 & p_3 y_2 & p_3 y_3 \\ p_2 y_1' & p_2 y_2' & p_2 y_3' \\ -p_1 y_1'' & -p_1 y_2'' & -p_1 y_3'' \end{vmatrix}.$$

It follows that $p_2(t)p_3(t)W' = -p_1(t)p_2(t)p_3(t)W$. As long as the coefficients are not zero, we obtain $W' = -p_1(t)W$.

(c). The first order equation $W' = -p_1(t)W$ is linear, with integrating factor $\mu(t) = \exp(\int p_1(t)dt)$. Hence $W(t) = c \exp(-\int p_1(t)dt)$. Furthermore, $W(t)$ is *zero* only if $c = 0$.

(d). It can be shown, by mathematical induction, that

$$W'(y_1, y_2, \dots, y_n) = \begin{vmatrix} y_1 & y_2 & \dots & y_{n-1} & y_n \\ y_1' & y_2' & \dots & y_{n-1}' & y_n' \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \dots & y_{n-1}^{(n-2)} & y_n^{(n-2)} \\ y_1^{(n)} & y_2^{(n)} & \dots & y_{n-1}^{(n)} & y_n^{(n)} \end{vmatrix}.$$

Based on the reasoning in Part(b), it follows that

$$p_2(t)p_3(t)\cdots p_n(t)W' = -p_1(t)p_2(t)p_3(t)\cdots p_n(t)W,$$

and hence $W' = -p_1(t)W$.

22. Inspection of the coefficients reveals that $p_1(t) = 0$. Based on Prob. 20, we find that $W' = 0$, and hence $W = c$.

23. After writing the equation in standard form, observe that $p_1(t) = 2/t$. Based on the results in Prob. 20, we find that $W' = (-2/t)W$, and hence $W = c/t^2$.

24. Writing the equation in standard form, we find that $p_1(t) = 1/t$. Using *Abel's formula*, the Wronskian has the form $W(t) = c \exp(-\int \frac{1}{t} dt) = c/t$.

25(a). Assuming that $c_1y_1(t) + c_2y_2(t) + \cdots + c_ny_n(t) = 0$, then taking the first $n - 1$ derivatives of this equation results in

$$c_1y_1^{(k)}(t) + c_2y_2^{(k)}(t) + \cdots + c_ny_n^{(k)}(t) = 0$$

for $k = 0, 1, \dots, n - 1$. Setting $t = t_0$, we obtain a system of n algebraic equations with unknowns c_1, c_2, \dots, c_n . The Wronskian, $W(y_1, y_2, \dots, y_n)(t_0)$, is the determinant of the coefficient matrix. Since system of equations is homogeneous, $W(y_1, y_2, \dots, y_n)(t_0) \neq 0$ implies that the only solution is the *trivial* solution, $c_1 = c_2 = \cdots = c_n = 0$.

(b). Suppose that $W(y_1, y_2, \dots, y_n)(t_0) = 0$ for some t_0 . Consider the system of algebraic equations

$$c_1y_1^{(k)}(t_0) + c_2y_2^{(k)}(t_0) + \cdots + c_ny_n^{(k)}(t_0) = 0,$$

$k = 0, 1, \dots, n - 1$, with unknowns c_1, c_2, \dots, c_n . Vanishing of the Wronskian, which is the determinant of the coefficient matrix, implies that there is a *nontrivial* solution of the system of homogeneous equations. That is, there exist constants c_1, c_2, \dots, c_n , not all zero, which satisfy the above equations. Now let

$$y(t) = c_1y_1(t) + c_2y_2(t) + \cdots + c_ny_n(t).$$

Since the ODE is linear, $y(t)$ is also a *nonzero* solution. Based on the system of algebraic equations above, $y(t_0) = y'(t_0) = \cdots = y^{(n-1)}(t_0) = 0$. This contradicts the uniqueness of the *identically zero* solution.

26. Let $y(t) = y_1(t)v(t)$. Then $y' = y_1'v + y_1v'$, $y'' = y_1''v + 2y_1'v' + y_1v''$, and $y''' = y_1'''v + 3y_1''v' + 3y_1'v'' + y_1v'''$. Substitution into the ODE results in

$$y_1'''v + 3y_1''v' + 3y_1'v'' + y_1v''' + p_1[y_1''v + 2y_1'v' + y_1v''] + p_2[y_1'v + y_1v'] + p_3y_1v = 0.$$

Since y_1 is assumed to be a solution, all terms containing the factor $v(t)$ vanish. Hence

$$y_1 v''' + [p_1 y_1 + 3y_1'] v'' + [3y_1'' + 2p_1 y_1' + p_2 y_1] v' = 0,$$

which is a *second order* ODE in the variable $u = v'$.

28. First write the equation in standard form:

$$y''' - 3 \frac{t+2}{t(t+3)} y'' + 6 \frac{t+1}{t^2(t+3)} y' - \frac{6}{t^2(t+3)} y = 0.$$

Let $y(t) = t^2 v(t)$. Substitution into the given ODE results in

$$t^2 v''' + 3 \frac{t(t+4)}{t+3} v'' = 0.$$

Set $w = v''$. Then w is a solution of the first order differential equation

$$w' + 3 \frac{t+4}{t(t+3)} w = 0.$$

This equation is *linear*, with integrating factor $\mu(t) = t^4/(t+3)$. The general solution is $w = c(t+3)/t^4$. Integrating twice, it follows that $v(t) = c_1 t^{-1} + c_1 t^{-2} + c_2 t + c_3$. Hence $y(t) = c_1 t + c_1 + c_2 t^3 + c_3 t^2$. Finally, since $y_1(t) = t^2$ and $y_2(t) = t^3$ are given solutions, the *third* independent solution is $y_3(t) = c_1 t + c_1$.

