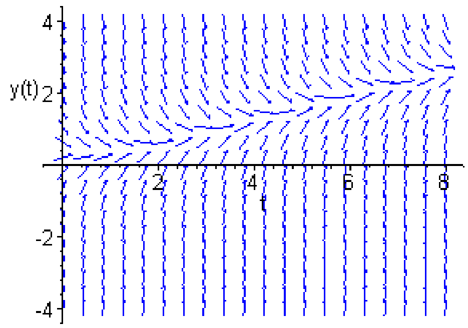
Chapter Two

Section 2.1

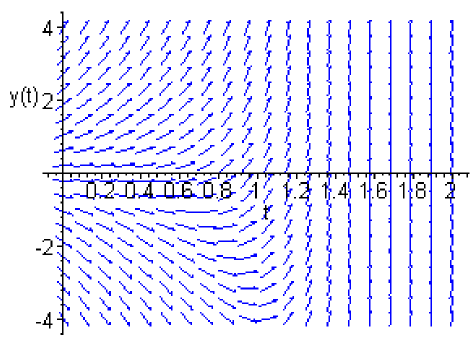
$1(a)$.



(b). Based on the direction field, all solutions seem to converge to a specific increasing function.

(c). The integrating factor is $\mu(t) = e^{3t}$, and hence $y(t) = t/3 - 1/9 + e^{-2t} + c\,e^{-3t}$. It follows that all solutions converge to the function $y_1(t) = t/3 - 1/9$.

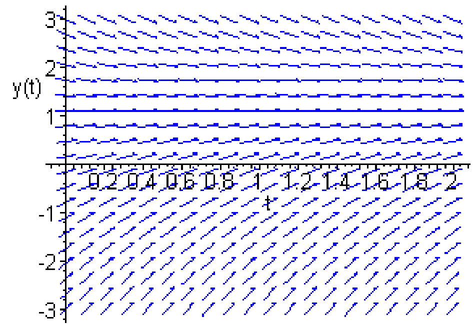
$2(a)$.



(b). All slopes *eventually* become positive, hence all solutions will increase without bound.

(c). The integrating factor is $\mu(t) = e^{-2t}$, and hence $y(t) = t^3 e^{2t}/3 + c\,e^{2t}$. It is evident that all solutions increase at an exponential rate.

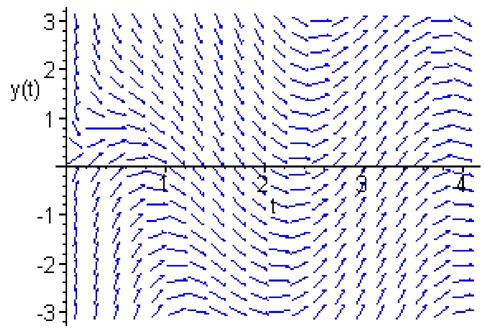
$3(a)$



(b). All solutions seem to converge to the function $y_0(t) = 1$.

(c). The integrating factor is $\mu(t) = e^{2t}$, and hence $y(t) = t^2e^{-t}/2 + 1 + c\,e^{-t}$. It is clear that all solutions converge to the specific solution $y_0(t) = 1$.

4(a).



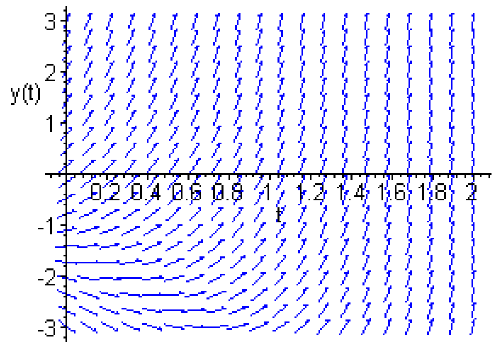
(b). Based on the direction field, the solutions eventually become oscillatory.

(c). The integrating factor is $\mu(t) = t$, and hence the general solution is

$$y(t) = \frac{3cos(2t)}{4t} + \frac{3}{2}sin(2t) + \frac{c}{t}$$

in which c is an arbitrary constant. As t becomes large, all solutions converge to the function $y_1(t) = 3sin(2t)/2$.

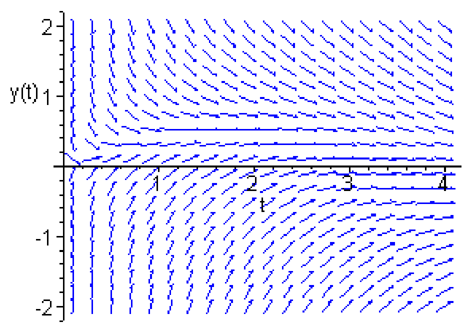
5(a).



(b). All slopes *eventually* become positive, hence all solutions will increase without bound.

(c). The integrating factor is $\mu(t) = exp(-\int 2dt) = e^{-2t}$. The differential equation can
be written as $e^{-2t}y' - 2e^{-2t}y = 3e^{-t}$, that is, $(e^{-2t}y)' = 3e^{-t}$. Integration of both sides of the equation results in the general solution $y(t) = -3e^{t} + c\,e^{2t}$. It follows that all solutions will increase exponentially.

$6(a)$



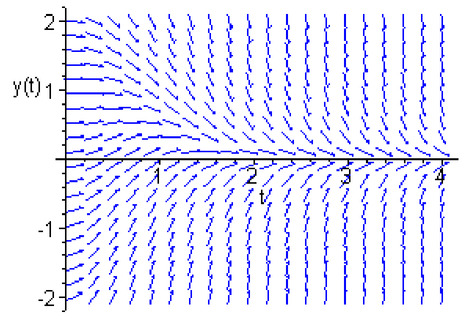
(b). All solutions seem to converge to the function $y_0(t) = 0$.

(c). The integrating factor is $\mu(t) = t^2$, and hence the general solution is

$$y(t) = -\frac{cos(t)}{t} + \frac{sin(2t)}{t^2} + \frac{c}{t^2}$$

in which c is an arbitrary constant. As t becomes large, all solutions converge to the function $y_0(t) = 0$.

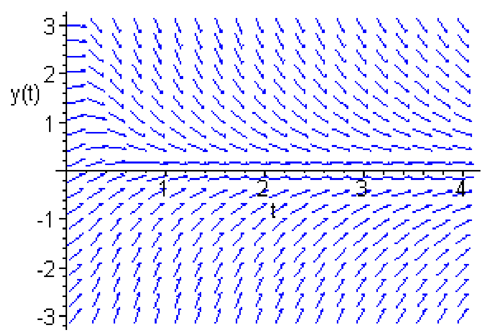
$7(a)$.



(b). All solutions seem to converge to the function $y_0(t) = 0$.

(c). The integrating factor is $\mu(t) = exp(t^2)$, and hence $y(t) = t^2 e^{-t^2} + c\, e^{-t^2}$. It is clear that all solutions converge to the function $y_0(t) = 0$.

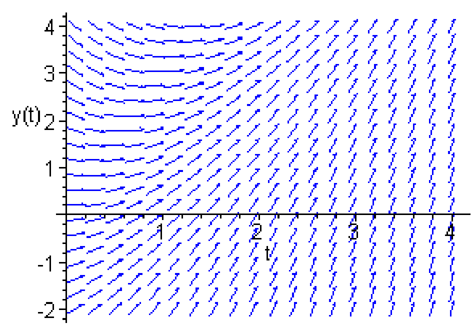
$8(a)$



(b). All solutions seem to converge to the function $y_0(t) = 0$.

(c). Since $\mu(t) = (1 + t^2)^2$, the general solution is $y(t) = [tan^{-1}(t) + C]/(1 + t^2)^2$. It follows that all solutions converge to the function $y_0(t) = 0$.

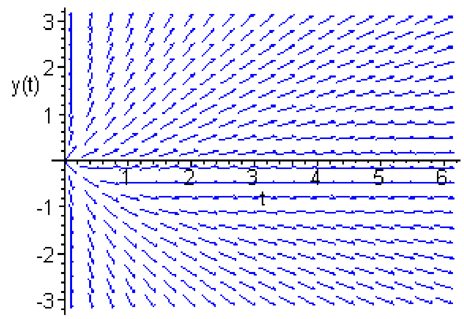
$9(a)$.



(b). All slopes *eventually* become positive, hence all solutions will increase without bound.

(c). The integrating factor is $\mu(t) = exp\left(\int \frac{1}{2}dt\right) = e^{t/2}$. The differential equation can be written as $e^{t/2}y' + e^{t/2}y/2 = 3t\,e^{t/2}/2$, that is, $(e^{t/2}\,y/2)' = 3t\,e^{t/2}/2$. Integration of both sides of the equation results in the general solution $y(t) = 3t - 6 + c\,e^{-t/2}$. All solutions approach the specific solution $y_0(t) = 3t - 6$.

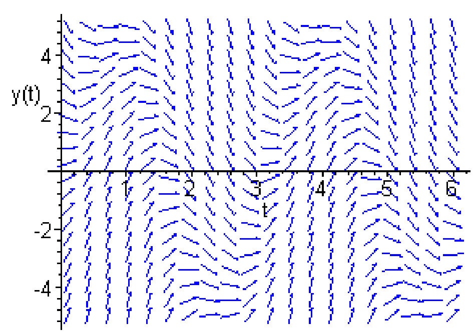
$10(a)$.



(b). For $y > 0$, the slopes are *all* positive, and hence the corresponding solutions increase
without bound. For $y < 0$, almost all solutions have negative slopes, and hence solutions tend to decrease without bound.

(c). First divide both sides of the equation by t. From the resulting *standard form*, the integrating factor is $\mu(t) = exp\left(-\int \frac{1}{t}dt\right) = 1/t$. The differential equation can be written as $y'/t - y/t^2 = t\,e^{-t}$, that is, $(y/t)' = t\,e^{-t}$. Integration leads to the general solution $y(t) = -te^{-t} + c\,t$. For $c \neq 0$, solutions *diverge*, as implied by the direction field. For the case $c = 0$, the specific solution is $y(t) = -te^{-t}$, which evidently approaches *zero* as $t \to \infty$.

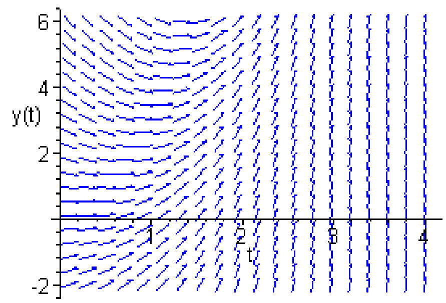
$11(a)$.



(b). The solutions appear to be oscillatory.

(c). The integrating factor is $\mu(t) = e^t$, and hence $y(t) = sin(2t) - 2\,cos(2t) + c\,e^{-t}$. It is evident that all solutions converge to the specific solution $y_0(t) = sin(2t) - 2$ $cos(2t)$.

$12(a)$.



(b). All solutions *eventually* have positive slopes, and hence increase without bound.

(c). The integrating factor is $\mu(t) = e^{2t}$. The differential equation can be written as $e^{t/2}y' + e^{t/2}y/2 = 3t^2/2$, that is, $\left(e^{t/2}\,y/2\right)' = 3t^2/2$. Integration of both sides of the equation results in the general solution $y(t) = 3t^2 - 12t + 24 + c\,e^{-t/2}$. It follows that all solutions converge to the specific solution $y_0(t) = 3t^2 - 12t + 24$.

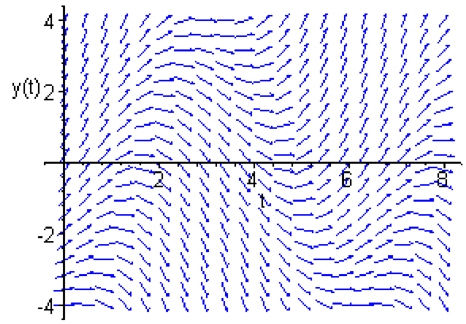
14. The integrating factor is $\mu(t) = e^{2t}$. After multiplying both sides by $\mu(t)$, the equation can be written as $\left(e^{2t}\,y\right)' = t$. Integrating both sides of the equation results in the general solution $y(t) = t^2e^{-2t}/2 + c\,e^{-2t}$. Invoking the specified condition, we require that $e^{-2}/2 + c\,e^{-2} = 0$. Hence $c = \, -1/2$, and the solution to the initial value problem is $y(t) = (t^2 - 1)e^{-2t}/2$.

16. The integrating factor is $\mu(t) = exp\left(\int \frac{2}{t}\,dt\right) = t^2$. Multiplying both sides by $\mu(t)$, the equation can be written as $(t^2\,y)' = cos(t)$. Integrating both sides of the equation results in the general solution $y(t) = sin(t)/t^2 + c\,t^{-2}$. Substituting $t = \pi$ and setting the value equal to *zero* gives $c = 0$. Hence the specific solution is $y(t) = sin(t)/t^2$.

17. The integrating factor is $\mu(t) = e^{-2t}$, and the differential equation can be written as $\left(e^{-2t}\,y\right)' = 1$. Integrating, we obtain $e^{-2t}\,y(t) = t + c$. Invoking the specified initial condition results in the solution $y(t) = (t + 2)e^{2t}$.

19. After writing the equation in *standard form*, we find that the integrating factor is $\mu(t) = exp\left(\int \frac{4}{t}\,dt\right) = t^4$. Multiplying both sides by $\mu(t)$, the equation can be written as $\left(t^4\,y\right)' = t\,e^{-t}$. Integrating both sides results in $t^4y(t) = \, -(t+1)e^{-t} + c$. Letting $t = \, -1$ and setting the value equal to *zero* gives $c = 0$. Hence the specific solution of the initial value problem is $y(t) = \, -\left(t^{-3} + t^{-4}\right)e^{-t}$.

$21(a)$.

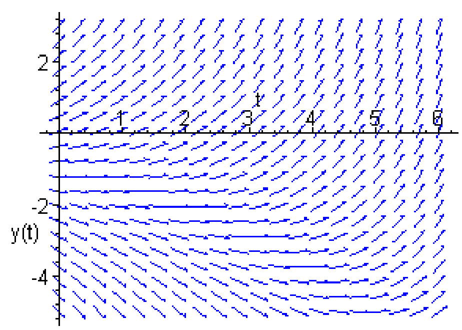


The solutions appear to diverge from an *apparent* oscillatory solution. From the direction
field, the critical value of the initial condition seems to be $a_0 = -1$. For $a > -1$, the solutions increase without bound. For $a < -1$, solutions decrease without bound.

(b). The integrating factor is $\mu(t) = e^{-t/2}$. The general solution of the differential equation is $y(t) = (8sin(t) - 4cos(t))/5 + c\,e^{t/2}$. The solution is sinusoidal as long as $c = 0$. The *initial value* of this sinusoidal solution is
$a_0 = (8sin(0) - 4cos(0))/5 = -4/5$.

(c). See part (b).

$22(a)$.



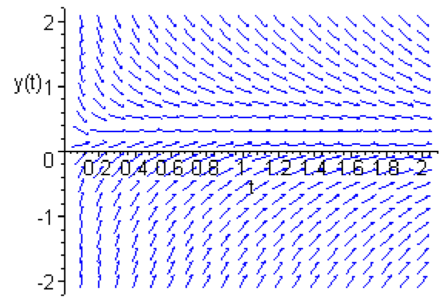
All solutions appear to *eventually* increase without bound. The solutions *initially* increase
or decrease, depending on the initial value a. The critical value seems to be $a_0 = -1$.

(b). The integrating factor is $\mu(t) = e^{-t/2}$, and the general solution of the differential equation is $y(t) = -3e^{t/3} + c\,e^{t/2}$. Invoking the initial condition $y(0) = a$, the solution
may also be expressed as $y(t) = -3e^{t/3} + (a+3)\,e^{t/2}$. Differentiating, follows that $y'(0) = -1 + (a+3)/2 = (a+1)/2$. The critical value is evidently $a_0 = -1$.

(c). For $a_0 = -1$, the solution is $y(t) = -3e^{t/3} + 2\,e^{t/2}$, which (for large t) is dominated by the term containing $e^{t/2}$.

is $y(t) = (8sin(t) - 4cos(t))/5 + c\,e^{t/2}$.

$23(a)$.

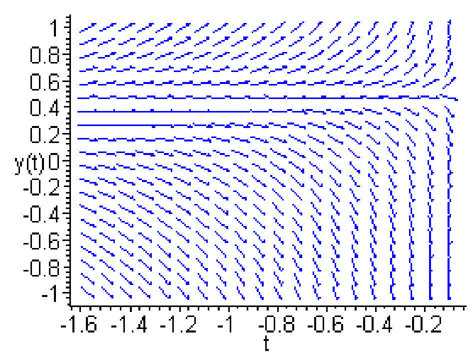


As $t \to 0$, solutions increase without bound if $y(1) = a > .4$, and solutions decrease without bound if $y(1) = a < .4$.

(b). The integrating factor is $\mu(t) = exp\left(\int \frac{t+1}{t}dt\right) = t\,e^t$. The general solution of the differential equation is $y(t) = t\,e^{-t} + c\,e^{-t}/t$. Invoking the specified value $y(1) = a$, we have $1 + c = a\,e$. That is, $c = a\,e - 1$. Hence the solution can also be expressed as $y(t) = t\,e^{-t} + (a\,e - 1)\,e^{-t}/t$. For *small* values of t, the second term is dominant. Setting $a\,e - 1 = 0$, critical value of the parameter is $a_0 = 1/e$.

(c). For $a > 1/e$, solutions increase without bound. For $a < 1/e$, solutions decrease without bound. When $a = 1/e$, the solution is $y(t) = t\,e^{-t}$, which approaches 0 as $t \to 0$.

$24(a)$.



As $t \to 0$, solutions increase without bound if $y(1) = a > .4$, and solutions decrease without bound if $y(1) = a < .4$.

(b). Given the initial condition, $y(-\pi/2) = a$, the solution is $y(t) = (a\,\pi^2/4 - cos\,t)/t$.

Since $\lim_{t\to 0} cos\,t = 1$, solutions increase without bound if $a > 4/\pi^2$, and solutions decrease without bound if $a < 4/\pi^2$. Hence the critical value is $a_0 = 4/\pi^2 = 0.452847....$

(c). For $a = 4/\pi^2$, the solution is $y(t) = (1 - cos\,t)/t$, and $\lim_{t\to 0} y(t) = 1/2$. Hence the solution is bounded.

25. The integrating factor is $\mu(t) = exp\left(\int \frac{1}{2}dt\right) = e^{t/2}$. Therefore general solution is $y(t) = [4cos(t) + 8sin(t)]/5 + c\,e^{-t/2}$. Invoking the initial condition, the specific solution is $y(t) = [4cos(t) + 8sin(t) - 9\,e^{t/2}]/5$. Differentiating, it follows that

$$y'(t) = \left[-4sin(t) + 8cos(t) + 4.5\,e^{-t/2}\right]/5$$
$$y''(t) = \left[-4cos(t) - 8sin(t) - 2.25\,e^{-t/2}\right]/5$$

Setting $y'(t) = 0$, the first solution is $t_1 = 1.3643$, which gives the location of the *first* stationary point. Since $y''(t_1) < 0$, the first stationary point in a local *maximum*. The coordinates of the point are $(1.3643\,,.82008)$.