Section 4.2

1. The *magnitude* of $1 + i$ is $R = \sqrt{2}$ and the *polar angle* is $\pi/4$. Hence the polar form is given by $1 + i = \sqrt{2} e^{i\pi/4}$.
3. The *magnitude* of -3 is $R = 3$ and the *polar angle* is π . Hence $-3 = 3 e^{i\pi}$.
4. The *magnitude* of $-i$ is $R = 1$ and the *polar angle* is $3\pi/2$. Hence $-i = e^{3\pi i/2}$.
5. The *magnitude* of $\sqrt{3} - i$ is $R = 2$ and the *polar angle* is $-\pi/6 = 11\pi/6$. Hence the polar form is given by $\sqrt{3} - i = 2 e^{11\pi i/6}$.
6. The *magnitude* of $-1 - i$ is $R = \sqrt{2}$ and the *polar angle* is $5\pi/4$. Hence the polar form is given by $-1 - i = \sqrt{2} e^{5\pi i/4}$.
7. Writing the complex number in polar form, $1 = e^{2m\pi i}$, where m may be any integer. Thus $1^{1/3} = e^{2m\pi i/3}$. Setting $m = 0, 1, 2$ successively, we obtain the three roots as $1^{1/3} = 1, 1^{1/3} = e^{2\pi i/3}, 1^{1/3} = e^{4\pi i/3}$. Equivalently, the roots can also be written as $1, \cos(2\pi/3) + i \sin(2\pi/3) = \frac{1}{2}(-1 + \sqrt{3}i), \cos(4\pi/3) + i \sin(4\pi/3) = \frac{1}{2}(-1 - \sqrt{3}i)$.
9. Writing the complex number in polar form, $1 = e^{2m\pi i/4}$, where m may be any integer. Thus $1^{1/4} = e^{2m\pi i/4}$. Setting $m = 0, 1, 2, 3$ successively, we obtain the three roots as $1^{1/4} = 1, 1^{1/4} = e^{\pi i/2}, 1^{1/4} = e^{\pi i}, 1^{1/4} = e^{3\pi i/2}$. Equivalently, the roots can also be written as $1, \cos(\pi/2) + i \sin(\pi/2) = i, \cos(\pi) + i \sin(\pi) = -1, \cos(3\pi/2) + i \sin(3\pi/2) = -i$.
10. In polar form, $2(\cos \pi/3 + i \sin \pi/3) = 2 e^{i\pi/3 + 2m\pi i}$, in which m is any integer. Thus $[2(\cos \pi/3 + i \sin \pi/3)]^{1/2} = 2^{1/2} e^{i\pi/6 + m\pi i}$. With $m = 0$, one square root is given by $2^{1/2} e^{i\pi/6} = (\sqrt{3} + i)/\sqrt{2}$. With $m = 1$, the other root is given by $2^{1/2} e^{i7\pi/6} = (-\sqrt{3} - i)/\sqrt{2}$.
11. The characteristic equation is $r^3 - r^2 - r + 1 = 0$. The roots are $r = -1, 1, 1$. One root is *repeated*, hence the general solution is $y = c_1 e^{-t} + c_2 e^t + c_3 t e^t$.
13. The characteristic equation is $r^3 - 2r^2 - r + 2 = 0$, with roots $r = -1, 1, 2$. The roots are real and *distinct*, hence the general solution is $y = c_1 e^{-t} + c_2 e^t + c_3 e^{2t}$.
14. The characteristic equation can be written as $r^2(r^2 - 4r + 4) = 0$. The roots are $r = 0, 0, 2, 2$. There are two repeated roots, and hence the general solution is given by $y = c_1 + c_2 t + c_3 e^{2t} + c_4 t e^{2t}$.
15. The characteristic equation is $r^6 + 1 = 0$. The roots are given by $r = (-1)^{1/6}$, that is, the six *sixth roots* of -1 . They are $e^{-\pi i/6 + m\pi i/3}, m = 0, 1, \dots, 5$. Explicitly,

$r = (\sqrt{3} - i)/2, (\sqrt{3} + i)/2, i, -i, (-\sqrt{3} + i)/2, (-\sqrt{3} - i)/2$. Hence the general solution is given by $y = e^{\sqrt{3}t/2}[c_1 \cos(t/2) + c_2 \sin(t/2)] + c_3 \cos t + c_4 \sin t + e^{-\sqrt{3}t/2}[c_5 \cos(t/2) + c_6 \sin(t/2)]$.

16. The characteristic equation can be written as $(r^2 - 1)(r^2 - 4) = 0$. The roots are given by $r = \pm 1, \pm 2$. The roots are real and *distinct*, hence the general solution is $y = c_1 e^{-t} + c_2 e^t + c_3 e^{-2t} + c_4 e^{2t}$.

17. The characteristic equation can be written as $(r^2 - 1)^3 = 0$. The roots are given by $r = \pm 1$, each with *multiplicity three*. Hence the general solution is

$$y = c_1 e^{-t} + c_2 t e^{-t} + c_3 t^2 e^{-t} + c_4 e^t + c_5 t e^t + c_6 t^2 e^t.$$

18. The characteristic equation can be written as $r^2(r^4 - 1) = 0$. The roots are given by $r = 0, 0, \pm 1, \pm i$. The general solution is $y = c_1 + c_2 t + c_3 e^{-t} + c_4 e^t + c_5 \cos t + c_6 \sin t$.

19. The characteristic equation can be written as $r(r^4 - 3r^3 + 3r^2 - 3r + 2) = 0$. Examining the coefficients, it follows that $r^4 - 3r^3 + 3r^2 - 3r + 2 = (r - 1)(r - 2) \times (r^2 + 1)$. Hence the roots are $r = 0, 1, 2, \pm i$. The general solution of the ODE is given by $y = c_1 + c_2 e^t + c_3 e^{2t} + c_4 \cos t + c_5 \sin t$.

20. The characteristic equation can be written as $r(r^3 - 8) = 0$, with roots $r = 0, 2 e^{2m\pi i/3}, m = 0, 1, 2$. That is, $r = 0, 2, -1 \pm i\sqrt{3}$. Hence the general solution is $y = c_1 + c_2 e^{2t} + e^{-t} [c_3 \cos \sqrt{3}t + c_4 \sin \sqrt{3}t]$.

21. The characteristic equation can be written as $(r^4 + 4)^2 = 0$. The roots of the equation $r^4 + 4 = 0$ are $r = 1 \pm i, -1 \pm i$. Each of these roots has *multiplicity two*. The general solution is $y = e^t [c_1 \cos t + c_2 \sin t] + t e^t [c_3 \cos t + c_4 \sin t] + e^{-t} [c_5 \cos t + c_6 \sin t] + t e^{-t} [c_7 \cos t + c_8 \sin t]$.

22. The characteristic equation can be written as $(r^2 + 1)^2 = 0$. The roots are given by $r = \pm i$, each with *multiplicity two*. The general solution is $y = c_1 \cos t + c_2 \sin t + t [c_3 \cos t + c_4 \sin t]$.

24. The characteristic equation is $r^3 + 5r^2 + 6r + 2 = 0$. Examining the coefficients, we find that $r^3 + 5r^2 + 6r + 2 = (r + 1)(r^2 + 4r + 2)$. Hence the roots are deduced as $r = -1, -2 \pm \sqrt{2}$. The general solution is $y = c_1 e^{-t} + c_2 e^{(-2+\sqrt{2})t} + c_3 e^{(-2-\sqrt{2})t}$.

25. The characteristic equation is $18r^3 + 21r^2 + 14r + 4 = 0$. By examining the first and last coefficients, we find that $18r^3 + 21r^2 + 14r + 4 = (2r + 1)(9r^2 + 6r + 4)$.

Hence the roots are $r = -1/2, (-1 \pm \sqrt{3})/3$. The general solution of the ODE is given by $y = c_1 e^{-t/2} + e^{-t/3} [c_2 \cos(t/\sqrt{3}) + c_3 \sin(t/\sqrt{3})]$.

26. The characteristic equation is $r^4 - 7r^3 + 6r^2 + 30r - 36 = 0$. By examining the first and last coefficients, we find that

$$r^4 - 7r^3 + 6r^2 + 30r - 36 = (r - 3)(r + 2)(r^2 - 6r + 6).$$

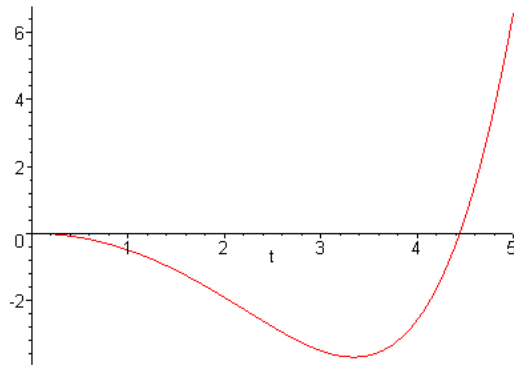
The roots are $r = -2, 3, 3 \pm \sqrt{3}$. The general solution is

$$y = c_1 e^{-2t} + c_2 e^{3t} + c_3 e^{(3-\sqrt{3})t} + c_4 e^{(3+\sqrt{3})t}.$$

28. The characteristic equation is $r^4 + 6r^3 + 17r^2 + 22r + 14 = 0$. It can be shown that $r^4 + 6r^3 + 17r^2 + 22r + 14 = (r^2 + 2r + 2)(r^2 + 4r + 7)$. Hence the roots are $r = -1 \pm i, -2 \pm i\sqrt{3}$. The general solution is

$$y = e^{-t} [c_1 \cos t + c_2 \sin t] + e^{-2t} [c_3 \cos \sqrt{3}t + c_4 \sin \sqrt{3}t].$$

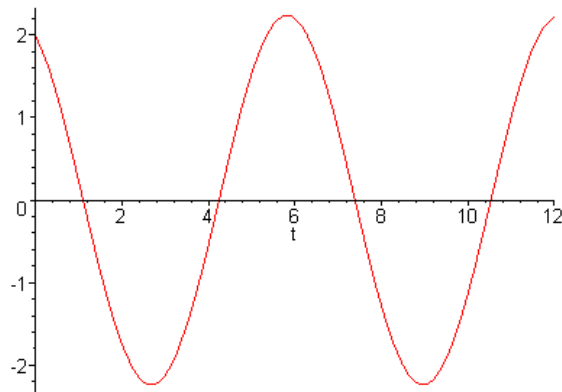
30. $y(t) = \frac{1}{2} e^{-t/\sqrt{2}} \sin(t/\sqrt{2}) - \frac{1}{2} e^{t/\sqrt{2}} \sin(t/\sqrt{2})$.



32. The characteristic equation is $r^3 - r^2 + r - 1 = 0$, with roots $r = 1, \pm i$. Hence the general solution is $y(t) = c_1 e^t + c_2 \cos t + c_3 \sin t$. Invoking the initial conditions, we obtain the system of equations

$$\begin{aligned} c_1 + c_2 &= 2 \\ c_1 + c_3 &= -1 \\ c_1 - c_2 &= -2 \end{aligned}$$

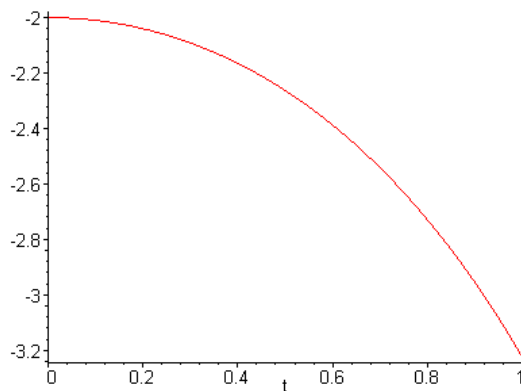
with solution $c_1 = 0, c_2 = 2, c_3 = -1$. Therefore the solution of the initial value problem is $y(t) = 2 \cos t - \sin t$.



33. The characteristic equation is $2r^4 - r^3 - 9r^2 + 4r + 4 = 0$, with roots $r = -1/2, 1, \pm 2$. Hence the general solution is $y(t) = c_1 e^{-t/2} + c_2 e^t + c_3 e^{-2t} + c_4 e^{2t}$. Applying the initial conditions, we obtain the system of equations

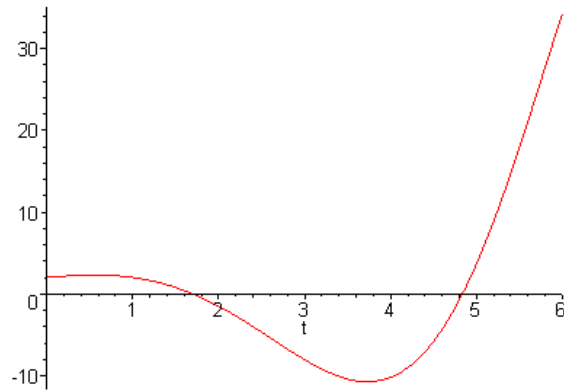
$$\begin{aligned} c_1 + c_2 + c_3 + c_4 &= -2 \\ -\frac{1}{2}c_1 + c_2 - 2c_3 + 2c_4 &= 0 \\ \frac{1}{4}c_1 + c_2 + 4c_3 + 4c_4 &= -2 \\ -\frac{1}{8}c_1 + c_2 - 8c_3 + 8c_4 &= 0 \end{aligned}$$

with solution $c_1 = -16/15, c_2 = -2/3, c_3 = -1/6, c_4 = -1/10$. Therefore the solution of the initial value problem is $y(t) = -\frac{16}{15}e^{-t/2} - \frac{2}{3}e^t - \frac{1}{6}e^{-2t} - \frac{1}{10}e^{2t}$.



The solution decreases without bound.

34. $y(t) = \frac{2}{13}e^{-t} + e^{t/2} \left[\frac{24}{13} \cos t + \frac{3}{13} \sin t \right]$.

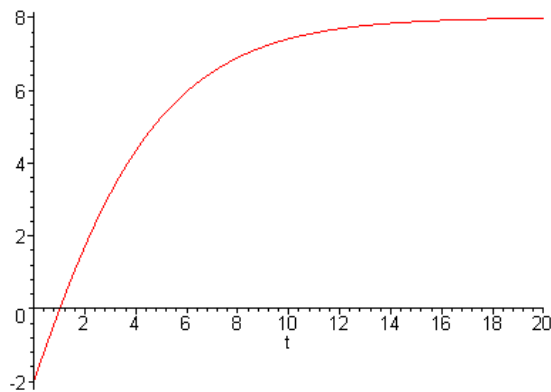


The solution is an oscillation with *increasing* amplitude.

35. The characteristic equation is $6r^3 + 5r^2 + r = 0$, with roots $r = 0, -1/3, -1/2$. The general solution is $y(t) = c_1 + c_2e^{-t/3} + c_3e^{-t/2}$. Invoking the initial conditions, we require that

$$\begin{aligned} c_1 + c_2 + c_3 &= -2 \\ -\frac{1}{3}c_2 - \frac{1}{2}c_3 &= 2 \\ \frac{1}{9}c_2 + \frac{1}{4}c_3 &= 0 \end{aligned}$$

with solution $c_1 = 8, c_2 = -18, c_3 = 8$. Therefore the solution of the initial value problem is $y(t) = 8 - 18e^{-t/3} + 8e^{-t/2}$.