26. The integrating factor is $\mu(t) = exp\left(\int \frac{2}{3}dt\right) = e^{2t/3}$, and the differential equation can
be written as $(e^{2t/3}\,y)' = e^{2t/3} - t\,e^{2t/3}/2$. The general solution is $y(t) = (21 - 6t)/8 +$ $+ c\,e^{-2t/3}$. Imposing the initial condition, we have $y(t) = (21 - 6t)/8 + (y_0 - 21/8)e^{-2t/3}$. Since the solution is smooth, the desired intersection will be a point of tangency. Taking the derivative, $y'(t) = -3/4 - (2y_0 - 21/4)e^{-2t/3}/3$. Setting $y'(t) = 0$, the solution is $t_1 = \frac{3}{2}ln[(21 - 8y_0)/9]$. Substituting into the solution, the respective *value* at the stationary point is $y(t_1) = \frac{3}{2} + \frac{9}{4}ln\,3 - \frac{9}{8}ln(21 - 8y_0)$. Setting this result equal to *zero*, we obtain the required initial value $y_0 = (21 - 9\,e^{4/3})/8 = -1.643$.

27. The integrating factor is $\mu(t) = e^{t/4}$, and the differential equation can be written as $(e^{t/4}\,y)' = 3\,e^{t/4} + 2\,e^{t/4}cos(2t)$. The general solution is

$$y(t) = 12 + [8cos(2t) + 64sin(2t)]/65 + c\;e^{-t/4}.$$

Invoking the initial condition, $y(0) = 0$, the specific solution is

$$y(t) = 12 + \left[8cos(2t) + 64sin(2t) - 788\,e^{-t/4}\right]/65\,.$$

As $t \to \infty$, the exponential term will decay, and the solution will oscillate about an *average*
value of 12, with an *amplitude* of $8/\sqrt{65}$.

29. The integrating factor is $\mu(t) = e^{-3t/2}$, and the differential equation can be written as $(e^{-3t/2}\, y)' = 3t\, e^{-3t/2} + 2\, e^{-t/2}$. The general solution is $y(t) = -2t - 4/3 - 4\, e^{t} + + c\, e^{3t/2}$. Imposing the initial condition, $y(t) = -2t - 4/3 - 4\, e^{t} + (y_0 + 16/3)\, e^{3t/2}$. As $t \to \infty$, the term containing $e^{3t/2}$ will *dominate* the solution. Its *sign* will determine the divergence properties. Hence the critical value of the initial condition is
$y_0 = -16/3$.
The corresponding solution, $y(t) = -2t - 4/3 - 4\, e^{t}$, will also decrease without bound.

Note on Problems 31-34 :

Let $g(t)$ be *given*, and consider the function $y(t) = y_1(t) + g(t)$, in which $y_1(t) \to \infty$ as $t \to \infty$. Differentiating, $y'(t) = y_1'(t) + g'(t)$. Letting a be a *constant*, it follows that $y'(t) + ay(t) = y_1'(t) + ay_1(t) + g'(t) + ag(t)$. Note that the hypothesis on the function $y_1(t)$ will be satisfied, if $y_1'(t) + ay_1(t) = 0$. That is, $y_1(t) = c\, e^{-at}$. Hence $y(t) = c\, e^{-at} + g(t)$, which is a solution of the equation $y' + ay = g'(t) + ag(t)$. For convenience, choose $a = 1$.

31. Here $g(t) = 3$, and we consider the linear equation $y' + y = 3$. The integrating factor is $\mu(t) = e^{t}$, and the differential equation can be written as $(e^{t}\, y)' = 3e^{t}$. The general solution is $y(t) = 3 + c\, e^{-t}$.

33. $g(t) = 3 - t$. Consider the linear equation $y' + y = -1 + 3 - t$.The integrating factor is $\mu(t) = e^{t}$, and the differential equation can be written as $(e^{t}\, y)' = (2 - t)e^{t}$. The general solution is $y(t) = 3 - t + c\, e^{-t}$.

34. $g(t) = 4 - t^2$. Consider the linear equation $y' + y = 4 - 2t - t^2$.The integrating factor is $\mu(t) = e^{t}$, and the equation can be written as $(e^{t}\, y)' = (4 - 2t - t^2)e^{t}$. The general solution is $y(t) = 4 - t^2 + c\, e^{-t}$.

Section 2.2

2. For $x \neq -1$, the differential equation may be written as $y\,dy = [x^2/(1+x^3)]dx$. Integrating both sides, with respect to the appropriate variables, we obtain the relation $y^2/2 = \frac{1}{3}ln|1+x^3| + c$. That is, $y(x) = \pm\sqrt{\frac{2}{3}ln|1+x^3| + c}$.

3. The differential equation may be written as $y^{-2}dy = -\sin x\,dx$. Integrating both sides of the equation, with respect to the appropriate variables, we obtain the relation $-y^{-1} = \cos x + c$. That is, $(C - \cos x)y = 1$, in which C is an arbitrary constant. Solving for the dependent variable, explicitly, $y(x) = 1/(C - \cos x)$.

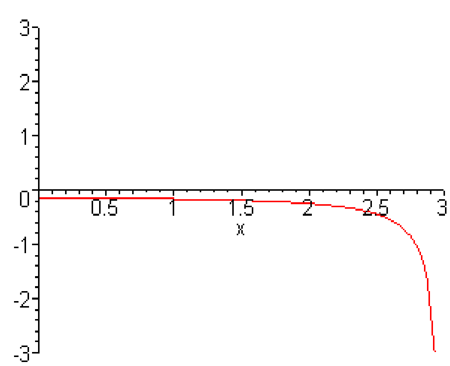
5. Write the differential equation as $\cos^{-2} 2y\,dy = \cos^2 x\,dx$, or $\sec^2 2y\,dy = \cos^2 x\,dx$. Integrating both sides of the equation, with respect to the appropriate variables, we obtain the relation $\tan 2y = \sin x \cos x + x + c$.

7. The differential equation may be written as $(y + e^y)dy = (x - e^{-x})dx$. Integrating both sides of the equation, with respect to the appropriate variables, we obtain the relation
$y^2 + 2\,e^y = x^2 + 2\,e^{-x} + c$.

8. Write the differential equation as $(1+y^2)dy = x^2\,dx$. Integrating both sides of the equation, we obtain the relation $y + y^3/3 = x^3/3 + c$, that is, $3y + y^3 = x^3 + C$.

9(a). The differential equation is separable, with $y^{-2}dy = (1-2x)dx$. Integration yields $-y^{-1} = x - x^2 + c$. Substituting $x = 0$ and $y = -1/6$, we find that $c = 6$. Hence the specific solution is $y^{-1} = x^2 - x - 6$. The *explicit form* is
$y(x) = 1/(x^2 - x - 6)$.

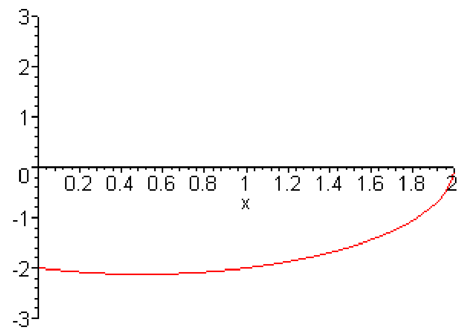
(b)



(c). Note that $x^2 - x - 6 = (x+2)(x-3)$. Hence the solution becomes *singular* at $x = -2$ and $x = 3$.

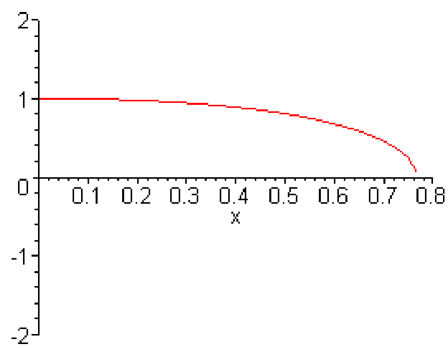
10(a). $y(x) = -\sqrt{2x - 2x^2 + 4}$.

$10(b)$.



$11(a)$. Rewrite the differential equation as $x\, e^x dx = -y\, dy$. Integrating both sides of the equation results in $x\, e^x - e^x = -y^2/2 + c$. Invoking the initial condition, we obtain $c = -1/2$. Hence $y^2 = 2e^x - 2x\, e^x - 1$. The *explicit form* of the solution is $y(x) = \sqrt{2e^x - 2x\, e^x - 1}$. The *positive* sign is chosen, since $y(0) = 1$.

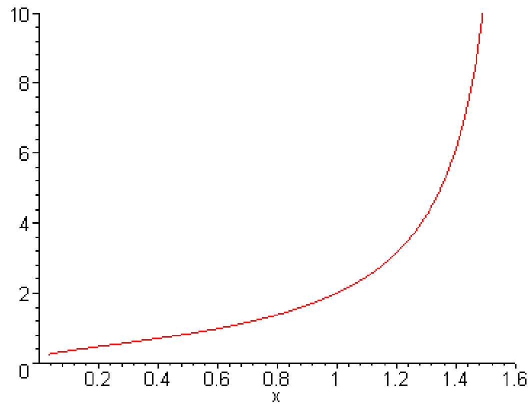
(b).



(c). The function under the radical becomes *negative* near $x = -1.7$ and $x = 0.76$.

$11(a)$. Write the differential equation as $r^{-2}dr = \theta^{-1}\, d\theta$. Integrating both sides of the equation results in the relation $-r^{-1} = ln\, \theta + c$. Imposing the condition $r(1) = 2$, we obtain $c = -1/2$. The *explicit form* of the solution is $r(\theta) = 2/(1 - 2\, ln\, \theta)$.

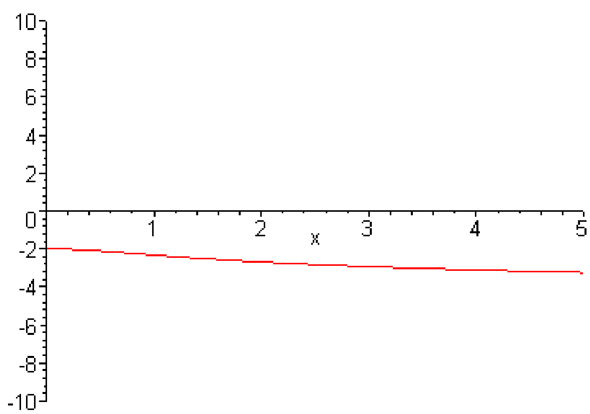
(b).



(c). Clearly, the solution makes sense only if $\theta > 0$. Furthermore, the solution becomes singular when $ln\,\theta = 1/2$, that is, $\theta = \sqrt{e}$.

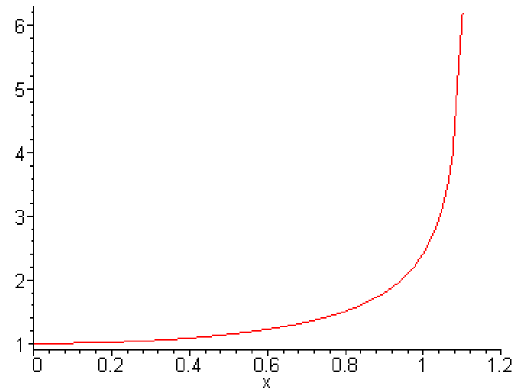
13(a). $y(x) = -\sqrt{2\,ln(1+x^2)+4}$.

(b).



14(a). Write the differential equation as $y^{-3}dy = x(1+x^2)^{-1/2}\,dx$. Integrating both sides of the equation, with respect to the appropriate variables, we obtain the relation $-y^{-2}/2 = \sqrt{1+x^2}+c$. Imposing the initial condition, we obtain $c = -3/2$. Hence the specific solution can be expressed as $y^{-2} = 3 - 2\sqrt{1+x^2}$. The *explicit form* of the solution is $y(x) = 1/\sqrt{3-2\sqrt{1+x^2}}$. The *positive* sign is chosen to satisfy the initial condition.

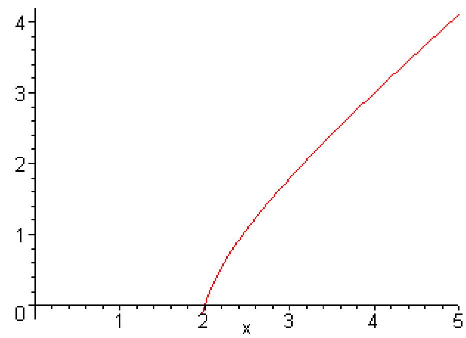
(b).



(c). The solution becomes singular when $2\sqrt{1+x^2}=3$. That is, at $x=\pm\sqrt{5}/2$.

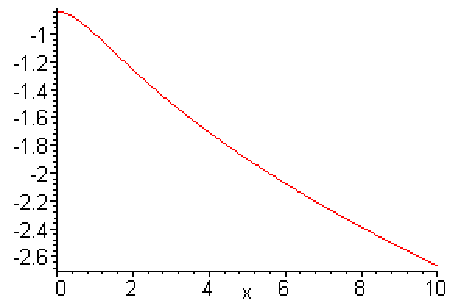
$15(a)$. $y(x)=-1/2+\sqrt{x^2-15/4}$.

(b).



$16(a)$. Rewrite the differential equation as $4y^3dy=x(x^2+1)dx$. Integrating both sides
of the equation results in $y^4=(x^2+1)^2/4+c$. Imposing the initial condition, we obtain $c=0$. Hence the solution may be expressed as $(x^2+1)^2-4y^4=0$. The *explicit* form of the solution is $y(x)=-\sqrt{(x^2+1)/2}$. The *sign* is chosen based on $y(0)=-1/\sqrt{2}$.

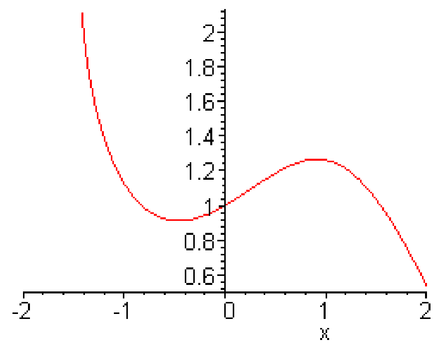
(b).



(c). The solution is valid for all $x \in \mathbb{R}$.

$17(a)$. $y(x) = -5/2 - \sqrt{x^3 - e^x + 13/4}$.

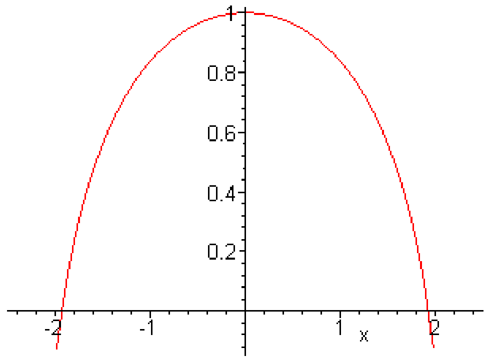
(b).



(c). The solution is valid for $x > -1.45$. This value is found by estimating the root of $4x^3 - 4e^x + 13 = 0$.

$18(a)$. Write the differential equation as $(3 + 4y)dy = (e^{-x} - e^x)dx$. Integrating both sides of the equation, with respect to the appropriate variables, we obtain the relation $3y + 2y^2 = -(e^x + e^{-x}) + c$. Imposing the initial condition, $y(0) = 1$, we obtain $c = 7$. Thus, the solution can be expressed as $3y + 2y^2 = -(e^x + e^{-x}) + 7$. Now by *completing the square* on the left hand side, $2(y + 3/4)^2 = -(e^x + e^{-x}) + 65/8$. Hence the *explicit* form of the solution is $y(x) = -3/4 + \sqrt{65/16 - cosh\, x}$.

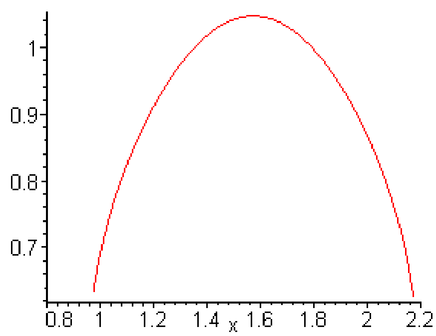
(b).



(c). Note the $65 - 16\,cosh\,x \geq 0$, as long as $|x| > 2.1$. Hence the solution is valid on the interval $-2.1 < x < 2.1$.

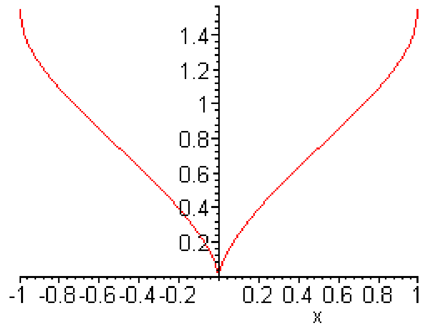
19(a). $y(x) = -\pi/3 + \frac{1}{3}sin^{-1}(3\,cos^2 x)$.

(b).



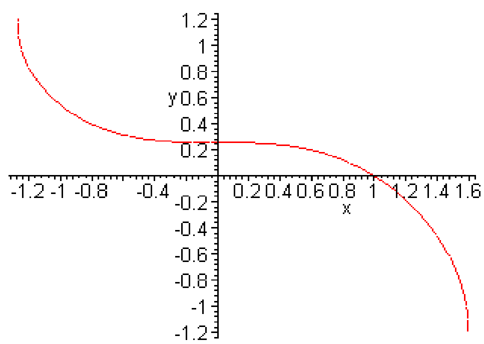
20(a). Rewrite the differential equation as $y^2dy = arcsin\,x/\sqrt{1-x^2}\,dx$. Integrating both sides of the equation results in $y^3/3 = (arcsin\,x)^2/2 + c$. Imposing the condition $y(0) = 0$, we obtain $c = 0$. The *explicit* form of the solution is $y(x) = \sqrt[3]{\frac{3}{2}}(arcsin\,x)^{2/3}$.

(b).



(c). Evidently, the solution is defined for $-1 \le x \le 1$.

22. The differential equation can be written as $(3y^2 - 4)dy = 3x^2 dx$. Integrating both sides, we obtain $y^3 - 4y = x^3 + c$. Imposing the initial condition, the specific solution is $y^3 - 4y = x^3 - 1$. Referring back to the differential equation, we find that $y' \to \infty$ as $y \to \pm 2/\sqrt{3}$. The respective values of the abscissas are $x = -1.276, 1.598$.



Hence the solution is valid for $-1.276 < x < 1.598$.

24. Write the differential equation as $(3 + 2y)dy = (2 - e^x)dx$. Integrating both sides, we obtain $3y + y^2 = 2x - e^x + c$. Based on the specified initial condition, the solution can be written as $3y + y^2 = 2x - e^x + 1$. *Completing the square*, it follows that $y(x) = -3/2 + \sqrt{2x - e^x + 13/4}$. The solution is defined if $2x - e^x + 13/4 \ge 0$, that is, $-1.5 \le x \le 2$ (*approximately*). In that interval, $y' = 0$, for $x = ln\,2$. It can be verified that $y''(ln\,2) < 0$. In fact, $y''(x) < 0$ on the interval of definition. Hence the solution attains a global maximum at $x = ln\,2$.