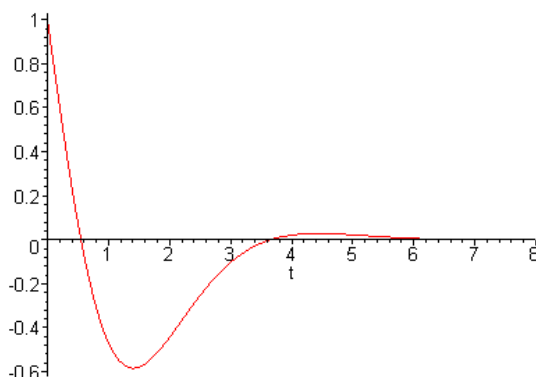
36. The general solution is derived in Prob.(28) as

$$y(t) = e^{-t}[c_1 \cos t + c_2 \sin t] + e^{-2t}[c_3 \cos \sqrt{3}t + c_4 \sin \sqrt{3}t].$$

Invoking the initial conditions, we obtain the system of equations

$$\begin{aligned}
 c_1 + c_3 &= 1 \\
 -c_1 + c_2 - 2c_3 + \sqrt{3}c_4 &= -2 \\
 -2c_2 + c_3 - 4\sqrt{3}c_4 &= 0 \\
 2c_1 + 2c_2 + 10c_3 + 9\sqrt{3}c_4 &= 3
 \end{aligned}$$

with solution $c_1 = 21/13$, $c_2 = -38/13$, $c_3 = -8/13$, $c_4 = 17\sqrt{3}/39$.



The solution is a rapidly-decaying oscillation.

38.

$$\begin{aligned}
 W(e^t, e^{-t}, \cos t, \sin t) &= -8 \\
 W(\cosh t, \sinh t, \cos t, \sin t) &= 4
 \end{aligned}$$

40. Suppose that $c_1e^{r_1t} + c_2e^{r_2t} + \dots + c_n e^{r_nt} = 0$, and each of the r_k are real and different. Multiplying this equation by e^{-r_1t} , $c_1 + c_2e^{(r_2-r_1)t} + \dots + c_n e^{(r_n-r_1)t} = 0$. Differentiation results in

$$c_2(r_2 - r_1)e^{(r_2-r_1)t} + \dots + c_n(r_n - r_1)e^{(r_n-r_1)t} = 0.$$

Now multiplying the latter equation by $e^{-(r_2-r_1)t}$, and differentiating, we obtain

$$c_3(r_3 - r_2)(r_3 - r_1)e^{(r_3-r_2)t} + \dots + c_n(r_n - r_2)(r_n - r_1)e^{(r_n-r_2)t} = 0.$$

Following the above steps in a similar manner, it follows that

$$c_n(r_n - r_{n-1}) \dots (r_n - r_1)e^{(r_n-r_{n-1})t} = 0.$$

Since these equations hold for all t , and all the r_k are different, we have $c_n = 0$. Hence

$$c_1e^{r_1t} + c_2e^{r_2t} + \dots + c_{n-1}e^{r_{n-1}t} = 0, \quad -\infty < t < \infty.$$

The same procedure can now be repeated, successively, to show that

$$c_1 = c_2 = \dots = c_n = 0.$$

Section 4.3

2. The general solution of the homogeneous equation is $y_c = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t$. Let $g_1(t) = 3t$ and $g_2(t) = \cos t$. By inspection, we find that $Y_1(t) = -3t$. Since $g_2(t)$ is a solution of the homogeneous equation, set $Y_2(t) = t(A \cos t + B \sin t)$. Substitution into the given ODE and comparing the coefficients of similar term results in $A = 0$ and $B = -1/4$. Hence the general solution of the nonhomogeneous problem is

$$y(t) = y_c(t) - 3t - \frac{t}{4} \sin t.$$

3. The characteristic equation corresponding to the homogeneous problem can be written as $(r+1)(r^2+1) = 0$. The solution of the homogeneous equation is $y_c = c_1 e^{-t} + c_2 \cos t + c_3 \sin t$. Let $g_1(t) = e^{-t}$ and $g_2(t) = 4t$. Since $g_1(t)$ is a solution of the homogeneous equation, set $Y_1(t) = A t e^{-t}$. Substitution into the ODE results in $A = 1/2$. Now let $Y_2(t) = B t + C$. We find that $B = -C = 4$. Hence the general solution of the nonhomogeneous problem is $y(t) = y_c(t) + t e^{-t}/2 + 4(t-1)$.

4. The characteristic equation corresponding to the homogeneous problem can be written as $r(r+1)(r-1) = 0$. The solution of the homogeneous equation is $y_c = c_1 + c_2 e^t + c_3 e^{-t}$. Since $g(t) = 2 \sin t$ is not a solution of the homogeneous problem, we can set $Y(t) = A \cos t + B \sin t$. Substitution into the ODE results in $A = 1$ and $B = 0$.

Thus

the general solution is $y(t) = c_1 + c_2 e^t + c_3 e^{-t} + \cos t$.

6. The characteristic equation corresponding to the homogeneous problem can be written as $(r^2+1)^2 = 0$. It follows that $y_c = c_1 \cos t + c_2 \sin t + t(c_3 \cos t + c_4 \sin t)$. Since $g(t)$ is not a solution of the homogeneous problem, set $Y(t) = A + B \cos 2t + C \sin 2t$. Substitution into the ODE results in $A = 3$, $B = 1/9$, $C = 0$. Thus the general solution is $y(t) = y_c(t) + 3 + \frac{1}{9} \cos 2t$.

7. The characteristic equation corresponding to the homogeneous problem can be written as $r^3(r^3+1) = 0$. Thus the homogeneous solution is

$$y_c = c_1 + c_2 t + c_3 t^2 + c_4 e^{-t} + e^{t/2} \left[c_5 \cos(\sqrt{3} t/2) + c_6 \sin(\sqrt{3} t/2) \right].$$

Note the $g(t) = t$ is a solution of the homogeneous problem. Consider a particular solution

of the form $Y(t) = t^3(At + B)$. Substitution into the ODE results in $A = 1/24$ and $B = 0$. Thus the general solution is $y(t) = y_c(t) + t^4/24$.

8. The characteristic equation corresponding to the homogeneous problem can be written as $r^3(r+1) = 0$. Hence the homogeneous solution is $y_c = c_1 + c_2 t + c_3 t^2 + c_4 e^{-t}$. Since $g(t)$ is not a solution of the homogeneous problem, set $Y(t) = A \cos 2t + B \sin 2t$. Substitution into the ODE results in $A = 1/40$ and $B = 1/20$. Thus the general solution

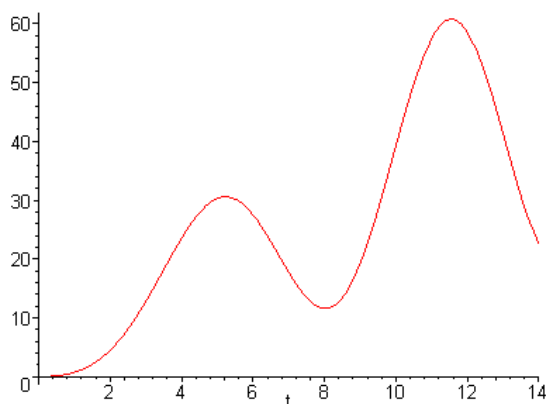
is $y(t) = y_c(t) + (\cos 2t + 2\sin 2t)/40$.

10. From Prob. 22 in Section 4.2, the homogeneous solution is

$$y_c = c_1 \cos t + c_2 \sin t + t[c_3 \cos t + c_4 \sin t].$$

Since $g(t)$ is *not* a solution of the homogeneous problem, substitute $Y(t) = At + B$ into the ODE to obtain $A = 3$ and $B = 4$. Thus the general solution is $y(t) = y_c(t) + 3t + 4$. Invoking the initial conditions, we find that $c_1 = -4$, $c_2 = -4$, $c_3 = 1$, $c_4 = -3/2$. Therefore the solution of the initial value problem is

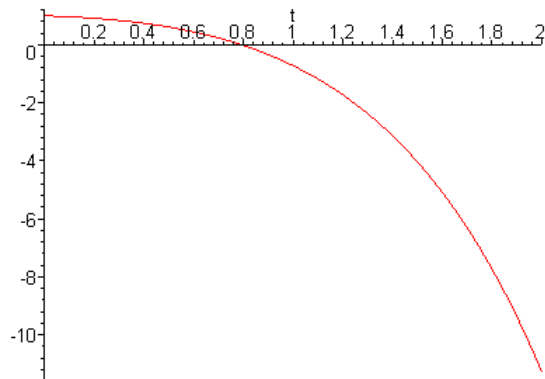
$$y(t) = (t - 4)\cos t - (3t/2 + 4)\sin t + 3t + 4.$$



11. The characteristic equation can be written as $r(r^2 - 3r + 2) = 0$. Hence the homogeneous solution is $y_c = c_1 + c_2 e^t + c_3 e^{2t}$. Let $g_1(t) = e^t$ and $g_2(t) = t$. Note that g_1 is a solution of the homogeneous problem. Set $Y_1(t) = Ate^t$. Substitution into the ODE results in $A = -1$. Now let $Y_2(t) = Bt^2 + Ct$. Substitution into the ODE results in $B = 1/4$ and $C = 3/4$. Therefore the general solution is

$$y(t) = c_1 + c_2 e^t + c_3 e^{2t} - te^t + (t^2 + 3t)/4.$$

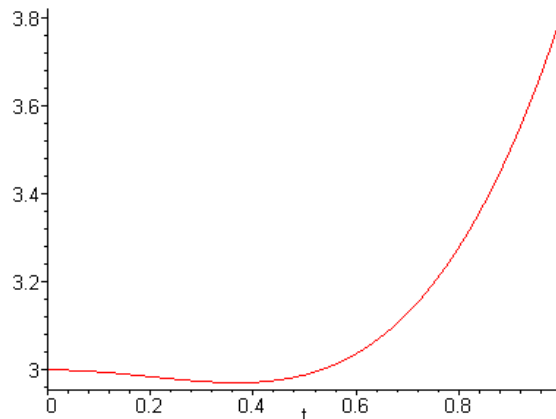
Invoking the initial conditions, we find that $c_1 = 1$, $c_2 = c_3 = 0$. The solution of the initial value problem is $y(t) = 1 - te^t + (t^2 + 3t)/4$.



12. The characteristic equation can be written as $(r - 1)(r + 3)(r^2 + 4) = 0$. Hence the homogeneous solution is $y_c = c_1 e^t + c_2 e^{-3t} + c_3 \cos 2t + c_4 \sin 2t$. None of the terms in $g(t)$ is a solution of the homogeneous problem. Therefore we can assume a form $Y(t) = Ae^{-t} + B \cos t + C \sin t$. Substitution into the ODE results in $A = 1/20$, $B = -2/5$, $C = -4/5$. Hence the general solution is

$$y(t) = c_1 e^t + c_2 e^{-3t} + c_3 \cos 2t + c_4 \sin 2t + e^{-t}/20 - (2 \cos t + 4 \sin t)/5.$$

Invoking the initial conditions, we find that $c_1 = 81/40$, $c_2 = 73/520$, $c_3 = 77/65$, $c_4 = -49/130$.



14. From Prob. 4, the homogeneous solution is $y_c = c_1 + c_2 e^t + c_3 e^{-t}$. Consider the terms $g_1(t) = t e^{-t}$ and $g_2(t) = 2 \cos t$. Note that since $r = -1$ is a *simple* root of the characteristic equation, Table 4.3.1 suggests that we set $Y_1(t) = t(At + B)e^{-t}$. The function $2 \cos t$ is *not* a solution of the homogeneous equation. We can simply choose $Y_2(t) = C \cos t + D \sin t$. Hence the particular solution has the form

$$Y(t) = t(At + B)e^{-t} + C \cos t + D \sin t.$$

15. The characteristic equation can be written as $(r^2 - 1)^2 = 0$. The roots are given

as $r = \pm 1$, each with *multiplicity two*. Hence the solution of the homogeneous problem is $y_c = c_1 e^t + c_2 t e^t + c_3 e^{-t} + c_4 t e^{-t}$. Let $g_1(t) = e^t$ and $g_2(t) = \sin t$. The function e^t is a solution of the homogeneous problem. Since $r = 1$ has *multiplicity two*, we set $Y_1(t) = At^2 e^t$. The function $\sin t$ is *not* a solution of the homogeneous equation. We can set $Y_2(t) = B \cos t + C \sin t$. Hence the particular solution has the form

$$Y(t) = At^2 e^t + B \cos t + C \sin t.$$

16. The characteristic equation can be written as $r^2(r^2 + 4) = 0$, with roots $r = 0, \pm 2i$. The root $r = 0$ has *multiplicity two*, hence the homogeneous solution is $y_c = c_1 + c_2 t + c_3 \cos 2t + c_4 \sin 2t$. The functions $g_1(t) = \sin 2t$ and $g_2(t) = 4$ are solutions of the homogeneous equation. The complex roots have *multiplicity one*, therefore we need to set $Y_1(t) = At \cos 2t + Bt \sin 2t$. Now $g_2(t) = 4$ is associated with the *double* root $r = 0$. Based on Table 4.3.1, set $Y_2(t) = Ct^2$. Finally, $g_3(t) = te^t$ (and its derivatives) is independent of the homogeneous solution. Therefore set $Y_3(t) = (Dt + E)e^t$. Conclude that the particular solution has the form

$$Y(t) = At \cos 2t + Bt \sin 2t + Ct^2 + (Dt + E)e^t.$$

18. The characteristic equation can be written as $r^2(r^2 + 2r + 2) = 0$, with roots $r = 0$, with *multiplicity two*, and $r = -1 \pm i$. The homogeneous solution is $y_c = c_1 + c_2 t + c_3 e^{-t} \cos t + c_4 e^{-t} \sin t$. The function $g_1(t) = 3e^t + 2te^{-t}$, and all of its derivatives, is independent of the homogeneous solution. Therefore set $Y_1(t) = Ae^t + (Bt + C)e^{-t}$. Now $g_2(t) = e^{-t} \sin t$ is a solution of the homogeneous equation, associated with the complex roots. We need to set $Y_2(t) = t(D e^{-t} \cos t + E e^{-t} \sin t)$. It follows that the particular solution has the form

$$Y(t) = Ae^t + (Bt + C)e^{-t} + t(D e^{-t} \cos t + E e^{-t} \sin t).$$

19. Differentiating $y = u(t)v(t)$, successively, we have

$$\begin{aligned} y' &= u'v + uv' \\ y'' &= u''v + 2u'v' + uv'' \\ &\vdots \\ y^{(n)} &= \sum_{j=0}^n \binom{n}{j} u^{(n-j)} v^{(j)} \end{aligned}$$