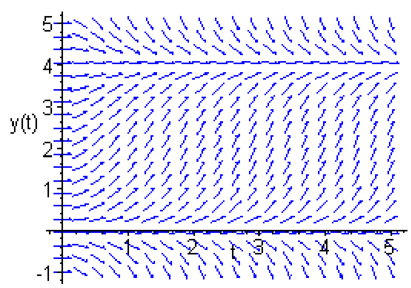
26. The differential equation can be written as $(1 + y^2)^{-1}dy = 2(1 + x)dx$. Integrating both sides of the equation, we obtain $arctany = 2x + x^2 + c$. Imposing the given initial
condition, the specific solution is $arctany = 2x + x^2$. Therefore, $y(x) = tan(2x + x^2)$. Observe that the solution is defined as long as $-\pi/2 < 2x + x^2 < \pi/2$. It is easy to see that $2x + x^2 \ge -1$. Furthermore, $2x + x^2 = \pi/2$ for $x = -2.6$ and 0.6. Hence the solution is valid on the interval $-2.6 < x < 0.6$. Referring back to the differential

equation, the solution is *stationary* at $x = -1$. Since $y''(x) > 0$ on the entire interval of
definition, the solution attains a global minimum at $x = -1$.

28(a). Write the differential equation as $y^{-1}(4-y)^{-1}dy = t(1+t)^{-1}dt$. Integrating both sides of the equation, we obtain $ln\,|y| - ln|y-4| = 4t - 4ln|1+t| + c$. Taking the *exponential* of both sides, it follows that $|y/(y-4)| = C\,e^{4t}/(1+t)^4$. It follows that as $t \to \infty$, $|y/(y-4)| = |1 + 4/(y-4)| \to \infty$. That is, $y(t) \to 4$.

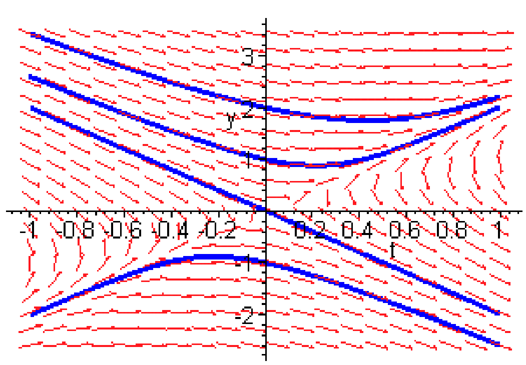
(b). Setting $y(0) = 2$, we obtain that $C = 1$. Based on the initial condition, the solution may be expressed as $y/(y-4) = -e^{4t}/(1+t)^4$. Note that $y/(y-4) < 0$, for all $t \geq 0$. Hence $y < 4$ for all $t \geq 0$. Referring back to the differential equation, it follows that y' is always *positive*. This means that the solution is *monotone increasing*. We find that the root of the equation $e^{4t}/(1+t)^4 = 399$ is near $t = 2.844$.

(c). Note the $y(t) = 4$ is an equilibrium solution. Examining the local direction field,

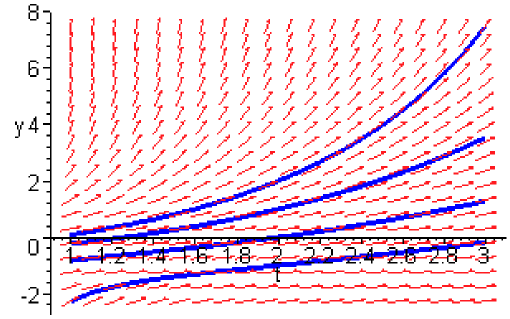


we see that if $y(0) > 0$, then the corresponding solutions converge to $y = 4$. Referring back to part (a), we have $y/(y-4) = [y_0/(y_0-4)]e^{4t}/(1+t)^4$, for $y_0 \neq 4$. Setting $t = 2$, we obtain $y_0/(y_0-4) = (3/e^2)^4 y(2)/(y(2)-4)$. Now since the function $f(y) = y/(y-4)$ is *monotone* for $y < 4$ and $y > 4$, we need only solve the equations $y_0/(y_0-4) = -399(3/e^2)^4$and $y_0/(y_0-4) = 401(3/e^2)^4$. The respective solutions are $y_0 = 3.6622$ and $y_0 = 4.4042$.

30(f).



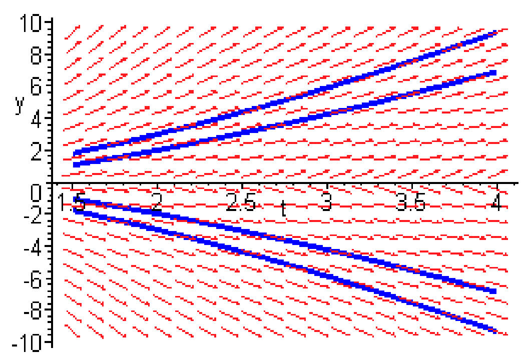
31(c)



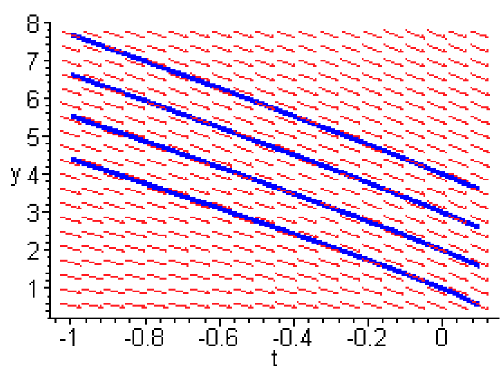
32(a). Observe that $(x^2+3y^2)/2xy = \frac{1}{2}\left(\frac{y}{x}\right)^{-1} + \frac{3}{2}\frac{y}{x}$. Hence the differential equation is *homogeneous*.

(b). The substitution $y = x\,v$ results in $v + x\,v' = (x^2+3x^2v^2)/2x^2v$. The transformed equation is $v' = (1+v^2)/2xv$. This equation is *separable*, with general solution $v^2+1 = c\,x$. In terms of the original dependent variable, the solution is $x^2+y^2 = c\,x^3$.

(c).



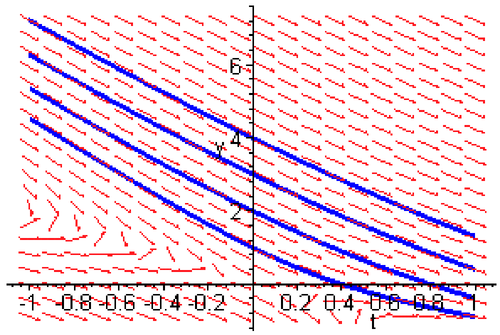
33(c).



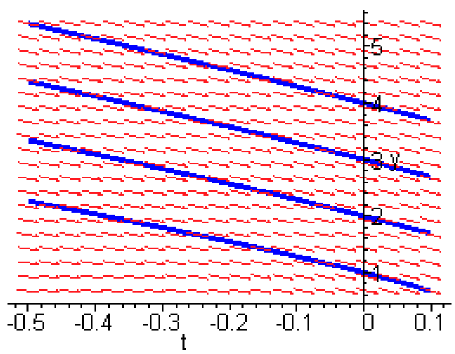
34(a). Observe that $-(4x+3y)/(2x+y) = -2 - \frac{y}{x}\left[2+\frac{y}{x}\right]^{-1}$. Hence the differential equation is *homogeneous*.

(b). The substitution $y = x\,v$ results in $v + x\,v' = -2 - v/(2+v)$. The transformed equation is $v' = -(v^2+5v+4)/(2+v)x$. This equation is *separable*, with general solution $(v{+}4)^2|v{+}1| = C/x^3$. In terms of the original dependent variable, the solution is $(4x+y)^2|x{+}y| = C$.

(c).



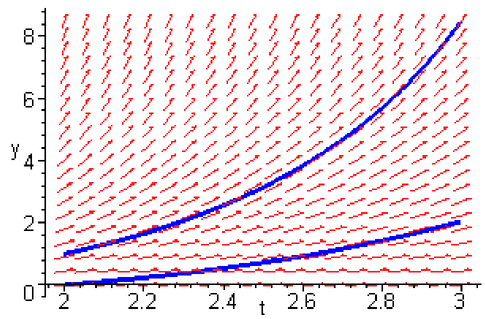
35(c).



36(a). Divide by x^2 to see that the equation is homogeneous. Substituting $y = x\,v$, we obtain $x\,v' = (1+v)^2$. The resulting differential equation is separable.

(b). Write the equation as $(1+v)^{-2}dv = x^{-1}dx$. Integrating both sides of the equation, we obtain the general solution $-1/(1+v) = ln|x| + c$. In terms of the original dependent variable, the solution is $y = x\,[C - ln|x|]^{-1} - x$.

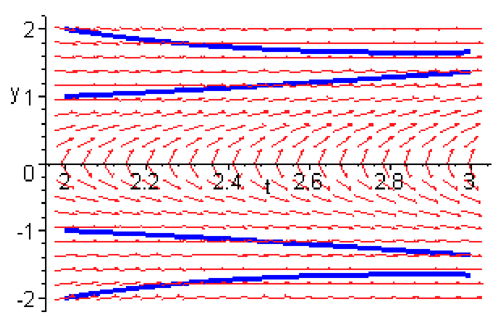
(c).



$37(a)$. The differential equation can be expressed as $y' = \frac{1}{2}\left(\frac{y}{x}\right)^{-1} - \frac{3}{2}\frac{y}{x}$. Hence the equation is homogeneous. The substitution $y = x\,v$ results in $x\,v' = (1-5v^2)/2v$. Separating variables, we have $\frac{2v}{1-5v^2}dv = \frac{1}{x}dx$.

(b). Integrating both sides of the transformed equation yields $-\frac{1}{5}$ $ln|1-5v^2| = ln|x| + c$,
that is, $1-5v^2 = C/|x|^5$. In terms of the original dependent variable, the general solution is $5y^2 = x^2 - C/|x|^3$.

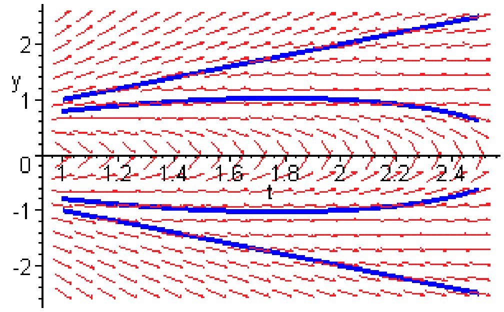
(c).



$38(a)$. The differential equation can be expressed as $y' = \frac{3}{2}\frac{y}{x} - \frac{1}{2}\left(\frac{y}{x}\right)^{-1}$. Hence the equation is homogeneous. The substitution $y = x\,v$ results in $x\,v' = (v^2-1)/2v$, that is, $\frac{2v}{v^2-1}dv = \frac{1}{x}dx$.

(b). Integrating both sides of the transformed equation yields $ln|v^2-1| = ln|x| + c$, that is, $v^2 - 1 = C|x|$. In terms of the original dependent variable, the general solution is $y^2 = C\,x^2|x| + x^2$.

(c).



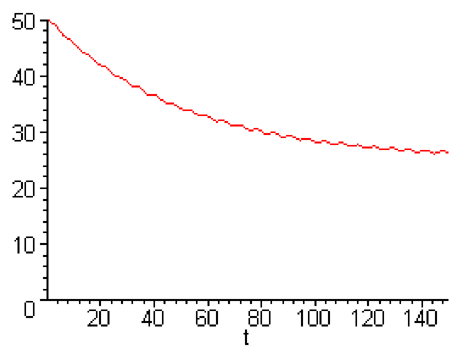
Section 2.3

$5(a)$. Let Q be the amount of salt in the tank. Salt enters the tank of water at a rate of $2\,\frac{1}{4}\left(1+\frac{1}{2}sin\,t\right)=\frac{1}{2}+\frac{1}{4}sin\,t\ \ oz/min$. It leaves the tank at a rate of $2\,Q/100\ \ oz/min$. Hence the differential equation governing the amount of salt at any time is

$$\frac{dQ}{dt}=\frac{1}{2}+\frac{1}{4}sin\,t-Q/50\,.$$

The initial amount of salt is $Q_0=50\ oz$. The governing ODE is *linear*, with integrating factor $\mu(t)=e^{t/50}$. Write the equation as $\left(e^{t/50}Q\right)'=e^{t/50}\left(\frac{1}{2}+\frac{1}{4}sin\,t\right)$. The specific solution is $Q(t)=25+\left[12.5sin\,t-625cos\,t+63150\,e^{-t/50}\right]/2501\ \ oz$.

(b).



(c). The amount of salt approaches a *steady state*, which is an oscillation of amplitude $1/4$ about a level of $25\ oz$.

$6(a)$. The equation governing the value of the investment is $dS/dt=r\,S$. The value of the investment, at any time, is given by $S(t)=S_0e^{rt}$. Setting $S(T)=2S_0$, the required time is $T=ln(2)/r$.

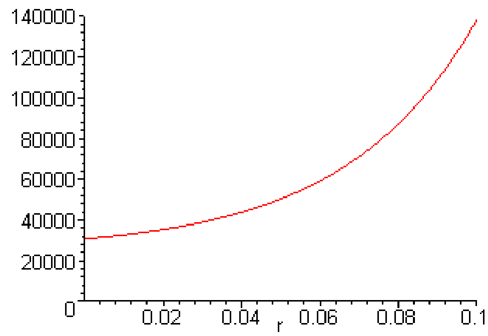
(b). For the case $r=7\%=.07$, $T\approx 9.9\ yrs$.

(c). Referring to Part(a), $r=ln(2)/T$. Setting $T=8$, the required interest rate is to be approximately $r=8.66\,\%$.

$8(a)$. Based on the solution in *Eq.*(16), with $S_0=0$, the value of the investments *with* contributions is given by $S(t)=25,000(e^{rt}-1)$. After *ten* years, person A has $S_A=\$25,000(1.226)=\$\,30,640$. Beginning at age 35, the investments can now be analyzed using the equations $S_A=30,640\,e^{.08t}$ and $S_B=25,000(e^{.08t}-1)$. After *thirty* years, the balances are $S_A=\$\,337,734$ and $S_B=\$\,250,579$.

(b). For an *unspecified* rate r, the balances after *thirty* years are $S_A=30,640\,e^{30r}$ and $S_B=25,000(e^{30r}-1)$.

(c).



(d). The two balances can *never* be equal.

$11(a)$. Let S be the value of the mortgage. The debt accumulates at a rate of rS, in which $r = .09$ is the *annual* interest rate. Monthly payments of $\$\,800$ are equivalent to $\$\,9,600$ *per year*. The differential equation governing the value of the mortgage is $dS/dt = .09\,S - 9,600$. Given that S_0 is the original amount borrowed, the debt is $S(t) = S_0 e^{.09t} - 106,667(e^{.09t} - 1)$. Setting $S(30) = 0$, it follows that $S_0 = \$\,99,500$.

(b). The *total* payment, over 30 years, becomes $\$\,288,000$. The interest paid on this purchase is $\$\,188,500$.

$13(a)$. The balance *increases* at a rate of $r\,S$ *\$/yr*, and *decreases* at a constant rate of k *\$ per year*. Hence the balance is modeled by the differential equation $dS/dt = rS - k$. The balance at any time is given by $S(t) = S_0 e^{rt} - \frac{k}{r}(e^{rt} - 1)$.

(b). The solution may also be expressed as $S(t) = (S_0 - \frac{k}{r})e^{rt} + \frac{k}{r}$. Note that if the withdrawal rate is $k_0 = r\,S_0$, the balance will remain at a constant level S_0.

(c). Assuming that $k > k_0$, $S(T_0) = 0$ for $T_0 = \frac{1}{r} ln\left[\frac{k}{k - k_0}\right]$.

(d). If $r = .08$ and $k = 2k_0$, then $T_0 = 8.66$ *years*.

(e). Setting $S(t) = 0$ and solving for e^{rt} in Part(b), $e^{rt} = \frac{k}{k - rS_0}$. Now setting $t = T$ results in $k = rS_0 e^{rT}/(e^{rT} - 1)$.

(f). In part(e), let $k = 12,000$, $r = .08$, and $T = 20$. The required investment becomes $S_0 = \$\,119,715$.

$14(a)$. Let $Q' = -r\,Q$. The general solution is $Q(t) = Q_0 e^{-rt}$. Based on the definition of *half-life*, consider the equation $Q_0/2 = Q_0 e^{-5730\,r}$. It follows that

$-5730\,r = ln(1/2)$, that is, $r = 1.2097 \times 10^{-4}$ *per year*.

(b). Hence the amount of carbon-14 is given by $Q(t) = Q_0\, e^{-1.2097\times 10^{-4}t}$.

(c). Given that $Q(T) = Q_0/5$, we have the equation $1/5 = e^{-1.2097\times 10^{-4}T}$. Solving for the *decay time*, the apparent age of the remains is approximately $T = 13,304.65$ *years*.

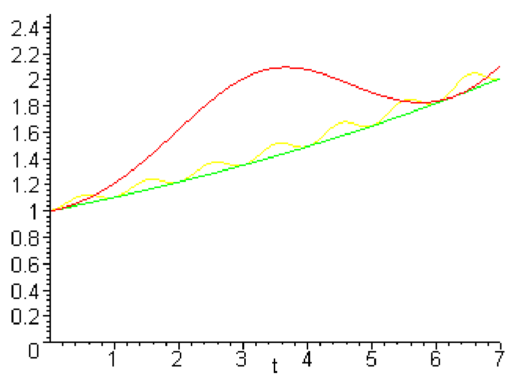
15. Let $P(t)$ be the population of mosquitoes at any time t. The rate of *increase* of the mosquito population is rP. The population *decreases* by $20,000$ *per day*. Hence the equation that models the population is given by $dP/dt = rP - 20,000$. Note that the variable t represents *days*. The solution is $P(t) = P_0e^{rt} - \frac{20,000}{r}(e^{rt} - 1)$. In the absence of predators, the governing equation is $dP_1/dt = rP_1$, with solution $P_1(t) = P_0e^{rt}$. Based on the data, set $P_1(7) = 2P_0$, that is, $2P_0 = P_0e^{7r}$. The growth rate is determined as $r = ln(2)/7 = .09902$ *per day*. Therefore the population, including the *predation* by birds, is $P(t) = 2\times 10^5 e^{.099t} - 201,997(e^{.099t} - 1) =$
$= 201,997.3 - 1977.3\, e^{.099t}$.

16(a). $y(t) = exp[2/10 + t/10 - 2cos(t)/10]$. The *doubling-time* is $\tau \approx 2.9632$.

(b). The differential equation is $dy/dt = y/10$, with solution $y(t) = y(0)e^{t/10}$. The *doubling-time* is given by $\tau = 10ln(2) \approx 6.9315$.

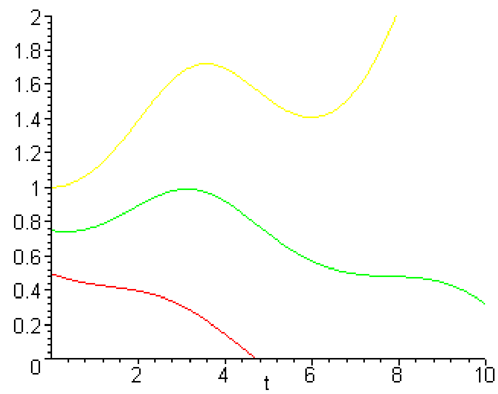
(c). Consider the differential equation $dy/dt = (0.5 + sin(2\pi t))\, y/5$. The equation is *separable*, with $\frac{1}{y}dy = \left(0.1 + \frac{1}{5}sin(2\pi t)\right)dt$. Integrating both sides, with respect to the appropriate variable, we obtain $ln\, y = (\pi t - cos(2\pi t))/10\pi + c$. Invoking the initial condition, the solution is $y(t) = exp[(1 + \pi t - cos(2\pi t))/10\pi]$. The *doubling-time* is $\tau \approx 6.3804$. The *doubling-time* approaches the value found in part(b).

(d).



17(a). The differential equation $dy/dt = r(t)\, y - k$ is *linear*, with integrating factor $\mu(t) = exp\left[-\int r(t)dt\right]$. Write the equation as $(\mu\, y)' = -k\,\mu(t)$. Integration of both

sides yields the general solution $y = \left[-k\int \mu(\tau)d\tau + y_0\,\mu(0)\right]/\mu(t)$. In this problem, the
integrating factor is $\mu(t) = exp[(cos\,t - t)/5]$.



(b). The population becomes *extinct*, if $y(t^*) = 0$, for some $t = t^*$. Referring to part(a),
we find that $y(t^*) = 0 \Rightarrow$

$$\int_0^{t^*} exp[(cos\,\tau - \tau)/5]d\tau = 5\,e^{1/5}y_c.$$

It can be shown that the integral on the left hand side increases *monotonically*, from *zero* to a limiting value of approximately 5.0893. Hence extinction can happen *only if* $5\,e^{1/5}y_c < 5.0893$, that is, $y_c < 0.8333$.

(c). Repeating the argument in part(b), it follows that $y(t^*) = 0 \Rightarrow$

$$\int_0^{t^*} exp[(cos\,\tau - \tau)/5]d\tau = \frac{1}{k}\,e^{1/5}y_c.$$

Hence extinction can happen *only if* $e^{1/5}y_c/k < 5.0893$, that is, $y_c < 4.1667\,k$.

(d). Evidently, y_c is a *linear* function of the parameter k.

19(a). Let $Q(t)$ be the *volume* of carbon monoxide in the room. The rate of *increase* of *CO* is $(.04)(0.1) = 0.004\ ft^3/min$. The amount of *CO leaves the room* at a rate of $(0.1)Q(t)/1200 = Q(t)/12000\ \ ft^3/min$. Hence the total rate of change is given by the differential equation $dQ/dt = 0.004 - Q(t)/12000$. This equation is *linear* and separable, with solution $Q(t) = 48 - 48\,exp(-t/12000)\ \ ft^3$. Note that $Q_0 = 0\ \ ft^3$. Hence the *concentration* at any time is given by $x(t) = Q(t)/1200 = Q(t)/12\ \%$.

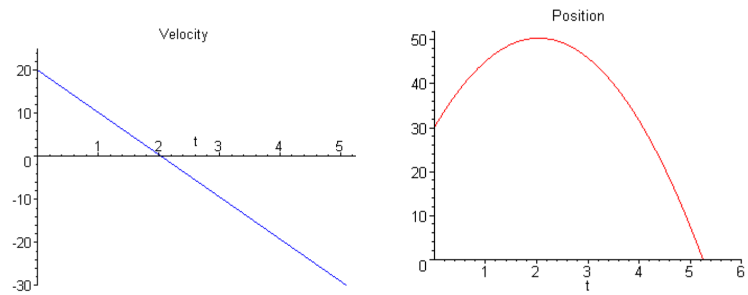
(b). The *concentration* of *CO* in the room is $x(t) = 4 - 4exp(-t/12000)\ \%$. A level of 0.00012 corresponds to $0.012\ \%$. Setting $x(\tau) = 0.012$, the solution of the equation $4 - 4exp(-t/12000) = 0.012$ is $\tau \approx 36\ \ minutes$.

$20(a)$. The concentration is $c(t) = k + P/r + (c_0 - k - P/r)e^{-rt/V}$. It is easy to see that $c(t \to \infty) = k + P/r$.

(b). $c(t) = c_0\, e^{-rt/V}$. The *reduction times* are $T_{50} = ln(2)V/r$ and $T_{10} = ln(10)V/r$.

(c). The *reduction times*, in *years*, are $T_S = ln(10)(65.2)/12,200 = 430.85$
$T_M = ln(10)(158)/4,900 = 71.4$; $T_E = ln(10)(175)/460 = 6.05$
$T_O = ln(10)(209)/16,000 = 17.63$.

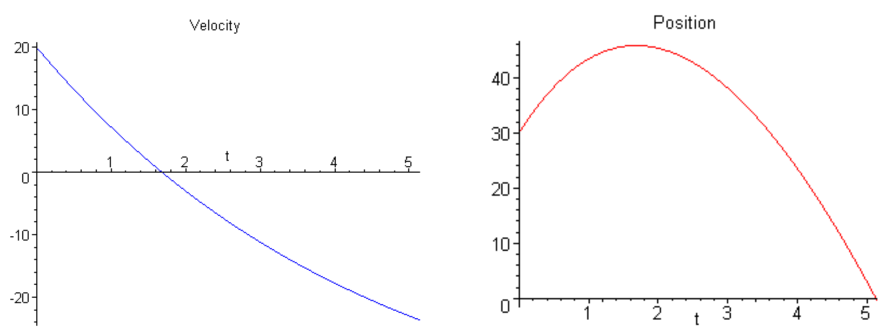
$21(c)$.



$22(a)$. The differential equation for the motion is $m\,dv/dt = -v/30 - mg$. Given the initial condition $v(0) = 20$ *m/s*, the solution is $v(t) = -44.1 + 64.1\,exp(-t/4.5)$. Setting $v(t_1) = 0$, the ball reaches the maximum height at $t_1 = 1.683$ *sec*. Integrating $v(t)$, the position is given by $x(t) = 318.45 - 44.1\,t - 288.45\,exp(-t/4.5)$. Hence the *maximum height* is $x(t_1) = 45.78$ *m*.

(b). Setting $x(t_2) = 0$, the ball hits the ground at $t_2 = 5.128$ *sec*.

(c).

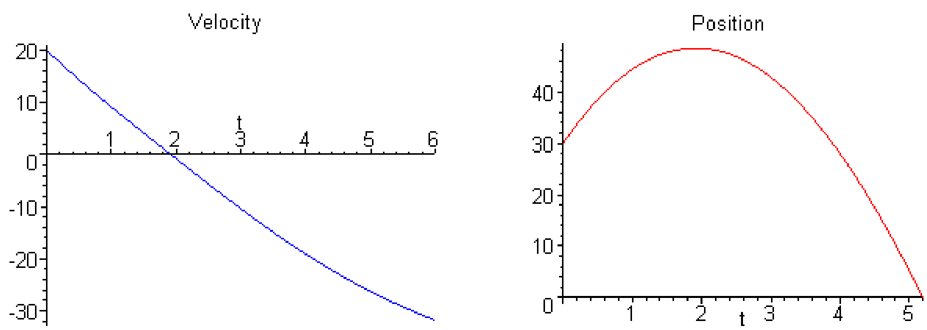


$23(a)$. The differential equation for the *upward* motion is $m\,dv/dt = -\mu v^2 - mg$, in which $\mu = 1/1325$. This equation is *separable*, with $\frac{m}{\mu v^2 + mg} dv = -dt$. Integrating both sides and invoking the initial condition, $v(t) = 44.133\, tan(.425 - .222\, t)$. Setting $v(t_1) = 0$, the ball reaches the maximum height at $t_1 = 1.916\, sec$. Integrating $v(t)$, the position is given by $x(t) = 198.75\, ln[cos(0.222\, t - 0.425)] + 48.57$. Therefore the *maximum height* is $x(t_1) = 48.56\ m$.

(b). The differential equation for the *downward* motion is $m\, dv/dt = +\mu v^2 - mg$. This equation is also separable, with $\frac{m}{mg - \mu v^2} dv = -dt$. For convenience, set $t = 0$ at the *top* of the trajectory. The new initial condition becomes $v(0) = 0$. Integrating both sides and invoking the initial condition, we obtain $ln[(44.13 - v)/(44.13 + v)] = t/2.25$.

Solving for the velocity, $v(t) = 44.13(1 - e^{t/2.25})/(1 + e^{t/2.25})$. Integrating $v(t)$, the position is given by $x(t) = 99.29\, ln\left[e^{t/2.25}/(1 + e^{t/2.25})^2\right] + 186.2$. To estimate the *duration* of the downward motion, set $x(t_2) = 0$, resulting in $t_2 = 3.276\, sec$. Hence the *total time* that the ball remains in the air is $t_1 + t_2 = 5.192\ sec$.

(c).



$24(a)$. Measure the positive direction of motion *downward*. Based on Newton's 2*nd* law,
the equation of motion is given by

$$m\frac{dv}{dt} = \begin{cases} -0.75\, v + mg & ,\ 0 < t < 10 \\ -12\, v + mg & ,\ t > 10 \end{cases} .$$

Note that gravity acts in the *positive* direction, and the drag force is *resistive*. During the first ten seconds of fall, the initial value problem is $dv/dt = -v/7.5 + 32$, with initial velocity $v(0) = 0\, fps$. This differential equation is separable and linear, with solution $v(t) = 240(1 - e^{-t/7.5})$. Hence $v(10) = 176.7\, fps$.

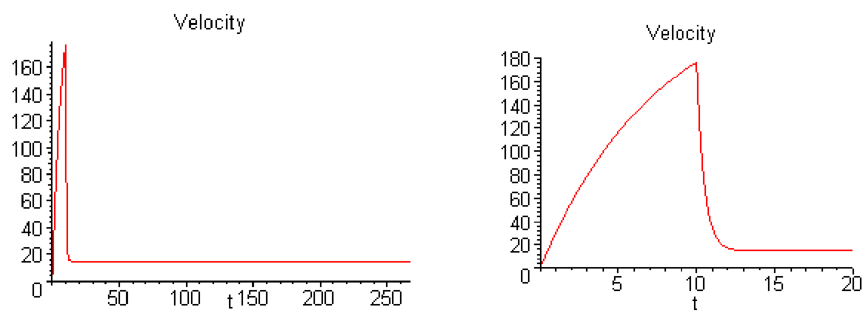
(b). Integrating the velocity, with $x(t) = 0$, the distance fallen is given by

$$x(t) = 240\, t + 1800\, e^{-t/7.5} - 1800 .$$

Hence $x(10) = 1074.5\, ft$.

(c). For computational purposes, reset time to $t = 0$. For the remainder of the motion, the initial value problem is $dv/dt = -32v/15 + 32$, with specified initial velocity $v(0) = 176.7\,fps$. The solution is given by $v(t) = 15 + 161.7\,e^{-32t/15}$. As $t \to \infty$, $v(t) \to v_L = 15\,fps$. Integrating the velocity, with $x(0) = 1074.5$, the distance fallen after the parachute is open is given by $x(t) = 15\,t - 75.8\,e^{-32t/15} + 1150.3$. To find the duration of the second part of the motion, estimate the root of the transcendental equation $15\,T - 75.8\,e^{-32T/15} + 1150.3 = 5000$. The result is $T = 256.6$ *sec*.

(d).



$25(a)$. Measure the positive direction of motion *upward*. The equation of motion is given by $mdv/dt = -k\,v - mg$. The initial value problem is $dv/dt = -kv/m - g$, with $v(0) = v_0$. The solution is $v(t) = -mg/k + (v_0 + mg/k)e^{-kt/m}$. Setting $v(t_m) = 0$, the maximum height is reached at time $t_m = (m/k)ln[(mg + k\,v_0)/mg]$. Integrating the velocity, the position of the body is

$$x(t) = -mg\,t/k + \left[\left(\frac{m}{k}\right)^2 g + \frac{m\,v_0}{k}\right](1 - e^{-kt/m}).$$

Hence the maximum height reached is

$$x_m = x(t_m) = \frac{m\,v_0}{k} - g\left(\frac{m}{k}\right)^2 ln\left[\frac{mg + k\,v_0}{mg}\right].$$

(b). Recall that for $\delta \ll 1$, $ln(1+\delta) = \delta - \frac{1}{2}\delta^2 + \frac{1}{3}\delta^3 - \frac{1}{4}\delta^4 + \ldots$

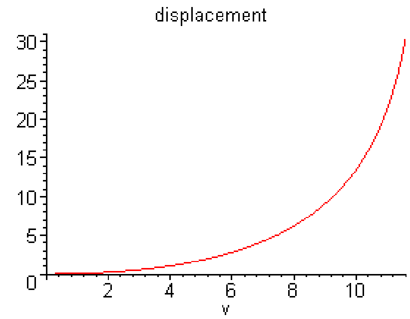
$26(b)$. $\lim_{k \to 0} \frac{-mg + (k\,v_0 + mg)e^{-kt/m}}{k} = \lim_{k \to 0} -\frac{t}{m}(k\,v_0 + mg)e^{-kt/m} = -gt$.

(c). $\lim_{m \to 0}\left[-\frac{mg}{k} + \left(\frac{mg}{k} + v_0\right)e^{-kt/m}\right] = 0$, since $\lim_{m \to 0} e^{-kt/m} = 0$.

$28(a)$. In terms of displacement, the differential equation is $mv\,dv/dx = -k\,v + mg$. This follows from the *chain rule*: $\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dt}$. The differential equation is separable, with

$$x(v) = -\frac{mv}{k} - \frac{m^2 g}{k^2} ln\left|\frac{mg - k\,v}{mg}\right|.$$

The inverse *exists*, since both x and v are monotone increasing. In terms of the given parameters, $x(v) = -1.25\,v - 15.31\,ln|0.0816\,v - 1|$.



(b). $x(10) = 13.45$ *meters*. The required value is $k = 0.24$.

(c). In part(a), set $v = 10$ *m/s* and $x = 10$ *meters*.

$29(a)$. Let x represent the height above the earth's surface. The equation of motion is given by $m\frac{dv}{dt} = -G\frac{Mm}{(R+x)^2}$, in which G is the universal gravitational constant. The symbols M and R are the *mass* and *radius* of the earth, respectively. By the chain rule,

$$mv\frac{dv}{dx} = -G\frac{Mm}{(R+x)^2}.$$

This equation is separable, with $v\,dv = -GM(R+x)^{-2}dx$. Integrating both sides, and
invoking the initial condition $v(0) = \sqrt{2gR}$, the solution is $v^2 = 2GM(R+x)^{-1} +$ $+ 2gR - 2GM/R$. From elementary physics, it follows that $g = GM/R^2$. Therefore $v(x) = \sqrt{2g}\left[R/\sqrt{R+x}\right]$. (*Note that* $g = 78,545$ *mi/hr*^{2}.)

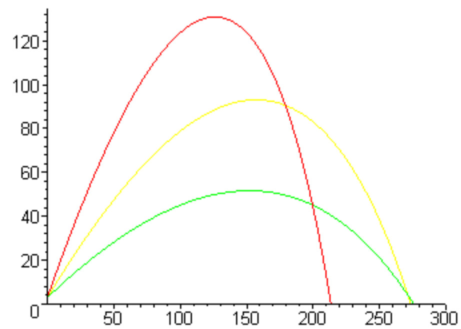
(b). We now consider $dx/dt = \sqrt{2g}\left[R/\sqrt{R+x}\right]$. This equation is also separable, with $\sqrt{R+x}\,dx = \sqrt{2g}\,R\,dt$. By definition of the variable x, the initial condition is $x(0) = 0$. Integrating both sides, we obtain $x(t) = \left[\frac{3}{2}\left(\sqrt{2g}\,R\,t + \frac{2}{3}R^{3/2}\right)\right]^{2/3} - R$. Setting the distance $x(T) + R = 240,000$, and solving for T, the duration of such a flight would be $T \approx 49$ *hours*.

$32(a)$. Both equations are linear and separable. The initial conditions are $v(0) = u\,cos$ A
and $w(0) = u\,sin\,A$. The two solutions are $v(t) = u\,cos\,A\,e^{-rt}$ and $w(t) = -g/r +$ $+ (u\,sin\,A + g/r)e^{-rt}$.

(b). Integrating the solutions in part(a), and invoking the initial conditions, the coordinates are $x(t) = \frac{u}{r} cos\, A(1 - e^{-rt})$ and

$$y(t) = -gt/r + \left(g + ur\, sin\, A + hr^2\right)/r^2 - \left(\frac{u}{r} sin\, A + g/r^2\right)e^{-rt}.$$

(c).



(d). Let T be the time that it takes the ball to go 350 *ft* horizontally. Then from above, $e^{-T/5} = (u\, cos\, A - 70)/u\, cos\, A$. At the same time, the height of the ball is given by $y(T) = -160\,T + 267 + 125usin\, A - (800 + 5u\, sin\, A)[(u\, cos\, A - 70)/u\, cos\, A]$. Hence A and u must satisfy the inequality

$$800ln\left[\frac{u\, cos\, A - 70}{u\, cos\, A}\right] + 267 + 125usin\, A - (800 + 5u\, sin\, A)[(u\, cos\, A - 70)/u\, cos\, A] \geq 10\,.$$

33(a). Solving equation (i), $y'(x) = [(k^2 - y)/y]^{1/2}$. The *positive* answer is chosen, since y is an *increasing* function of x .

(b). Let $y = k^2 sin^2 t$. Then $dy = 2k^2 sin\, t\, cos\, t\, dt$. Substituting into the equation in part(a), we find that

$$\frac{2k^2 sin\, t\, cos\, t\, dt}{dx} = \frac{cos\, t}{sin\, t}\,.$$

Hence $2k^2 sin^2 t\, dt = dx$.

(c). Letting $\theta = 2t$, we further obtain $k^2 sin^2 \frac{\theta}{2}\, d\theta = dx$. Integrating both sides of the equation and noting that $t = \theta = 0$ corresponds to the *origin*, we obtain the solutions $x(\theta) = k^2(\theta - sin\,\theta)/2$ and [from part(b)] $y(\theta) = k^2(1 - cos\,\theta)/2$.

(d). Note that $y/x = (1 - cos\,\theta)/(\theta - sin\,\theta)$. Setting $x = 1\,, y = 2$, the solution of the equation $(1 - cos\,\theta)/(\theta - sin\,\theta) = 2$ is $\theta \approx 1.401$. Substitution into either of the expressions yields $k \approx 2.193$.

Section 2.4

2. Considering the roots of the coefficient of the leading term, the ODE has unique solutions on intervals *not* containing 0 or 4 . Since $2 \in (0\,,4)$, the initial value problem has a unique solution on the interval $(0\,,4)$.

3. The function $tan\,t$ is discontinuous at *odd multiples* of $\frac{\pi}{2}$. Since $\frac{\pi}{2} < \pi < \frac{3\pi}{2}$, the initial value problem has a unique solution on the interval $\left(\frac{\pi}{2}\,,\,\frac{3\pi}{2}\right)$.