Setting $v(t) = e^{\alpha t}$, $v^{(j)} = \alpha^j e^{\alpha t}$. So for any $p = 1, 2, \dots, n$,

$$y^{(p)} = e^{\alpha t} \sum_{j=0}^p \binom{p}{j} \alpha^j u^{(p-j)}.$$

It follows that

$$L[e^{\alpha t}u] = e^{\alpha t} \sum_{p=0}^n \left[a_{n-p} \sum_{j=0}^p \binom{p}{j} \alpha^j u^{(p-j)} \right] \quad (*).$$

It is evident that the right hand side of Eq. (*) is of the form

$$e^{\alpha t} [k_0 u^{(n)} + k_1 u^{(n-1)} + \cdots + k_{n-1} u' + k_n u].$$

Hence operator equation $L[e^{\alpha t}u] = e^{\alpha t}(b_0 t^m + b_1 t^{m-1} + \cdots + b_{m-1}t + b_m)$ can be written as

$$\begin{aligned} k_0 u^{(n)} + k_1 u^{(n-1)} + \cdots + k_{n-1} u' + k_n u &= \\ &= b_0 t^m + b_1 t^{m-1} + \cdots + b_{m-1}t + b_m. \end{aligned}$$

The coefficients $k_i, i = 0, 1, \dots, n$ can be determined by collecting the like terms in the double summation in Eq. (*). For example, k_0 is the coefficient of $u^{(n)}$. The *only* term that contains $u^{(n)}$ is when $p = n$ and $j = 0$. Hence $k_0 = a_0$. On the other hand, k_n is the coefficient of $u(t)$. The inner summation in (*) contains terms with u , given by $\alpha^p u$ (when $j = p$), for each $p = 0, 1, \dots, n$. Hence

$$k_n = \sum_{p=0}^n a_{n-p} \alpha^p.$$

21(a). Clearly, e^{2t} is a solution of $y' - 2y = 0$, and te^{-t} is a solution of the differential equation $y'' + 2y' + y = 0$. The latter ODE has characteristic equation $(r + 1)^2 = 0$. Hence $(D - 2)[3e^{2t}] = 3(D - 2)[e^{2t}] = 0$ and $(D + 1)^2[te^{-t}] = 0$. Furthermore, we have $(D - 2)(D + 1)^2[te^{-t}] = (D - 2)[0] = 0$, and $(D - 2)(D + 1)^2[3e^{2t}] = (D + 1)^2(D - 2)[3e^{2t}] = (D + 1)^2[0] = 0$.

(b). Based on Part (a),

$$\begin{aligned} (D - 2)(D + 1)^2[(D - 2)^3(D + 1)Y] &= (D - 2)(D + 1)^2[3e^{2t} - te^{-t}] \\ &= 0, \end{aligned}$$

since the operators are linear. The implied operations are associative and commutative. Hence

$$(D - 2)^4(D + 1)^3Y = 0.$$

The operator equation corresponds to the solution of a linear homogeneous ODE with characteristic equation $(r - 2)^4(r + 1)^3 = 0$. The roots are $r = 2$, with multiplicity 4 and $r = -1$, with multiplicity 3. It follows that the given homogeneous solution is

$$Y(t) = c_1 e^{2t} + c_2 t e^{2t} + c_3 t^2 e^{2t} + c_4 t^3 e^{2t} + c_5 e^{-t} + c_6 t e^{-t} + c_7 t^2 e^{-t},$$

which is a linear combination of seven independent solutions.

22(15). Observe that $(D - 1)[e^t] = 0$ and $(D^2 + 1)[\sin t] = 0$. Hence the operator $H(D) = (D - 1)(D^2 + 1)$ is an annihilator of $e^t + \sin t$. The operator corresponding to the left hand side of the given ODE is $(D^2 - 1)^2$. It follows that

$$(D + 1)^2(D - 1)^3(D^2 + 1)Y = 0.$$

The resulting ODE is homogeneous, with solution

$$Y(t) = c_1e^{-t} + c_2te^{-t} + c_3e^t + c_4te^t + c_5t^3e^t + c_6\cos t + c_7\sin t.$$

After examining the homogeneous solution of Prob. 15, and eliminating duplicate terms, we have

$$Y(t) = c_5t^3e^t + c_6\cos t + c_7\sin t.$$

22(16). We find that $D[4] = 0$, $(D - 1)^2[te^t] = 0$, and $(D^2 + 4)[\sin 2t] = 0$. The operator $H(D) = D(D - 1)^2(D^2 + 4)$ is an annihilator of $t^2 + te^t + \sin 2t$. The operator corresponding to the left hand side of the ODE is $D^2(D^2 + 4)$. It follows that

$$D^3(D - 1)^2(D^2 + 4)^2Y = 0.$$

The resulting ODE is homogeneous, with solution

$$Y(t) = c_1 + c_2t + c_3t^2 + c_4e^t + c_5te^t + c_6\cos 2t + c_7\sin 2t + c_8t\cos 2t + c_9t\sin 2t.$$

After examining the homogeneous solution of Prob. 16, and eliminating duplicate terms, we have

$$Y(t) = c_3t^2 + c_4e^t + c_5te^t + c_8t\cos 2t + c_9t\sin 2t.$$

22(18). Observe that $(D - 1)[e^t] = 0$, $(D + 1)^2[te^{-t}] = 0$. The function $e^{-t}\sin t$ is a solution of a second order ODE with characteristic roots $r = -1 \pm i$. It follows that $(D^2 + 2D + 2)[e^{-t}\sin t] = 0$. Therefore the operator

$$H(D) = (D - 1)(D + 1)^2(D^2 + 2D + 2)$$

is an annihilator of $3e^t + 2te^{-t} + e^{-t}\sin t$. The operator corresponding to the left hand side of the given ODE is $D^2(D^2 + 2D + 2)$. It follows that

$$D^2(D - 1)(D + 1)^2(D^2 + 2D + 2)^2Y = 0.$$

The resulting ODE is homogeneous, with solution

$$Y(t) = c_1 + c_2t + c_3e^t + c_4e^{-t} + c_5te^{-t} + e^{-t}(c_6\cos t + c_7\sin t) + te^{-t}(c_8\cos t + c_9\sin t).$$

After examining the homogeneous solution of Prob. 18, and eliminating duplicate terms,

we have

$$Y(t) = c_3 e^t + c_4 e^{-t} + c_5 t e^{-t} + t e^{-t} (c_8 \cos t + c_9 \sin t).$$

Section 4.4

2. The characteristic equation is $r(r^2 - 1) = 0$. Hence the homogeneous solution is $y_c(t) = c_1 + c_2e^t + c_3e^{-t}$. The Wronskian is evaluated as $W(1, e^t, e^{-t}) = 2$. Now compute the three determinants

$$W_1(t) = \begin{vmatrix} 0 & e^t & e^{-t} \\ 0 & e^t & -e^{-t} \\ 1 & e^t & e^{-t} \end{vmatrix} = -2$$

$$W_2(t) = \begin{vmatrix} 1 & 0 & e^{-t} \\ 0 & 0 & -e^{-t} \\ 0 & 1 & e^{-t} \end{vmatrix} = e^{-t}$$

$$W_3(t) = \begin{vmatrix} 1 & e^t & 0 \\ 0 & e^t & 0 \\ 0 & e^t & 1 \end{vmatrix} = e^t$$