5. $p(t) = 2t/(4-t^2)$ and $g(t) = 3t^2/(4-t^2)$. These functions are discontinuous at $x = \pm 2$. The initial value problem has a unique solution on the interval $(\,-2\,,2)$.

6. The function $ln\,t$ is defined and continuous on the interval $(0\,,\infty)$. Therefore the initial value problem has a unique solution on the interval $(0\,,\infty)$.

7. The function $f(t\,,y)$ is continuous everywhere on the plane, *except* along the straight line $y = \,-2t/5$. The partial derivative $\partial f/\partial y = \,-7t/(2t+5y)^2$ has the *same* region of continuity.

9. The function $f(t\,,y)$ is discontinuous along the coordinate axes, and on the hyperbola $t^2 - y^2 = 1$. Furthermore,

$$\frac{\partial f}{\partial y} = \frac{\pm 1}{y(1-t^2+y^2)} - 2\frac{y\,ln|ty|}{(1-t^2+y^2)^2}$$

has the *same* points of discontinuity.

10. $f(t\,,y)$ is continuous everywhere on the plane. The partial derivative $\partial f/\partial y$ is also continuous everywhere.

12. The function $f(t\,,y)$ is discontinuous along the lines $t = \pm k\,\pi$ and $y = \,-1$. The partial derivative $\partial f/\partial y = cot(t)/(1+y)^2$ has the *same* region of continuity.

14. The equation is separable, with $dy/y^2 = 2t\,dt$. Integrating both sides, the solution is given by $y(t) = y_0/(1-y_0t^2)$. For $y_0 > 0$, solutions exist as long as $t^2 < 1/y_0$. For $y_0 \leq 0$, solutions are defined for *all* t .

15. The equation is separable, with $dy/y^3 = \,-\,dt$. Integrating both sides and invoking the initial condition, $y(t) = y_0/\sqrt{2y_0t+1}$. Solutions exist as long as $2y_0t + 1 > 0$, that is, $2y_0t > \,-1$. If $y_0 > 0$, solutions exist for $t > \,-1/2y_0$. If $y_0 = 0$, then the solution $y(t) = 0$ exists for all t . If $y_0 < 0$, solutions exist for $t < \,-1/2y_0$.

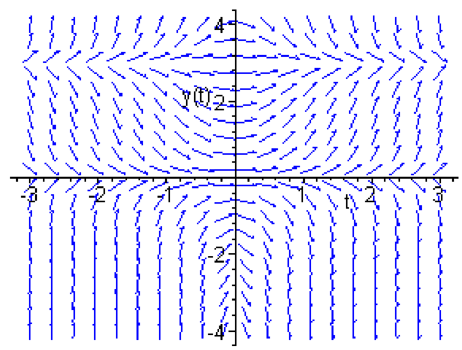
16. The function $f(t\,,y)$ is discontinuous along the straight lines $t = \,-1$ and $y = 0$. The partial derivative $\partial f/\partial y$ is discontinuous along the same lines. The equation is

separable, with $y\,dy = t^2\,dt/(1+t^3)$. Integrating and invoking the initial condition, the solution is $y(t) = \left[\frac{2}{3} ln|1+t^3| + y_0^2\right]^{1/2}$. Solutions exist as long as

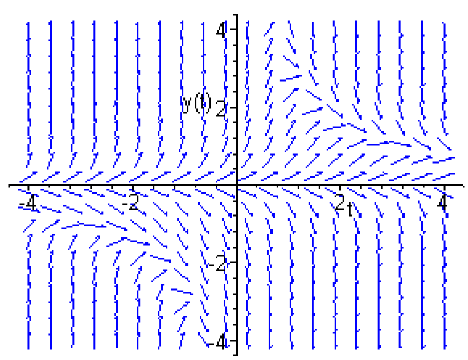
$$\frac{2}{3} ln|1+t^3| + y_0^2 \geq 0\,,$$

that is, $y_0^2 \geq -\frac{2}{3} ln|1+t^3|$. For all y_0 (it can be verified that $y_0 = 0$ yields a valid solution, even though Theorem 2.4.2 does not guarantee one) , solutions exists as long as $|1+t^3| \geq exp(-3y_0^2/2)$. From above, we must have $t > -1$. Hence the inequality may be written as $t^3 \geq exp(-3y_0^2/2) - 1$. It follows that the solutions are valid for $\left[exp(-3y_0^2/2) - 1\right]^{1/3} < t < \infty$.

17.

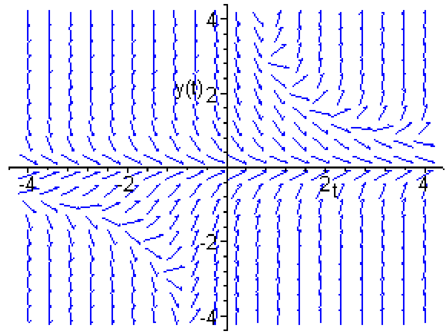


18.



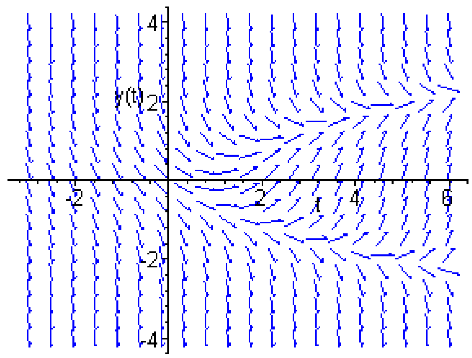
Based on the direction field, and the differential equation, for $y_0 < 0$, the slopes *eventually* become negative, and hence solutions tend to $-\infty$. For $y_0 < 0$, solutions increase without bound if $t_0 < 0$. Otherwise, the slopes *eventually* become negative, and solutions tend to *zero*. Furthermore, $y_0 = 0$ is an *equilibrium solution*. Note that slopes are *zero* along the curves $y = 0$ and $ty = 3$.

19.



For initial conditions (t_0, y_0) satisfying $ty < 3$, the respective solutions all tend to *zero*. Solutions with initial conditions *above or below* the hyperbola $ty = 3$ eventually tend to $\pm\infty$. Also, $y_0 = 0$ is an *equilibrium solution*.

20.



Solutions with $t_0 < 0$ all tend to $-\infty$. Solutions with initial conditions (t_0, y_0) to the *right* of the parabola $t = 1 + y^2$ asymptotically approach the parabola as $t \to \infty$. Integral curves with initial conditions *above* the parabola (and $y_0 > 0$) also approach the curve. The slopes for solutions with initial conditions *below* the parabola (and $y_0 < 0$) are all negative. These solutions tend to $-\infty$.

21. Define $y_c(t) = \frac{2}{3}(t-c)^{3/2}u(t-c)$, in which $u(t)$ is the Heaviside step function. Note that $y_c(c) = y_c(0) = 0$ and $y_c\left(c + (3/2)^{2/3}\right) = 1$.

(a). Let $c = 1 - (3/2)^{2/3}$.

(b). Let $c = 2 - (3/2)^{2/3}$.

(c). Observe that $y_0(2) = \frac{2}{3}(2)^{3/2}$, $y_c(t) < \frac{2}{3}(2)^{3/2}$ for $0 < c < 2$, and that $y_c(2) = 0$ for $c \geq 2$. So for any $c \geq 0$, $\pm y_c(2) \in [-2, 2]$.

26(a). Recalling Eq. (35) in Section 2.1,

$$y = \frac{1}{\mu(t)} \int \mu(s)g(s)\,ds + \frac{c}{\mu(t)}.$$

It is evident that $y_1(t) = \frac{1}{\mu(t)}$ and $y_2(t) = \frac{1}{\mu(t)} \int \mu(s)g(s)\,ds$.

(b). By definition, $\frac{1}{\mu(t)} = exp\big(-\int p(t)dt\big)$. Hence $y_1' = -p(t)\,\frac{1}{\mu(t)} = -p(t)y_1$. That is, $y_1' + p(t)y_1 = 0$.

(c). $y_2' = \left(-p(t)\,\frac{1}{\mu(t)}\right)\int_0^t \mu(s)g(s)\,ds + \left(\frac{1}{\mu(t)}\right)\mu(t)g(t) = -p(t)y_2 + g(t)$. That is, $y_2' + p(t)y_2 = g(t)$.

30. Since $n = 3$, set $v = y^{-2}$. It follows that $\frac{dv}{dt} = -2y^{-3}\frac{dy}{dt}$ and $\frac{dy}{dt} = -\frac{y^3}{2}\frac{dv}{dt}$. Substitution into the differential equation yields $-\frac{y^3}{2}\frac{dv}{dt} - \varepsilon y = -\sigma y^3$, which further results in $v' + 2\varepsilon v = 2\sigma$. The latter differential equation is linear, and can be written as $(e^{2\varepsilon t})' = 2\sigma$. The solution is given by $v(t) = 2\sigma t\, e^{-2\varepsilon t} + ce^{-2\varepsilon t}$. Converting back to the original dependent variable, $y = \pm v^{-1/2}$.

31. Since $n = 3$, set $v = y^{-2}$. It follows that $\frac{dv}{dt} = -2y^{-3}\frac{dy}{dt}$ and $\frac{dy}{dt} = -\frac{y^3}{2}\frac{dv}{dt}$. The differential equation is written as $-\frac{y^3}{2}\frac{dv}{dt} - (\Gamma cos\, t + T)y = \sigma y^3$, which upon further substitution is $v' + 2(\Gamma cos\, t + T)v = 2$. This ODE is linear, with integrating factor $\mu(t) = exp\big(2\int(\Gamma cos\, t + T)dt\big) = exp(-2\Gamma sin\, t + 2Tt)$. The solution is

$$v(t) = 2exp(2\Gamma sin\, t - 2Tt)\int_0^t exp(-2\Gamma sin\,\tau + 2T\tau)d\tau + c\,exp(-2\Gamma sin\, t + 2Tt).$$

Converting back to the original dependent variable, $y = \pm v^{-1/2}$.

33. The solution of the initial value problem $y_1' + 2y_1 = 0$, $y_1(0) = 1$ is $y_1(t) = e^{-2t}$. Therefore y$(1^-) = y_1(1) = e^{-2}$. On the interval $(1, \infty)$, the differential equation is $y_2' + y_2 = 0$, with $y_2(t) = ce^{-t}$. Therefore $y(1^+) = y_2(1) = ce^{-1}$. Equating the limits $y(1^-) = y(1^+)$, we require that $c = e^{-1}$. Hence the global solution of the initial value problem is

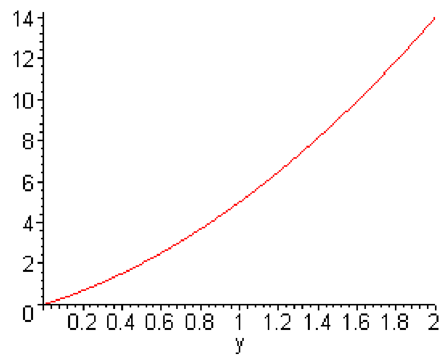
$$y(t) = \begin{cases} e^{-2t}, & 0 \le t \le 1 \\ e^{-1-t}, & t > 1 \end{cases}.$$

Note the discontinuity of the derivative

$$y(t) = \begin{cases} -2e^{-2t}, & 0 < t < 1 \\ -e^{-1-t}, & t > 1 \end{cases}.$$

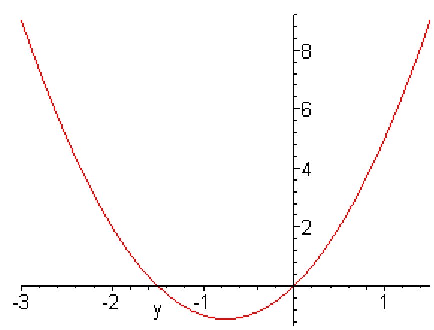
Section 2.5

1.



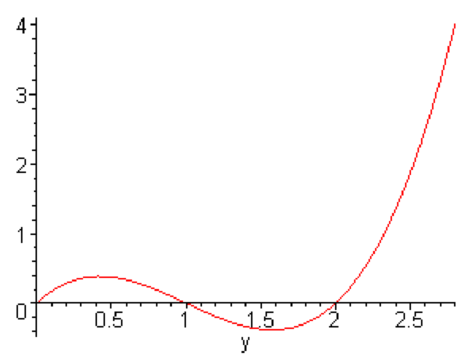
For $y_0 \geq 0$, the only equilibrium point is $y^* = 0$. $f'(0) = a > 0$, hence the equilibrium solution $\phi(t) = 0$ is *unstable*.

2.

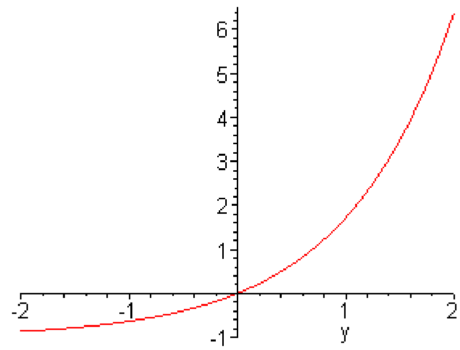


The equilibrium points are $y^* = -a/b$ and $y^* = 0$. $f'(-a/b) < 0$, therefore the equilibrium solution $\phi(t) = -a/b$ is *asymptotically stable*.

3.

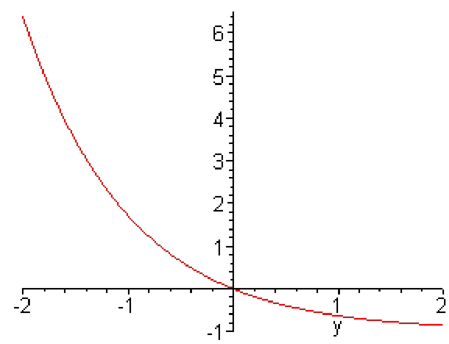


4.



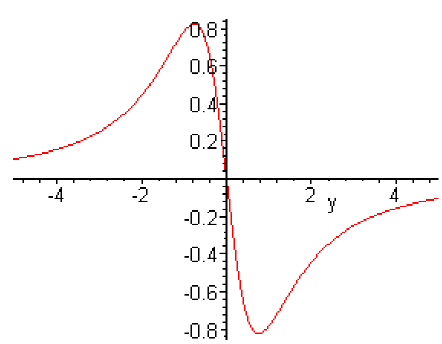
The only equilibrium point is $y^* = 0$. $f'(0) > 0$, hence the equilibrium solution $\phi(t) = 0$
is *unstable*.

5.

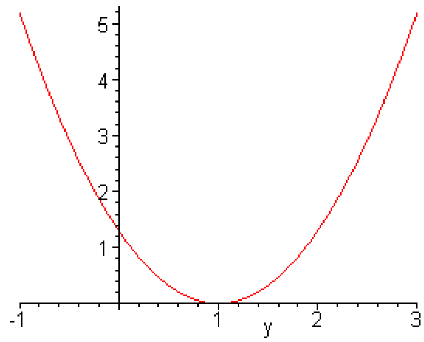


The only equilibrium point is $y^* = 0$. $f'(0) < 0$, hence the equilibrium solution $\phi(t) = 0$
is *asymptotically stable*.

6.



7(*b*).



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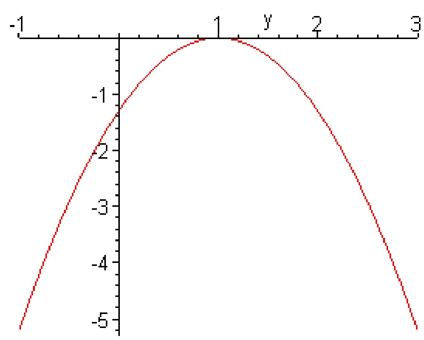
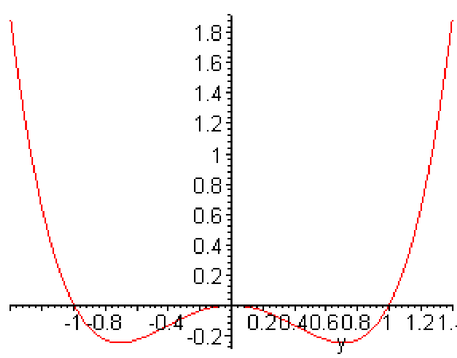
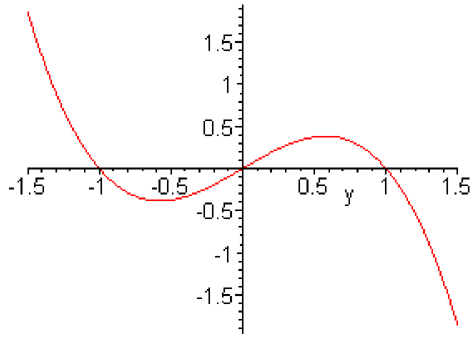


The only equilibrium point is $y^* = 1$. Note that $f'(1) = 0$, and that $y' < 0$ for $y \neq 1$. As long as $y_0 \neq 1$, the corresponding solution is *monotone decreasing*. Hence the equilibrium solution $\phi(t) = 1$ is *semistable*.

9.

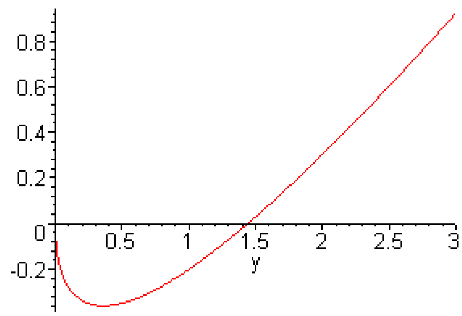


10.

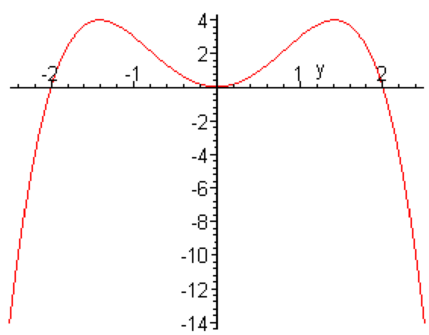


The equilibrium points are $y^* = 0, \pm 1$. $f'(y) = 1 - 3y^2$. The equilibrium solution $\phi(t) = 0$ is *unstable*, and the remaining two are *asymptotically stable*.

11.

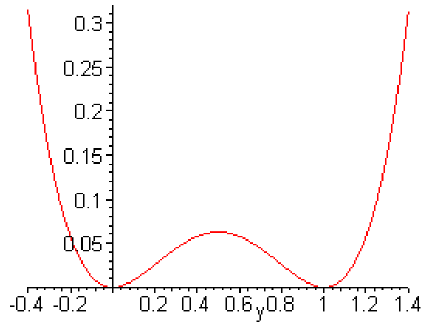


12.



The equilibrium points are $y^* = 0, \pm 2$. $f'(y) = 8y - 4y^3$. The equilibrium solutions $\phi(t) = -2$ and $\phi(t) = +2$ are *unstable* and *asymptotically stable*, respectively. The equilibrium solution $\phi(t) = 0$ is *semistable*.

13.



The equilibrium points are $y^* = 0$ and 1. $f'(y) = 2y - 6y^2 + 4y^3$. Both equilibrium solutions are *semistable*.

$15(a)$. Inverting the Solution (11), Eq. (13) shows t as a function of the population y and
the carrying capacity K. With $y_0 = K/3$,

$$t = -\frac{1}{r} ln \left| \frac{(1/3)[1-(y/K)]}{(y/K)[1-(1/3)]} \right| .$$

Setting $y = 2y_0$,

$$\tau = -\frac{1}{r} ln \left| \frac{(1/3)[1-(2/3)]}{(2/3)[1-(1/3)]} \right| .$$

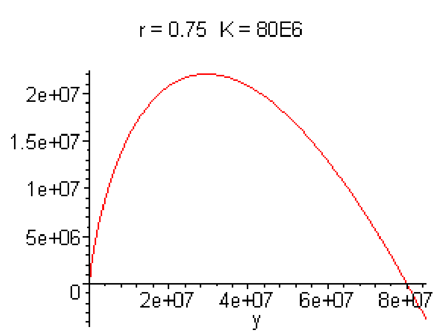
That is, $\tau = \frac{1}{r} ln\, 4$. If $r = 0.025$ *per year*, $\tau = 55.45$ *years*.

(b). In Eq. (13), set $y_0/K = \alpha$ and $y/K = \beta$. As a result, we obtain

$$T = -\frac{1}{r} ln \left| \frac{\alpha[1-\beta]}{\beta[1-\alpha]} \right| .$$

Given $\alpha = 0.1$, $\beta = 0.9$ and $r = 0.025$ *per year*, $\tau = 175.78$ *years*.

$16(a)$.



17. Consider the change of variable $u = ln(y/K)$. Differentiating both sides with respect
to t, $u' = y'/y$. Substitution into the Gompertz equation yields $u' = -ru$, with solution $u = u_0 e^{-rt}$. It follows that $ln(y/K) = ln(y_0/K)e^{-rt}$. That is,

$$\frac{y}{K} = exp\big[ln(y_0/K)e^{-rt}\big].$$

(a). Given $K = 80.5 \times 10^6$, $y_0/K = 0.25$ and $r = 0.71$ *per year*, $y(2) = 57.58 \times 10^6$.

(b). Solving for t,

$$t = -\frac{1}{r} ln\left[\frac{ln(y/K)}{ln(y_0/K)}\right].$$

Setting $y(\tau) = 0.75K$, the corresponding time is $\tau = 2.21$ *years*.

$19(a)$. The rate of *increase* of the volume is given by rate of *flow in* $-$ rate of *flow out*. That is, $dV/dt = k - \alpha a\sqrt{2gh}$. Since the cross section is *constant*, $dV/dt = Adh/dt$. Hence the governing equation is $dh/dt = \big(k - \alpha a\sqrt{2gh}\big)/A$.

(b). Setting $dh/dt = 0$, the equilibrium height is $h_e = \frac{1}{2g}\left(\frac{k}{\alpha a}\right)^2$. Furthermore, since $f'(h_e) < 0$, it follows that the equilibrium height is *asymptotically stable*.

(c). Based on the answer in part(b), the water level will intrinsically tend to approach h_e. Therefore the height of the tank must be *greater* than h_e; that is, $h_e < V/A$.

$22(a)$. The equilibrium points are at $y^* = 0$ and $y^* = 1$. Since $f'(y) = \alpha - 2\alpha y$, the equilibrium solution $\phi = 0$ is *unstable* and the equilibrium solution $\phi = 1$ is *asymptotically stable*.

(b). The ODE is separable, with $[y(1-y)]^{-1}dy = \alpha\, dt$. Integrating both sides and invoking the initial condition, the solution is

$$y(t) = \frac{y_0\, e^{\alpha t}}{1 - y_0 + y_0\, e^{\alpha t}}.$$

It is evident that (independent of y_0) $\lim_{t\to-\infty} y(t) = 0$ and $\lim_{t\to\infty} y(t) = 1$.

$23(a)$. $y(t) = y_0\, e^{-\beta t}$.

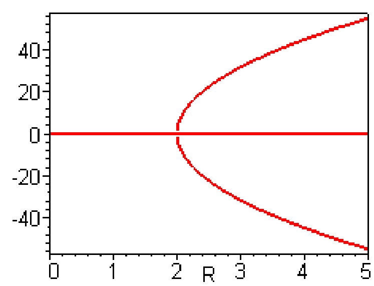
(b). From part(a), $dx/dt = \alpha\, x\, y_0\, e^{-\beta t}$. Separating variables, $dx/x = \alpha\, y_0\, e^{-\beta t}dt$. Integrating both sides, the solution is $x(t) = x_0\, exp\big[\alpha\, y_0/\beta\big(1 - e^{-\beta t}\big)\big]$.

(c). As $t \to \infty$, $y(t) \to 0$ and $x(t) \to x_0\, exp(\alpha\, y_0/\beta)$. Over a *long* period of time, the proportion of carriers *vanishes*. Therefore the proportion of the population that escapes the epidemic is the proportion of *susceptibles* left at that time, $x_0\, exp(\alpha\, y_0/\beta)$.

25(a). Note that $f(x) = x[(R - R_c) - a\,x^2]$, and $f'(x) = (R - R_c) - 3a\,x^2$. So if $(R - R_c) < 0$, the only equilibrium point is $x^* = 0$. $f'(0) < 0$, and hence the solution $\phi(t) = 0$ is *asymptotically stable*.

(b). If $(R - R_c) > 0$, there are *three* equilibrium points $x^* = 0\,, \pm\sqrt{(R - R_c)/a}$. Now $f'(0) > 0$, and $f'\left(\pm\sqrt{(R - R_c)/a}\right) < 0$. Hence the solution $\phi = 0$ is *unstable*, and the solutions $\phi = \pm\sqrt{(R - R_c)/a}$ are *asymptotically stable*.

(c).



Section 2.6

1. $M(x,y) = 2x + 3$ and $N(x,y) = 2y - 2$. Since $M_y = N_x = 0$, the equation is *exact*. Integrating M with respect to x, while holding y constant, yields $\psi(x,y) =$ $= x^2 + 3x + h(y)$. Now $\psi_y = h'(y)$, and equating with N results in the possible function $h(y) = y^2 - 2y$. Hence $\psi(x,y) = x^2 + 3x + y^2 - 2y$, and the solution is defined *implicitly* as $x^2 + 3x + y^2 - 2y = c$.

2. $M(x,y) = 2x + 4y$ and $N(x,y) = 2x - 2y$. Note that $M_y \neq N_x$, and hence the differential equation is *not* exact.

4. First divide both sides by $(2xy + 2)$. We now have $M(x,y) = y$ and $N(x,y) = x$. Since $M_y = N_x = 0$, the resulting equation is *exact*. Integrating M with respect to x, while holding y constant, results in $\psi(x,y) = xy + h(y)$. Differentiating with respect to y, $\psi_y = x + h'(y)$. Setting $\psi_y = N$, we find that $h'(y) = 0$, and hence $h(y) = 0$ is acceptable. Therefore the solution is defined *implicitly* as $xy = c$. Note that if $xy + 1 = 0$, the equation is trivially satisfied.

6. Write the given equation as $(ax - by)dx + (bx - cy)dy$. Now $M(x,y) = ax - by$ and $N(x,y) = bx - cy$. Since $M_y \neq N_x$, the differential equation is *not* exact.

8. $M(x,y) = e^x sin\, y + 3y$ and $N(x,y) = -3x + e^x sin\, y$. Note that $M_y \neq N_x$, and hence the differential equation is *not* exact.

10. $M(x,y) = y/x + 6x$ and $N(x,y) = ln\, x - 2$. Since $M_y = N_x = 1/x$, the given equation is *exact*. Integrating N with respect to y, while holding x constant, results in $\psi(x,y) = y\, ln\, x - 2y + h(x)$. Differentiating with respect to x, $\psi_x = y/x + h'(x)$. Setting $\psi_x = M$, we find that $h'(x) = 6x$, and hence $h(x) = 3x^2$. Therefore the solution
is defined *implicitly* as $3x^2 + y\, ln\, x - 2y = c$.

11. $M(x,y) = x\, ln\, y + xy$ and $N(x,y) = y\, ln\, x + xy$. Note that $M_y \neq N_x$, and hence the differential equation is *not* exact.

13. $M(x,y) = 2x - y$ and $N(x,y) = 2y - x$. Since $M_y = N_x = -1$, the equation is *exact*. Integrating M with respect to x, while holding y constant, yields $\psi(x,y) =$ $= x^2 - xy + h(y)$. Now $\psi_y = -x + h'(y)$. Equating ψ_y with N results in $h'(y) = 2y$, and hence $h(y) = y^2$. Thus $\psi(x,y) = x^2 - xy + y^2$, and the solution is given *implicitly* as $x^2 - xy + y^2 = c$. Invoking the initial condition $y(1) = 3$, the specific solution is $x^2 - xy + y^2 = 7$. The *explicit* form of the solution is $y(x) = \frac{1}{2}\left[x + \sqrt{28 - 3x^2}\,\right]$. Hence the solution is valid as long as $3x^2 \leq 28$.

16. $M(x,y) = y\,e^{2xy} + x$ and $N(x,y) = bx\,e^{2xy}$. Note that $M_y = e^{2xy} + 2xy\,e^{2xy}$, and $N_x = b\,e^{2xy} + 2bxy\,e^{2xy}$. The given equation is *exact*, as long as $b = 1$. Integrating

N with respect to y, while holding x constant, results in $\psi(x,y)=e^{2xy}/2+h(x)$. Now differentiating with respect to x, $\psi_x = y\,e^{2xy}+h'(x)$. Setting $\psi_x = M$, we find that $h'(x)=x$, and hence $h(x)=x^2/2$. Conclude that $\psi(x,y)=e^{2xy}/2+x^2/2$. Hence the solution is given *implicitly* as $e^{2xy}+x^2=c$.

17. Integrating $\psi_y = N$, while holding x constant, yields
$\psi(x,y)=\int N(x,y)dy+h(x)$.
Taking the partial derivative with respect to x, $\psi_x=\int\frac{\partial}{\partial x}N(x,y)dy+h'(x)$. Now set $\psi_x = M(x,y)$ and therefore $h'(x)=M(x,y)-\int\frac{\partial}{\partial x}N(x,y)dy$. Based on the fact that $M_y = N_x$, it follows that $\frac{\partial}{\partial y}[h'(x)]=0$. Hence the expression for $h'(x)$ can be integrated to obtain

$$h(x)=\int M(x,y)dx-\int\left[\int\frac{\partial}{\partial x}N(x,y)dy\right]dx\,.$$

18. Observe that $\frac{\partial}{\partial y}[M(x)]=\frac{\partial}{\partial x}[N(y)]=0$.

20. $M_y = y^{-1}cos\,y - y^{-2}sin\,y$ and $N_x = -2\,e^{-x}(cos\,x+sin\,x)/y$. Multiplying both sides by the integrating factor $\mu(x,y)=y\,e^x$, the given equation can be written as $(e^x sin\,y-2y\,sin\,x)dx+(e^x cos\,y+2cos\,x)dy=0$. Let $\overline{M}=\mu M$ and $\overline{N}=\mu N$. Observe that $\overline{M}_y=\overline{N}_x$, and hence the latter ODE is *exact*. Integrating $\overline{N}$ with respect to y, while holding x constant, results in $\psi(x,y)=e^x sin\,y+2y\,cos\,x+h(x)$. Now differentiating with respect to x, $\psi_x = e^x sin\,y-2y\,sin\,x+h'(x)$. Setting $\psi_x=\overline{M}$, we find that $h'(x)=0$, and hence $h(x)=0$ is feasible. Hence the solution of the given equation is defined *implicitly* by $e^x sin\,y+2y\,cos\,x=\beta$.

21. $M_y = 1$ and $N_x = 2$. Multiply both sides by the integrating factor $\mu(x,y)=y$ to obtain $y^2dx+(2xy-y^2e^y)dy=0$. Let $\overline{M}=yM$ and $\overline{N}=yN$. It is easy to see that $\overline{M}_y=\overline{N}_x$, and hence the latter ODE is *exact*. Integrating $\overline{M}$ with respect to x yields $\psi(x,y)=xy^2+h(y)$. Equating ψ_y with $\overline{N}$ results in $h'(y)=-y^2e^y$, and hence $h(y)=-e^y(y^2-2y+2)$. Thus $\psi(x,y)=xy^2-e^y(y^2-2y+2)$, and the solution is defined *implicitly* by $xy^2-e^y(y^2-2y+2)=c$.

24. The equation $\mu M+\mu Ny'=0$ has an integrating factor if $(\mu M)_y=(\mu N)_x$, that is, $\mu_y M-\mu_x N=\mu N_x-\mu M_y$. Suppose that $N_x - M_y = R\,(xM-yN)$, in which R is some function depending *only* on the quantity $z=xy$. It follows that the modified form of the equation is *exact*, if $\mu_y M-\mu_x N=\mu R\,(xM-yN)=R\,(\mu\,xM-\mu\,yN)$. This relation is satisfied if $\mu_y=(\mu\,x)R$ and $\mu_x=(\mu\,y)R$. Now consider $\mu=\mu(xy)$. Then the partial derivatives are $\mu_x=\mu' y$ and $\mu_y=\mu' x$. Note that $\mu'=d\mu/dz$. Thus μ must satisfy $\mu'(z)=R(z)$. The latter equation is *separable*, with $d\mu=R(z)dz$, and $\mu(z)=\int R(z)dz$. Therefore, given $R=R(xy)$, it is possible to determine $\mu=\mu(xy)$ which becomes an integrating factor of the differential equation.

28. The equation is not exact, since $N_x - M_y = 2y - 1$. However, $(N_x - M_y)/M =$ $= (2y-1)/y$ is a function of y alone. Hence there exists $\mu = \mu(y)$, which is a solution of the differential equation $\mu' = (2 - 1/y)\mu$. The latter equation is *separable*, with $d\mu/\mu = 2 - 1/y$. One solution is $\mu(y) = exp(2y - ln\, y) = e^{2y}/y$. Now rewrite the given ODE as $e^{2y}dx + (2x\,e^{2y} - 1/y)dy = 0$. This equation is *exact*, and it is easy to see that $\psi(x,y) = x\,e^{2y} - ln\, y$. Therefore the solution of the given equation is defined implicitly by $x\,e^{2y} - ln\, y = c$.

30. The given equation is not exact, since $N_x - M_y = 8x^3/y^3 + 6/y^2$. But note that $(N_x - M_y)/M = 2/y$ is a function of y alone, and hence there is an integrating factor $\mu = \mu(y)$. Solving the equation $\mu' = (2/y)\mu$, an integrating factor is $\mu(y) = y^2$. Now rewrite the differential equation as $(4x^3 + 3y)dx + (3x + 4y^3)dy = 0$. By inspection, $\psi(x,y) = x^4 + 3xy + y^4$, and the solution of the given equation is defined implicitly by $x^4 + 3xy + y^4 = c$.

32. Multiplying both sides of the ODE by $\mu = [xy(2x+y)]^{-1}$, the given equation is equivalent to $[(3x+y)/(2x^2+xy)]dx + [(x+y)/(2xy+y^2)]dy = 0$. Rewrite the differential equation as

$$\left[\frac{2}{x} + \frac{2}{2x+y}\right]dx + \left[\frac{1}{y} + \frac{1}{2x+y}\right]dy = 0\,.$$

It is easy to see that $M_y = N_x$. Integrating M with respect to x, while keeping y constant, results in $\psi(x,y) = 2ln|x| + ln|2x+y| + h(y)$. Now taking the partial derivative with respect to y , $\psi_y = (2x+y)^{-1} + h'(y)$. Setting $\psi_y = N$, we find that $h'(y) = 1/y$, and hence $h(y) = ln\,|y|$. Therefore

$$\psi(x,y) = 2ln|x| + ln|2x+y| + ln\,|y|\,,$$

and the solution of the given equation is defined implicitly by $2x^3y + x^2y^2 = c$.

Section 2.7

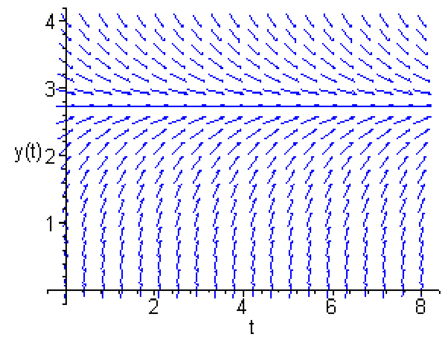
$2(a)$. The Euler formula is $y_{n+1} = y_n + h(2y_n - 1) = (1+2h)y_n - h$.

(d). The differential equation is *linear*, with solution $y(t) = (1 + e^{2t})/2$.

$4(a)$. The Euler formula is $y_{n+1} = (1-2h)y_n + 3h\,cos\,t_n$.

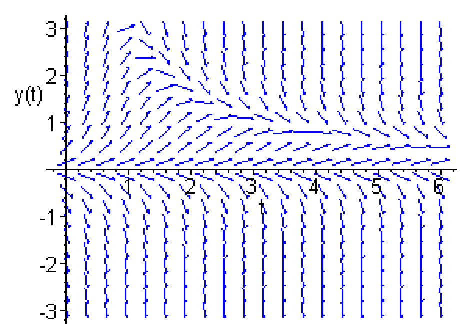
(d). The exact solution is $y(t) = (6cos\,t + 3sin\,t - 6\,e^{-2t})/5$.

5.



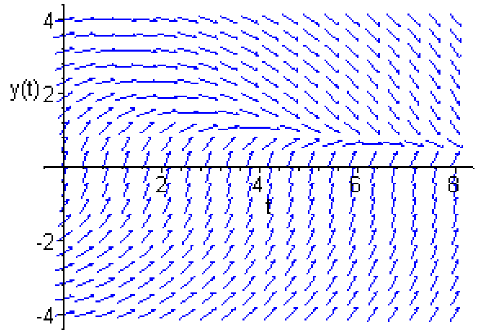
All solutions seem to converge to $\phi(t) = 25/9$.

6.



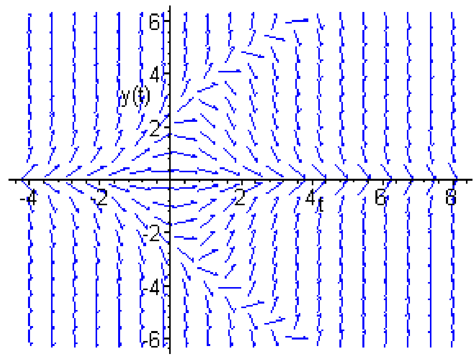
Solutions with *positive* initial conditions seem to converge to a specific function. On the other hand, solutions with *negative* coefficients decrease without bound. $\phi(t) = 0$ is an equilibrium solution.

7.



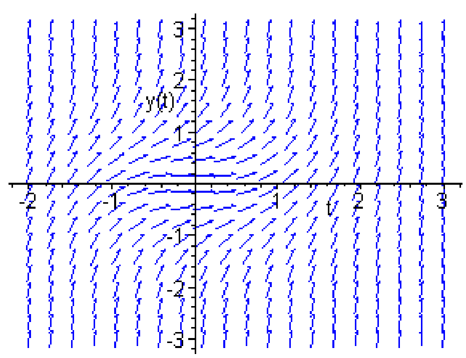
All solutions seem to converge to a specific function.

8.