The solution of the system of equations (10) is

$$u_1'(t) = \frac{t W_1(t)}{W(t)} = -t$$

$$u_2'(t) = \frac{t W_2(t)}{W(t)} = te^{-t}/2$$

$$u_3'(t) = \frac{t W_3(t)}{W(t)} = te^t/2$$

Hence $u_1(t) = -t^2/2$, $u_2(t) = -e^{-t}(t+1)/2$, $u_3(t) = e^t(t-1)/2$. The particular solution becomes $Y(t) = -t^2/2 - (t+1)/2 + (t-1)/2 = -t^2/2 - 1$. The constant is a solution of the homogeneous equation, therefore the general solution is

$$y(t) = c_1 + c_2e^t + c_3e^{-t} - t^2/2.$$

3. From Prob. 13 in Section 4.2, $y_c(t) = c_1e^{-t} + c_2e^t + c_3e^{2t}$. The Wronskian is evaluated as $W(e^{-t}, e^t, e^{2t}) = 6e^{2t}$. Now compute the three determinants

$$W_1(t) = \begin{vmatrix} 0 & e^t & e^{2t} \\ 0 & e^t & 2e^{2t} \\ 1 & e^t & 4e^{2t} \end{vmatrix} = e^{3t}$$

$$W_2(t) = \begin{vmatrix} e^{-t} & 0 & e^{2t} \\ -e^{-t} & 0 & 2e^{2t} \\ e^{-t} & 1 & 4e^{2t} \end{vmatrix} = -3e^t$$

$$W_3(t) = \begin{vmatrix} e^{-t} & e^t & 0 \\ -e^{-t} & e^t & 0 \\ e^{-t} & e^t & 1 \end{vmatrix} = 2$$

Hence $u_1'(t) = e^{5t}/6$, $u_2'(t) = -e^{3t}/2$, $u_3'(t) = e^{2t}/3$. Therefore the particular solution can be expressed as

$$\begin{aligned} Y(t) &= e^{-t}[e^{5t}/30] - e^t[e^{3t}/6] + e^{2t}[e^{2t}/6] \\ &= e^{4t}/30. \end{aligned}$$

6. From Prob. 22 in Section 4.2, $y_c(t) = c_1 \cos t + c_2 \sin t + t[c_3 \cos t + c_4 \sin t]$. The Wronskian is evaluated as $W(\cos t, \sin t, t \cos t, t \sin t) = 4$. Now compute the four auxiliary determinants

$$W_1(t) = \begin{vmatrix} 0 & \sin t & t \cos t & t \sin t \\ 0 & \cos t & \cos t - t \sin t & \sin t + t \cos t \\ 0 & -\sin t & -2\sin t - t \cos t & 2\cos t - t \sin t \\ 1 & -\cos t & -3\cos t + t \sin t & -3\sin t - t \cos t \end{vmatrix} = -2\sin t + 2t \cos t$$

$$W_2(t) = \begin{vmatrix} \cos t & 0 & t \cos t & t \sin t \\ -\sin t & 0 & \cos t - t \sin t & \sin t + t \cos t \\ -\cos t & 0 & -2\sin t - t \cos t & 2\cos t - t \sin t \\ \sin t & 1 & -3\cos t + t \sin t & -3\sin t - t \cos t \end{vmatrix} = 2t \sin t + 2\cos t$$

$$W_3(t) = \begin{vmatrix} \cos t & \sin t & 0 & t \sin t \\ -\sin t & \cos t & 0 & \sin t + t \cos t \\ -\cos t & -\sin t & 0 & 2\cos t - t \sin t \\ \sin t & -\cos t & 1 & -3\sin t - t \cos t \end{vmatrix} = -2\cos t$$

$$W_4(t) = \begin{vmatrix} \cos t & \sin t & t \cos t & 0 \\ -\sin t & \cos t & \cos t - t \sin t & 0 \\ -\cos t & -\sin t & -2\sin t - t \cos t & 0 \\ \sin t & -\cos t & -3\cos t + t \sin t & 1 \end{vmatrix} = -2\sin t$$

It follows that $u_1'(t) = [-\sin^2 t + t \sin t \cos t]/2$, $u_2'(t) = [t \sin^2 t + \sin t \cos t]/2$, $u_3'(t) = -\sin t \cos t/2$, $u_4'(t) = -\sin^2 t/2$. Hence

$$u_1(t) = [3\sin t \cos t - 2t \cos^2 t - t]/8$$

$$u_2(t) = [\sin^2 t - 2\cos^2 t - 2t \sin t \cos t + t^2]/8$$

$$u_3(t) = -\sin^2 t/4$$

$$u_4(t) = [\cos t \sin t - t]/4$$

Therefore the particular solution can be expressed as

$$\begin{aligned} Y(t) &= \cos t [u_1(t)] + \sin t [u_2(t)] + t \cos t [u_3(t)] + t \sin t [u_4(t)] \\ &= [\sin t - 3t \cos t - t^2 \sin t]/8. \end{aligned}$$

Note that only the *last term* is not a solution of the homogeneous equation. Hence the general solution is

$$y(t) = c_1 \cos t + c_2 \sin t + t[c_3 \cos t + c_4 \sin t] - t^2 \sin t / 8.$$

8. Based on the results in Prob. 2, $y_c(t) = c_1 + c_2 e^t + c_3 e^{-t}$. It was also shown that $W(1, e^t, e^{-t}) = 2$, with $W_1(t) = -2$, $W_2(t) = e^{-t}$, $W_3(t) = e^t$. Therefore we have $u_1'(t) = -\csc t$, $u_2'(t) = e^{-t} \csc t / 2$, $u_3'(t) = e^t \csc t / 2$. The particular solution can be expressed as $Y(t) = [u_1(t)] + e^{-t}[u_2(t)] + e^t [u_3(t)]$. More specifically,

$$\begin{aligned} Y(t) &= \ln|\csc(t) + \cot(t)| + \frac{e^t}{2} \int_{t_0}^t e^{-s} \csc(s) ds + \frac{e^{-t}}{2} \int_{t_0}^t e^s \csc(s) ds \\ &= \ln|\csc(t) + \cot(t)| + \int_{t_0}^t \cosh(t-s) \csc(s) ds. \end{aligned}$$

9. Based on Prob. 4, $u_1'(t) = \sec t$, $u_2'(t) = -1$, $u_3'(t) = -\tan t$. The particular solution can be expressed as $Y(t) = [u_1(t)] + \cos t [u_2(t)] + \sin t [u_3(t)]$. That is,

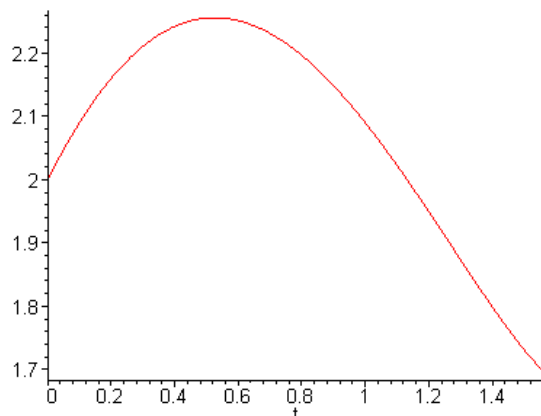
$$Y(t) = \ln|\sec(t) + \tan(t)| - t \cos t + \sin t \ln|\cos(t)|.$$

Hence the general solution of the initial value problem is

$$y(t) = c_1 + c_2 \cos t + c_3 \sin t + \ln|\sec(t) + \tan(t)| - t \cos t + \sin t \ln|\cos(t)|.$$