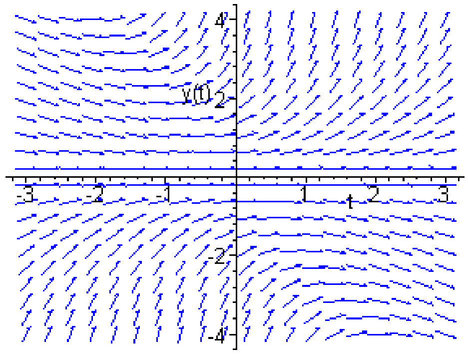
Solutions with initial conditions to the *'left'* of the curve $t = 0.1y^2$ seem to diverge. On the other hand, solutions to the *'right'* of the curve seem to converge to *zero*. Also, $\phi(t)$ is an equilibrium solution.

9.



All solutions seem to diverge.

10.



Solutions with *positive* initial conditions increase without bound. Solutions with *negative*
initial conditions decrease without bound. Note that $\phi(t) = 0$ is an equilibrium solution.

11. The Euler formula is $y_{n+1} = y_n - 3h\sqrt{y_n} + 5h$. The initial value is $y_0 = 2$.

12. The iteration formula is $y_{n+1} = (1+3h)y_n - h\,t_n\,y_n^2$. $(t_0, y_0) = (0\,, 0.5)$.

14. The iteration formula is $y_{n+1} = (1 - h\,t_n\,)y_n + h\,y_n^3\,/10$. $(t_0, y_0) = (0\,, 1)$.

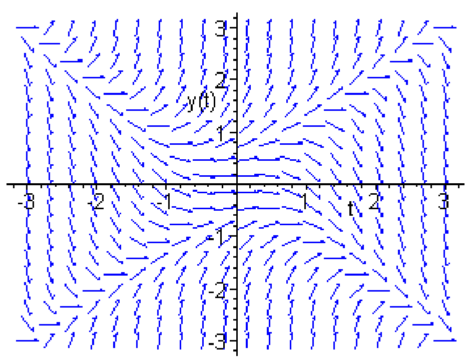
17. The Euler formula is

$$y_{n+1} = y_n + \frac{h(y_n^2 + 2t_n\,y_n)}{3 + t_n^2}.$$

The initial point is $(t_0, y_0) = (1\,, 2)$.

$18(a)$. See Problem 8.

$19(a)$.



(b). The iteration formula is $y_{n+1} = y_n + h\,y_n^2 - h\,t_n^2$. The critical value of α appears to be near $\alpha_0 \approx 0.6815$. For $y_0 > \alpha_0$, the iterations diverge.

$20(a)$. The ODE is *linear*, with general solution $y(t) = t + c\,e^t$. Invoking the specified initial condition, $y(t_0) = y_0$, we have $y_0 = t_0 + c\,e^{t_0}$. Hence $c = (y_0 - t_0)e^{-t_0}$. Thus the solution is given by $\phi(t) = (y_0 - t_0)e^{t-t_0} + t$.

(b). The Euler formula is $y_{n+1} = (1+h)y_n + h - h\,t_n$. Now set $k = n+1$.

(c). We have $y_1 = (1+h)y_0 + h - h\,t_0 = (1+h)y_0 + (t_1 - t_0) - h\,t_0$. Rearranging the terms, $y_1 = (1+h)(y_0 - t_0) + t_1$. Now suppose that $y_k = (1+h)^k(y_0 - t_0) + t_k$, for some $k \geq 1$. Then $y_{k+1} = (1+h)y_k + h - h\,t_k$. Substituting for y_k, we find that $y_{k+1} = (1+h)^{k+1}(y_0 - t_0) + (1+h)\,t_k + h - h\,t_k = (1+h)^{k+1}(y_0 - t_0) + t_k + h$. Noting that $t_{k+1} = t_k + k$, the result is verified.

(d). Substituting $h = (t - t_0)/n$, with $t_n = t$,

$$y_n = \left(1 + \frac{t - t_0}{n}\right)^n (y_0 - t_0) + t\,.$$

Taking the limit of both sides, as $n \to \infty$, and using the fact that $\lim_{n\to\infty}(1 + a/n)^n = e^a$, pointwise convergence is proved.

21. The exact solution is $\phi(t) = e^t$. The Euler formula is $y_{n+1} = (1+h)y_n$. It is easy to see that $y_n = (1+h)^n y_0 = (1+h)^n$. Given $t > 0$, set $h = t/n$. Taking the limit, we find that $\lim_{n\to\infty} y_n = \lim_{n\to\infty}(1 + t/n)^n = e^t$.

23. The exact solution is $\phi(t) = t/2 + e^{2t}$. The Euler formula is $y_{n+1} = (1+2h)y_n + $ $+\,h/2 - h\,t_n$. Since $y_0 = 1$, $y_1 = (1+2h) + h/2 = (1+2h) + t_1/2$. It is easy to show by mathematical induction, that $y_n = (1+2h)^n + t_n/2$. For $t > 0$, set $h = t/n$ and thus $t_n = t$. Taking the limit, we find that $\lim_{n\to\infty} y_n = \lim_{n\to\infty}[(1 + 2t/n)^n + t/2] =$ $= e^{2t} + t/2$. Hence pointwise convergence is proved.

Section 2.8

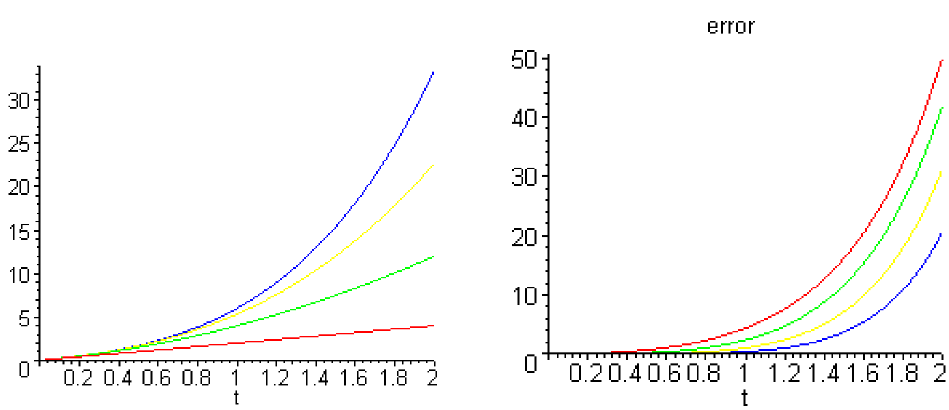
2. Let $z = y - 3$ and $\tau = t + 1$. It follows that $dz/d\tau = (dz/dt)(dt/d\tau) = dz/dt$. Furthermore, $dz/dt = dy/dt = 1 - y^3$. Hence $dz/d\tau = 1 - (z+3)^3$. The new initial condition is $z(\tau = 0) = 0$.

3. The approximating functions are defined recursively by $\phi_{n+1}(t) = \int_0^t 2[\phi_n(s) + 1]ds$. Setting $\phi_0(t) = 0$, $\phi_1(t) = 2t$. Continuing, $\phi_2(t) = 2t^2 + 2t$, $\phi_3(t) = \frac{4}{3}t^3 + 2t^2 + 2t$, $\phi_4(t) = \frac{2}{3}t^4 + \frac{4}{3}t^3 + 2t^2 + 2t$, $\cdots$. Given convergence, set

$$\begin{aligned}\phi(t) &= \phi_1(t) + \sum_{k=1}^{\infty}[\phi_{k+1}(t) - \phi_k(t)] \\ &= 2t + \sum_{k=2}^{\infty}\frac{a_k}{k\,!}t^k\,.\end{aligned}$$

Comparing coefficients, $a_3/3! = 4/3$, $a_4/4! = 2/3$, $\cdots$. It follows that $a_3 = 8$, $a_4 = 16$,
and so on. We find that in general, that $a_k = 2^k$. Hence

$$\begin{aligned}\phi(t) &= \sum_{k=1}^{\infty}\frac{2^k}{k\,!}t^k \\ &= e^{2t} - 1\,.\end{aligned}$$



5. The approximating functions are defined recursively by

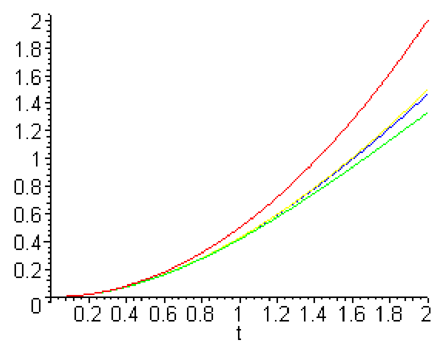
$$\phi_{n+1}(t) = \int_0^t [-\phi_n(s)/2 + s]ds\,.$$

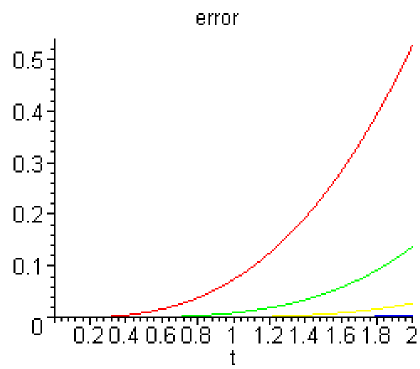
Setting $\phi_0(t) = 0$, $\phi_1(t) = t^2/2$. Continuing, $\phi_2(t) = t^2/2 - t^3/12$, $\phi_3(t) = t^2/2 - t^3/12 + t^4/96$, $\phi_4(t) = t^2/2 - t^3/12 + t^4/96 - t^5/960$, $\cdots$. Given convergence, set

$$\phi(t) = \phi_1(t) + \sum_{k=1}^{\infty}[\phi_{k+1}(t) - \phi_k(t)]$$
$$= t^2/2 + \sum_{k=3}^{\infty} \frac{a_k}{k\,!} t^k \,.$$

Comparing coefficients, $a_3/3! = -1/12\,,\, a_4/4! = 1/96\,,\, a_5/5! = -1/960\,, \cdots$. We find that $a_3 = -1/2\,,\, a_4 = 1/4,\, a_5 = -1/8\,, \cdots$. In general, $a_k = 2^{-k+1}$. Hence

$$\phi(t) = \sum_{k=2}^{\infty} \frac{2^{-k+2}}{k\,!}(-t)^k$$
$$= 4\,e^{-t/2} + 2t - 4\,.$$





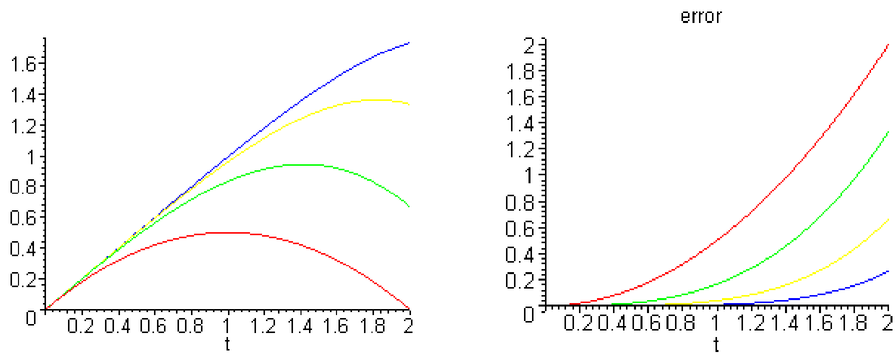
6. The approximating functions are defined recursively by

$$\phi_{n+1}(t) = \int_0^t [\phi_n(s) + 1 - s]ds\,.$$

Setting $\phi_0(t) = 0\,,\ \phi_1(t) = t - t^2/2\,,\, \phi_2(t) = t - t^3/6\,,\, \phi_3(t) = t - t^4/24\,,\, \phi_4(t) =$ $= t - t^5/120\,, \cdots$. Given convergence, set

$$\phi(t) = \phi_1(t) + \sum_{k=1}^{\infty}[\phi_{k+1}(t) - \phi_k(t)]$$
$$= t - t^2/2 + \left[t^2/2 - t^3/6\right] + \left[t^3/6 - t^4/24\right] + \cdots$$
$$= t + 0 + 0 + \cdots\,.$$

Note that the terms can be rearranged, as long as the series converges *uniformly*.



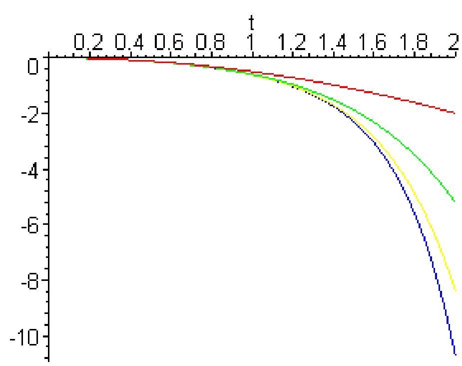
$8(a)$. The approximating functions are defined recursively by

$$\phi_{n+1}(t) = \int_0^t \left[s^2\phi_n(s) - s\right]ds\,.$$

Set $\phi_0(t) = 0$. The iterates are given by $\phi_1(t) = -t^2/2$, $\phi_2(t) = -t^2/2 - t^5/10$, $\phi_3(t) = -t^2/2 - t^5/10 - t^8/80$, $\phi_4(t) = -t^2/2 - t^5/10 - t^8/80 - t^{11}/880, \cdots$. Upon inspection, it becomes apparent that

$$\begin{aligned}\phi_n(t) &= -t^2\left[\frac{1}{2} + \frac{t^3}{2\cdot 5} + \frac{t^6}{2\cdot 5\cdot 8} + \cdots + \frac{(t^3)^{n-1}}{2\cdot 5\cdot 8\cdots[2+3(n-1)]}\right] \\ &= -t^2\sum_{k=1}^{n}\frac{(t^3)^{k-1}}{2\cdot 5\cdot 8\cdots[2+3(k-1)]}\,.\end{aligned}$$

(b).



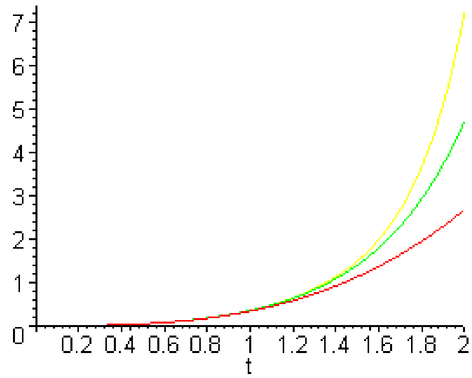
The iterates appear to be converging.

$9(a)$. The approximating functions are defined recursively by

$$\phi_{n+1}(t) = \int_0^t \left[s^2 + \phi_n^2(s)\right]ds\,.$$

Set $\phi_0(t) = 0$. The first three iterates are given by $\phi_1(t) = t^3/3$, $\phi_2(t) = t^3/3 + t^7/63$, $\phi_3(t) = t^3/3 + t^7/63 + 2t^{11}/2079 + t^{15}/59535$.

(b).



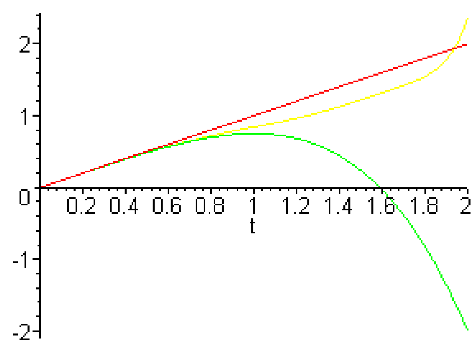
The iterates appear to be converging.

$10(a)$. The approximating functions are defined recursively by

$$\phi_{n+1}(t) = \int_0^t \left[1 - \phi_n^3(s)\right] ds\,.$$

Set $\phi_0(t) = 0$. The first three iterates are given by $\phi_1(t) = t$, $\phi_2(t) = t - t^4/4$, $\phi_3(t) = t - t^4/4 + 3t^7/28 - 3t^{10}/160 + t^{13}/833$.

(b).



The approximations appear to be diverging.

$12(a)$. The approximating functions are defined recursively by

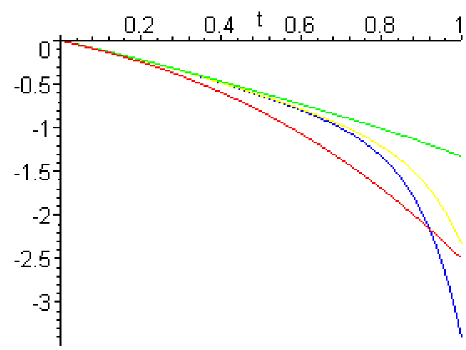
$$\phi_{n+1}(t) = \int_0^t \left[\frac{3s^2 + 4s + 2}{2(\phi_n(s) - 1)}\right] ds\,.$$

Note that $1/(2y-2) = -\frac{1}{2}\sum_{k=0}^{6} y^k + O(y^7)$. For computational purposes, replace the above iteration formula by

$$\phi_{n+1}(t) = -\frac{1}{2}\int_0^t \left[\left(3s^2+4s+2\right)\sum_{k=0}^{6} \phi_n^k(s) \right] ds\,.$$

Set $\phi_0(t) = 0$. The first four approximations are given by $\phi_1(t) = -t - t^2 - t^3/2$,
$\phi_2(t) = -t - t^2/2 + t^3/6 + t^4/4 - t^5/5 - t^6/24 + \cdots$,
$\phi_3(t) = -t - t^2/2 + t^4/12 - 3t^5/20 + 4t^6/45 + \cdots$,
$\phi_4(t) = -t - t^2/2 + t^4/8 - 7t^5/60 + t^6/15 + \cdots$

(b).



The approximations appear to be converging to the exact solution,

$$\phi(t) = 1 - \sqrt{1+2t+2t^2+t^3}\,.$$

13. Note that $\phi_n(0) = 0$ and $\phi_n(1) = 1\,,\forall\, n \geq 1$. Let $a \in (0\,,1)$. Then $\phi_n(a) = a^n$. Clearly, $\lim_{n\to\infty} a^n = 0$. Hence the assertion is true.

14(a). $\phi_n(0) = 0\,,\forall\, n \geq 1$. Let $a \in (0\,,1]$. Then $\phi_n(a) = 2na\,e^{-na^2} = 2na/e^{na^2}$. Using l'Hospital's rule, $\lim_{z\to\infty} 2az/e^{az^2} = \lim_{z\to\infty} 1/ze^{az^2} = 0$. Hence $\lim_{n\to\infty} \phi_n(a) = 0$.

(b). $\int_0^1 2nx\,e^{-nx^2}dx = -e^{-nx^2}\big|_0^1 = 1 - e^{-n}$. Therefore,

$$\lim_{n\to\infty}\int_0^1 \phi_n(x)dx \neq \int_0^1 \lim_{n\to\infty} \phi_n(x)dx\,.$$

15. Let t be fixed, such that $(t\,,y_1), (t\,,y_2) \in D$. Without loss of generality, assume that $y_1 < y_2$. Since f is differentiable with respect to y, the mean value theorem asserts that $\exists\, \xi \in (y_1\,,y_2)$ such that $f(t\,,y_1) - f(t\,,y_2) = f_y(t\,,\xi)(y_1 - y_2)$. Taking the absolute value of both sides, $|f(t\,,y_1) - f(t\,,y_2)| = |f_y(t\,,\xi)|\,|y_1 - y_2|$. Since, by assumption, $\partial f/\partial y$ is continuous in D, f_y attains a *maximum* on any closed and bounded subset of D
.

Hence $|f(t,y_1)-f(t,y_2)| \le K\,|y_1-y_2|$.

16. For a *sufficiently small* interval of t, $\phi_{n-1}(t)\,,\phi_n(t)\in D$. Since f satisfies a Lipschitz condition, $|f(t,\phi_n(t))-f(t,\phi_{n-1}(t))| \le K\,|\phi_n(t)-\phi_{n-1}(t)|$. Here $K = max\,|f_y|$.

17(a). $\phi_1(t)=\int_0^t f(s,0)ds$. Hence $|\phi_1(t)| \le \int_0^{|t|}|f(s,0)|ds \le \int_0^{|t|}Mds = M|t|$, in which M is the maximum value of $|f(t,y)|$ on D.

(b). By definition, $\phi_2(t)-\phi_1(t)=\int_0^t[f(s,\phi_1(s))-f(s,0)]ds$. Taking the absolute value of both sides, $|\phi_2(t)-\phi_1(t)| \le \int_0^{|t|}|[f(s,\phi_1(s))-f(s,0)]|ds$. Based on the results in Problems 16 and 17, $|\phi_2(t)-\phi_1(t)| \le \int_0^{|t|}K|\phi_1(s)-0|ds \le KM\int_0^{|t|}|s|ds$. Evaluating the last integral, we obtain $|\phi_2(t)-\phi_1(t)| \le MK|t|^2/2$.

(c). Suppose that

$$|\phi_i(t)-\phi_{i-1}(t)| \le \frac{MK^{i-1}|t|^i}{i\,!}$$

for some $i \ge 1$. By definition, $\phi_{i+1}(t)-\phi_i(t)=\int_0^t[f(t,\phi_i(s))-f(s,\phi_{i-1}(s))]ds$. It follows that

$$\begin{aligned}|\phi_{i+1}(t)-\phi_i(t)| &\le \int_0^{|t|}|f(s,\phi_i(s))-f(s,\phi_{i-1}(s))|ds\\ &\le \int_0^{|t|}K|\phi_i(s)-\phi_{i-1}(s)|ds\\ &\le \int_0^{|t|}K\frac{MK^{i-1}|s|^i}{i\,!}ds\\ &= \frac{MK^i|t|^{i+1}}{(i+1)\,!} \le \frac{MK^ih^{i+1}}{(i+1)\,!}\,.\end{aligned}$$

Hence, by mathematical induction, the assertion is true.

18(a). Use the triangle inequality, $|a+b| \le |a|+|b|$.

(b). For $|t| \le h$, $|\phi_1(t)| \le Mh$, and $|\phi_n(t)-\phi_{n-1}(t)| \le MK^{n-1}h^n/(n\,!)$. Hence

$$\begin{aligned}|\phi_n(t)| &\le M\sum_{i=1}^{n}\frac{K^{i-1}h^i}{i\,!}\\ &= \frac{M}{K}\sum_{i=1}^{n}\frac{(Kh)^i}{i\,!}\,.\end{aligned}$$

(c). The sequence of partial sums in (b) converges to $\frac{M}{K}\left(e^{Kh}-1\right)$. By the *comparison test*, the sums in (a) also converge. Furthermore, the sequence $|\phi_n(t)|$ is *bounded*, and hence has a convergent subsequence. Finally, since individual terms of the series must tend to zero, $|\phi_n(t)-\phi_{n-1}(t)| \to 0$, and it follows that the sequence $|\phi_n(t)|$ is convergent.

19(a). Let $\phi(t)=\int_0^t f(s\,,\phi(s))ds$ and $\psi(t)=\int_0^t f(s\,,\psi(s))ds$. Then by *linearity* of the integral, $\phi(t)-\psi(t)=\int_0^t [f(s\,,\phi(s))-f(s\,,\psi(s))]ds$.

(b). It follows that $|\phi(t)-\psi(t)| \leq \int_0^t |f(s\,,\phi(s))-f(s\,,\psi(s))|ds$.

(c). We know that f satisfies a Lipschitz condition,

$$|f(t\,,y_1)-f(t\,,y_2)| \leq K\,|y_1-y_2|\,,$$

based on $|\partial f/\partial y| \leq K$ in D. Therefore,

$$\begin{aligned} |\phi(t)-\psi(t)| &\leq \int_0^t |f(s\,,\phi(s))-f(s\,,\psi(s))|ds \\ &\leq \int_0^t K|\phi(s)-\psi(s)|ds\,. \end{aligned}$$

Section 2.9

1. Writing the equation for each $n \geq 0$, $y_1 = -0.9\, y_0$, $y_2 = -0.9\, y_1$, $y_3 = -0.9\, y_2$ and so on, it is apparent that $y_n = (-0.9)^n\, y_0$. The terms constitute an *alternating series*, which converge to *zero*, regardless of y_0.

3. Write the equation for each $n \geq 0$, $y_1 = \sqrt{3}\, y_0$, $y_2 = \sqrt{4/2}\, y_1$, $y_3 = \sqrt{5/3}\, y_2, \cdots$. Upon substitution, we find that $y_2 = \sqrt{(4 \cdot 3)/2}\, y_1$, $y_3 = \sqrt{(5 \cdot 4 \cdot 3)/(3 \cdot 2)}\, y_0, \cdots$. It can be proved by mathematical induction, that

$$\begin{aligned} y_n &= \frac{1}{\sqrt{2}} \sqrt{\frac{(n+2)!}{n!}}\, y_0 \\ &= \frac{1}{\sqrt{2}} \sqrt{(n+1)(n+2)}\, y_0 . \end{aligned}$$

This sequence is *divergent*, except for $y_0 = 0$.

4. Writing the equation for each $n \geq 0$, $y_1 = -y_0$, $y_2 = y_1$, $y_3 = -y_2$, $y_4 = y_3$, and so on, it can be shown that

$$y_n = \begin{cases} y_0 & , \text{ for } n = 4k \text{ or } n = 4k-1 \\ -y_0 & , \text{ for } n = 4k-2 \text{ or } n = 4k-3 \end{cases}$$

The sequence is convergent *only* for $y_0 = 0$.

6. Writing the equation for each $n \geq 0$,

$$\begin{aligned} y_1 &= 0.5\, y_0 + 6 \\ y_2 &= 0.5\, y_1 + 6 = 0.5(0.5\, y_0 + 6) + 6 = (0.5)^2 y_0 + 6 + (0.5)6 \\ y_3 &= 0.5\, y_2 + 6 = 0.5(0.5\, y_1 + 6) + 6 = (0.5)^3 y_0 + 6[1 + (0.5) + (0.5)^2] \\ &\vdots \\ y_n &= (0.5)^n y_0 + 12[1 - (0.5)^n\,] \end{aligned}$$

which can be verified by mathematical induction. The sequence is convergent for all y_0, and in fact $y_n \rightarrow 12$.

7. Let y_n be the balance at the end of the *n-th* day. Then $y_{n+1} = (1 + r/356)\, y_n$. The solution of this difference equation is $y_n = (1 + r/365)^n\, y_0$, in which y_0 is the initial balance. At the end of *one year*, the balance is $y_{365} = (1 + r/365)^{365}\, y_0$. Given that $r = .07$, $y_{365} = (1 + r/365)^{365}\, y_0 = 1.0725\, y_0$. Hence the effective annual yield is $(1.0725\, y_0 - y_0)/y_0 = 7.25\,\%$.

8. Let y_n be the balance at the end of the *n-th* month. Then $y_{n+1} = (1 + r/12)\, y_n + 25$. As in the previous solutions, we have

$$y_n = \rho^n \left[y_0 - \frac{25}{1-\rho} \right] + \frac{25}{1-\rho} \,,$$

in which $\rho = (1 + r/12)$. Here r is the annual interest rate, given as $8\,\%$. Therefore $y_{36} = (1.0066)^{36} \left[1000 + \frac{(12)25}{r} \right] - \frac{(12)25}{r} = 2,283.63$ *dollars*.

9. Let y_n be the balance due at the end of the n-*th* month. The appropriate difference equation is $y_{n+1} = (1 + r/12)\, y_n - P$. Here r is the annual interest rate and P is the monthly payment. The solution, in terms of the amount borrowed, is given by

$$y_n = \rho^n \left[y_0 + \frac{P}{1-\rho} \right] - \frac{P}{1-\rho} \,,$$

in which $\rho = (1 + r/12)$ and $y_0 = 8,000$. To figure out the monthly payment, P, we require that $y_{36} = 0$. That is,

$$\rho^{36} \left[y_0 + \frac{P}{1-\rho} \right] = \frac{P}{1-\rho} \,.$$

After the specified amounts are substituted, we find the $P = \$\,258.14$.

11. Let y_n be the balance due at the end of the n-*th* month. The appropriate difference equation is $y_{n+1} = (1 + r/12)\, y_n - P$, in which $r = .09$ and P is the monthly payment. The initial value of the mortgage is $y_0 = 100,000$ *dollars*. Then the balance due at the end of the n-*th* month is

$$y_n = \rho^n \left[y_0 + \frac{P}{1-\rho} \right] - \frac{P}{1-\rho} \,.$$

where $\rho = (1 + r/12)$. In terms of the specified values,

$$y_n = (0.0075)^n \left[10^5 - \frac{12P}{r} \right] + \frac{12P}{r} \,.$$

Setting $n = 30(12) = 360$, and $y_{360} = 0$, we find that $P = 804.62$ *dollars*. For the monthly payment corresponding to a 20 year mortgage, set $n = 240$ and $y_{240} = 0$.

12. Let y_n be the balance due at the end of the n-*th* month, with y_0 the initial value of the mortgage. The appropriate difference equation is $y_{n+1} = (1 + r/12)\, y_n - P$, in which $r = 0.1$ and $P = 900$ *dollars* is the *maximum* monthly payment. Given that the life of the mortgage is 20 *years*, we require that $y_{240} = 0$. The balance due at the end of the n-*th* month is

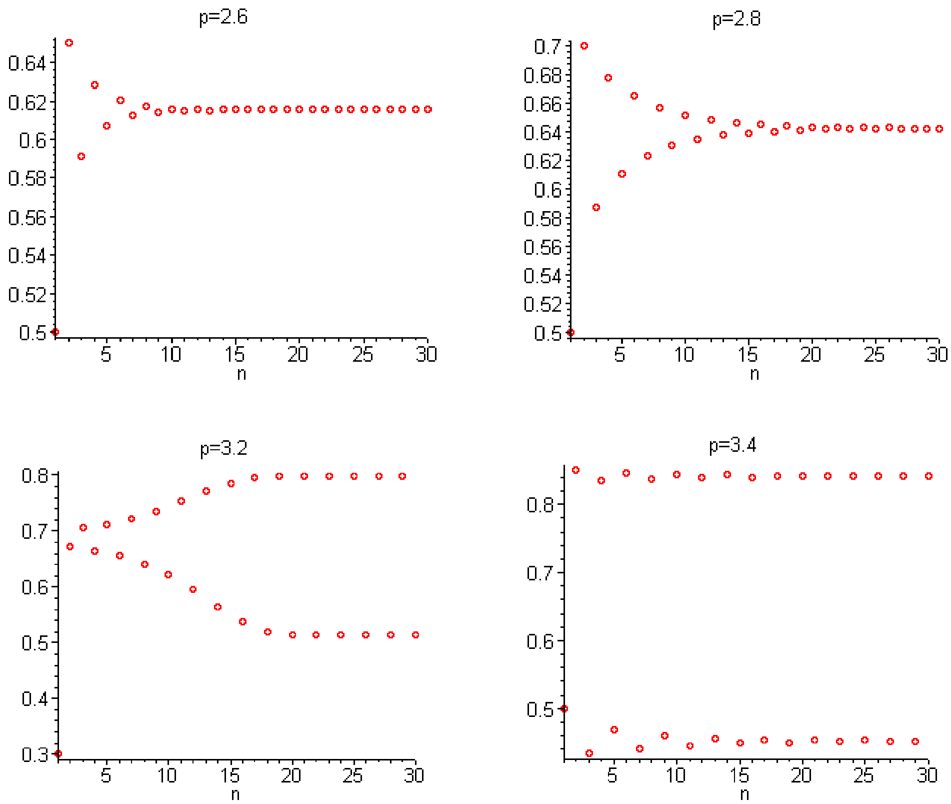
$$y_n = \rho^n \left[y_0 + \frac{P}{1-\rho} \right] - \frac{P}{1-\rho} \,.$$

In terms of the specified values for the parameters, the solution of

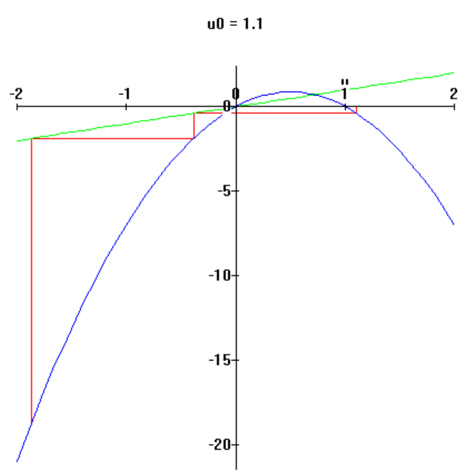
$$(.00833)^{240}\left[y_0 - \frac{12(1000)}{0.1}\right] = -\frac{12(1000)}{0.1}$$

is $y_0 = 103,624.62$ *dollars*.

15.



16. For example, take $\rho = 3.5$ and $u_0 = 1.1$:

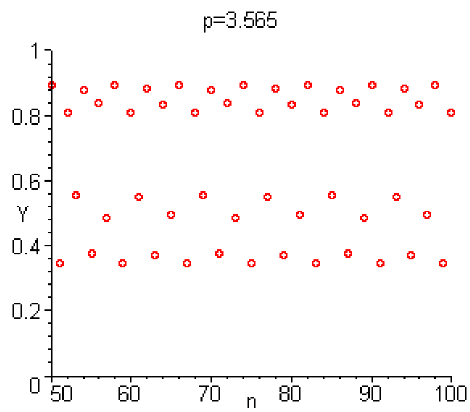


$19(a)$. $\delta_2 = (\rho_2 - \rho_1)/(\rho_3 - \rho_2) = (3.449 - 3)/(3.544 - 3.449) = 4.7263$.

(b). $\%\ diff = \frac{|\delta - \delta_2|}{\delta} \times 100 = \frac{|4.6692 - 4.7363|}{4.6692} \times 100 \approx 1.22\ \%$.

(c). Assuming $(\rho_3 - \rho_2)/(\rho_4 - \rho_3) = \delta$, $\rho_4 \approx 3.5643$

(d). A period 16 solutions appears near $\rho \approx 3.565$.



(e). Note that $(\rho_{n+1} - \rho_n) = \delta_n^{-1}(\rho_n - \rho_{n-1})$. With the assumption that $\delta_n = \delta$, we have $(\rho_{n+1} - \rho_n) = \delta^{-1}(\rho_n - \rho_{n-1})$, which is of the form $y_{n+1} = \alpha\, y_n$, $n \geq 3$. It follows that $(\rho_k - \rho_{k-1}) = \delta^{3-k}(\rho_3 - \rho_2)$ for $k \geq 4$. Then

$$\begin{aligned}\rho_n &= \rho_1 + (\rho_2 - \rho_1) + (\rho_3 - \rho_2) + (\rho_4 - \rho_3) + \cdots + (\rho_n - \rho_{n-1}) \\ &= \rho_1 + (\rho_2 - \rho_1) + (\rho_3 - \rho_2)[1 + \delta^{-1} + \delta^{-2} + \cdots + \delta^{3-n}] \\ &= \rho_1 + (\rho_2 - \rho_1) + (\rho_3 - \rho_2)\left[\frac{1 - \delta^{4-n}}{1 - \delta^{-1}}\right].\end{aligned}$$

Hence $\lim_{n\to\infty} \rho_n = \rho_2 + (\rho_3 - \rho_2)\left[\frac{\delta}{\delta - 1}\right]$. Substitution of the appropriate values yields

$$\lim_{n\to\infty} \rho_n = 3.5699$$

Miscellaneous Problems

1. Linear $[\, y = c/x^2 + x^3/5\,]$.
2. Homogeneous $[arctan(y/x) - ln\sqrt{x^2+y^2} = c\,]$.
3. Exact $[\, x^2 + xy - 3y - y^3 = 0\,]$.
4. Linear in $x(y)$ $[\, x = c\, e^y + y\, e^y\,]$.
5. Exact $[\, x^2y + xy^2 + x = c\,]$.
6. Linear $[\, y = x^{-1}(1 - e^{1-x})\,]$.
7. Let $u = x^2$ $[\, x^2 + y^2 + 1 = c\, e^{y^2}\,]$.
8. Linear $[\, y = (4 + cos\, 2 - cos\, x)/x^2\,]$.
9. Exact $[\, x^2y + x + y^2 = c\,]$.
10. $\mu = \mu(x)$ $[\, y^2/x^3 + y/x^2 = c\,]$.
11. Exact $[\, x^3/3 + xy + e^y = c\,]$.
12. Linear $[\, y = c\, e^{-x} + e^{-x} ln(1 + e^x)\,]$.
13. Homogeneous $[\, 2\sqrt{y/x} - ln\,|x| = c\,]$.
14. Exact/Homogeneous $[\, x^2 + 2xy + 2y^2 = 34\,]$.
15. Separable $[\, y = c/cosh^2(x/2)\,]$.
16. Homogeneous $[\, \left(2/\sqrt{3}\right) arctan\left[(2y - x)/\sqrt{3}\, x\right] - ln\,|x| = c\,]$.
17. Linear $[\, y = c\, e^{3x} - e^{2x}\,]$.
18. Linear/Homogeneous $[\, y = c\, x^{-2} - x\,]$.
19. $\mu = \mu(x)$ $[\, 3y - 2xy^3 - 10x = 0\,]$.
20. Separable $[\, e^x + e^{-y} = c\,]$.
21. Homogeneous $[\, e^{-y/x} + ln\,|x| = c\,]$.
22. Separable $[\, y^3 + 3y - x^3 + 3x = 2\,]$.
23. Bernoulli $[\, 1/y = -x\int x^{-2}e^{2x}\, dx + cx\,]$.
24. Separable $[\, sin^2x\, sin\, y = c\,]$.
25. Exact $[\, x^2/y + arctan(y/x) = c\,]$.
26. $\mu = \mu(x)$ $[\, x^2 + 2x^2y - y^2 = c\,]$.
27. $\mu = \mu(x)$ $[\, sin\, x\, cos\, 2y - \frac{1}{2}\, sin^2x = c\,]$.
28. Exact $[\, 2xy + xy^3 - x^3 = c\,]$.
29. Homogeneous $[\, arcsin(y/x) - ln\,|x| = c\,]$.
30. Linear in $x(y)$ $[\, xy^2 - ln\,|y| = 0\,]$.
31. Separable $[\, x + ln\,|x| + x^{-1} + y - 2\, ln\,|y| = c\,]$.
32. $\mu = \mu(y)$ $[\, x^3y^2 + xy^3 = -4\,]$.