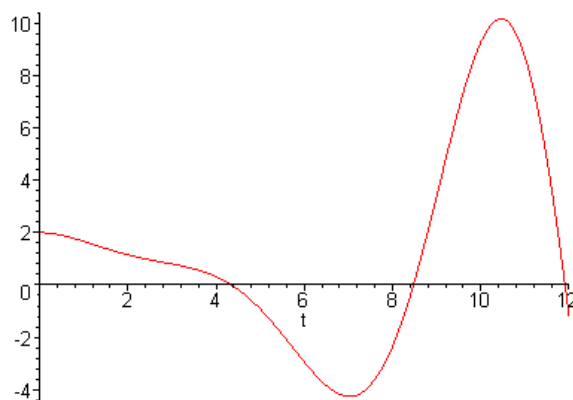
Invoking the initial conditions, we require that $c_1 + c_2 = 2$, $c_3 = 1$, $-c_2 = -2$. Therefore

$$y(t) = 2 \cos t + \sin t + \ln|\sec(t) + \tan(t)| - t \cos t + \sin t \ln|\cos(t)|$$



10. From Prob. 6, $y(t) = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t - t^2 \sin t / 8$. In order to satisfy the initial conditions, we require that $c_1 = 2$, $c_2 + c_3 = 0$, $-c_1 + 2c_4 = -1$, $-3/4 - c_2 - 3c_3 = 1$. Therefore

$$y(t) = 2 \cos t + [7 \sin t - 7t \cos t + 4t \sin t - t^2 \sin t] / 8.$$



12. From Prob. 8, the general solution of the initial value problem is

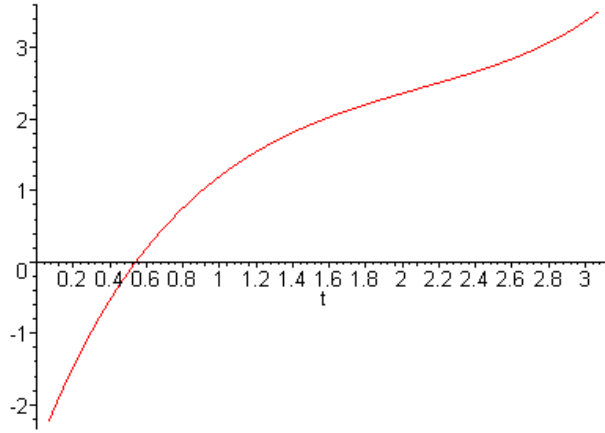
$$y(t) = c_1 + c_2 e^t + c_3 e^{-t} + \ln |c \sec(t) + \cot(t)| + \frac{e^t}{2} \int_{t_0}^t e^{-s} \csc(s) ds + \frac{e^{-t}}{2} \int_{t_0}^t e^s \csc(s) ds.$$

In this case, $t_0 = \pi/2$. Observe that $y(\pi/2) = y_c(\pi/2)$, $y'(\pi/2) = y'_c(\pi/2)$, and $y''(\pi/2) = y''_c(\pi/2)$. Therefore we obtain the system of equations

$$\begin{aligned} c_1 + c_2 e^{\pi/2} + c_3 e^{-\pi/2} &= 2 \\ c_2 e^{\pi/2} - c_3 e^{-\pi/2} &= 1 \\ c_2 e^{\pi/2} + c_3 e^{-\pi/2} &= -1 \end{aligned}$$

Hence the solution of the initial value problem is

$$y(t) = 3 - e^{-t+\pi/2} + \ln|\csc(t) + \cot(t)| + \int_{t_0}^t \cosh(t-s)\csc(s)ds.$$



13. First write the equation as $y''' + x^{-1}y'' - 2x^{-2}y' + 2x^{-3}y = 2x$. The Wronskian is evaluated as $W(x, x^2, 1/x) = 6/x$. Now compute the three determinants

$$W_1(x) = \begin{vmatrix} 0 & x^2 & 1/x \\ 0 & 2x & -1/x^2 \\ 1 & 2 & 2/x^3 \end{vmatrix} = -3$$

$$W_2(x) = \begin{vmatrix} x & 0 & 1/x \\ 1 & 0 & -1/x^2 \\ 0 & 1 & 2/x^3 \end{vmatrix} = 2/x$$

$$W_3(x) = \begin{vmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & 1 \end{vmatrix} = x^2$$

Hence $u_1'(x) = -x^2$, $u_2'(x) = 2x/3$, $u_3'(x) = x^4/3$. Therefore the particular solution can be expressed as

$$\begin{aligned} Y(x) &= x[-x^3/3] + x^2[x^2/3] + \frac{1}{x}[x^5/15] \\ &= x^4/15. \end{aligned}$$

15. The homogeneous solution is $y_c(t) = c_1 \cos t + c_2 \sin t + c_3 \cosh t + c_4 \sinh t$. The Wronskian is evaluated as $W(\cos t, \sin t, \cosh t, \sinh t) = 4$. Now the four additional determinants are given by $W_1(t) = 2 \sin t$, $W_2(t) = -2 \cos t$, $W_3(t) = -2 \sinh t$, $W_4(t) = 2 \cosh t$. It follows that $u_1'(t) = g(t) \sin(t)/2$, $u_2'(t) = -g(t) \cos(t)/2$, $u_3'(t) = -g(t) \sinh(t)/2$, $u_4'(t) = g(t) \cosh(t)/2$. Therefore the particular solution

can be expressed as

$$Y(t) = \frac{\cos(t)}{2} \int_{t_0}^t g(s) \sin(s) ds - \frac{\sin(t)}{2} \int_{t_0}^t g(s) \cos(s) ds - \\ - \frac{\cosh(t)}{2} \int_{t_0}^t g(s) \sinh(s) ds + \frac{\sinh(t)}{2} \int_{t_0}^t g(s) \cosh(s) ds.$$

Using the appropriate identities, the integrals can be combined to obtain

$$Y(t) = \frac{1}{2} \int_{t_0}^t g(s) \sinh(t-s) ds - \frac{1}{2} \int_{t_0}^t g(s) \sin(t-s) ds.$$

17. First write the equation as $y''' - 3x^{-1}y'' + 6x^{-2}y' - 6x^{-3}y = g(x)/x^3$. It can be shown that $y_c(x) = c_1x + c_2x^2 + c_3x^3$ is a solution of the homogeneous equation. The Wronskian of this fundamental set of solutions is $W(x, x^2, x^3) = 2x^3$. The three additional determinants are given by $W_1(x) = x^4$, $W_2(x) = -2x^3$, $W_3(x) = x^2$. Hence $u_1'(x) = g(x)/2x^2$, $u_2'(x) = -g(x)/x^3$, $u_3'(x) = g(x)/2x^4$. Therefore the particular solution can be expressed as

$$Y(x) = x \int_{x_0}^x \frac{g(t)}{2t^2} dt - x^2 \int_{x_0}^x \frac{g(t)}{t^3} dt + x^3 \int_{x_0}^x \frac{g(t)}{2t^4} dt \\ = \frac{1}{2} \int_{x_0}^x \left[\frac{x}{t^2} - \frac{2x^2}{t^3} + \frac{x^3}{t^4} \right] g(t) dt.$$