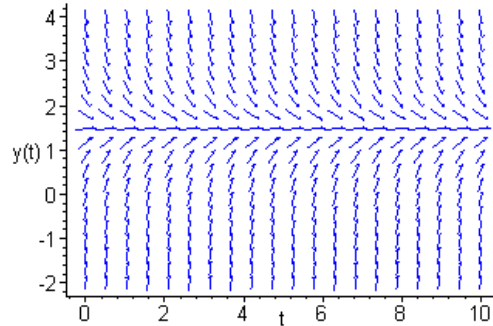


## Chapter One

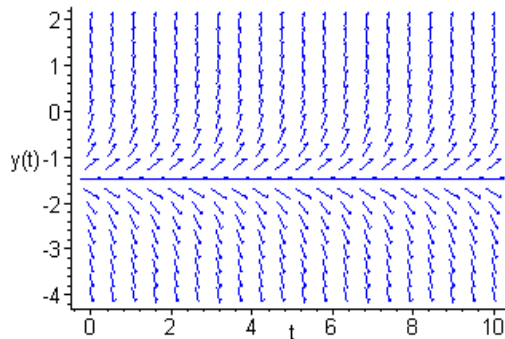
### Section 1.1

1.



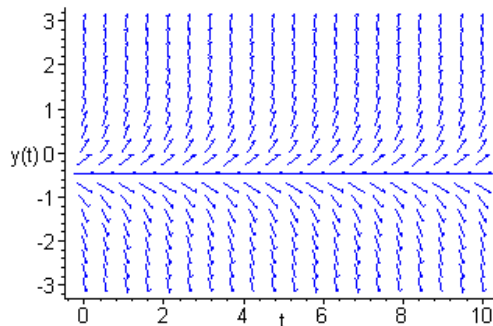
For  $y > 1.5$ , the slopes are *negative*, and hence the solutions decrease. For  $y < 1.5$ , the slopes are *positive*, and hence the solutions increase. The equilibrium solution appears to be  $y(t) = 1.5$ , to which all other solutions converge.

3.



For  $y > -1.5$ , the slopes are *positive*, and hence the solutions increase. For  $y < -1.5$ , the slopes are *negative*, and hence the solutions decrease. All solutions appear to diverge away from the equilibrium solution  $y(t) = -1.5$ .

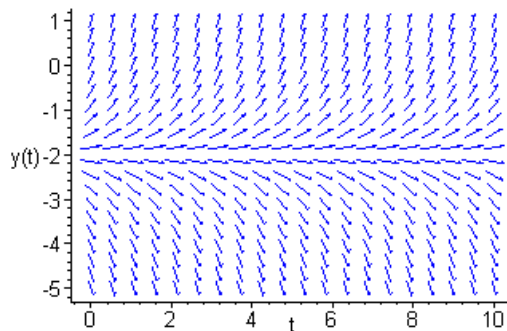
5.



For  $y > -1/2$ , the slopes are *positive*, and hence the solutions increase. For  $y < -1/2$ , the slopes are *negative*, and hence the solutions decrease. All solutions diverge away from

the equilibrium solution  $y(t) = -1/2$ .

6.



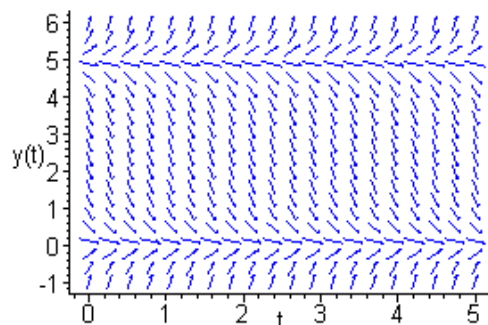
For  $y > -2$ , the slopes are *positive*, and hence the solutions increase. For  $y < -2$ , the slopes are *negative*, and hence the solutions decrease. All solutions diverge away from the equilibrium solution  $y(t) = -2$ .

8. For *all* solutions to approach the equilibrium solution  $y(t) = 2/3$ , we must have  $y' < 0$  for  $y > 2/3$ , and  $y' > 0$  for  $y < 2/3$ . The required rates are satisfied by the differential equation  $y' = 2 - 3y$ .

9. For solutions *other* than  $y(t) = 2$  to diverge from  $y = 2$ ,  $y(t)$  must be an *increasing* function for  $y > 2$ , and a *decreasing* function for  $y < 2$ . The simplest differential equation whose solutions satisfy these criteria is  $y' = y - 2$ .

10. For solutions *other* than  $y(t) = 1/3$  to diverge from  $y = 1/3$ , we must have  $y' < 0$  for  $y < 1/3$ , and  $y' > 0$  for  $y > 1/3$ . The required rates are satisfied by the differential equation  $y' = 3y - 1$ .

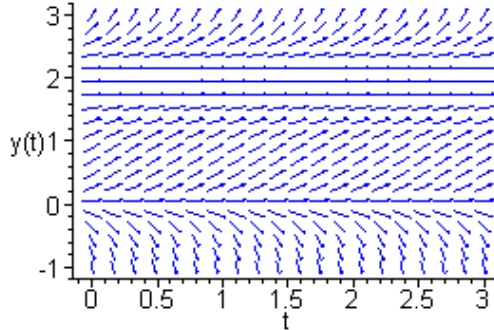
12.



Note that  $y' = 0$  for  $y = 0$  and  $y = 5$ . The two equilibrium solutions are  $y(t) = 0$  and  $y(t) = 5$ . Based on the direction field,  $y' > 0$  for  $y > 5$ ; thus solutions with initial values *greater* than 5 diverge from the solution  $y(t) = 5$ . For  $0 < y < 5$ , the slopes are *negative*, and hence solutions with initial values *between* 0 and 5 all decrease toward the

solution  $y(t) = 0$ . For  $y < 0$ , the slopes are all *positive*; thus solutions with initial values less than 0 approach the solution  $y(t) = 0$ .

14.



Observe that  $y' = 0$  for  $y = 0$  and  $y = 2$ . The two equilibrium solutions are  $y(t) = 0$  and  $y(t) = 2$ . Based on the direction field,  $y' > 0$  for  $y > 2$ ; thus solutions with initial values *greater* than 2 diverge from  $y(t) = 2$ . For  $0 < y < 2$ , the slopes are also *positive*, and hence solutions with initial values *between* 0 and 2 all increase toward the solution

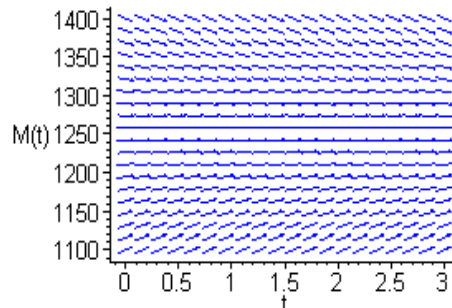
$y(t) = 2$ . For  $y < 0$ , the slopes are all *negative*; thus solutions with initial values less than 0 diverge from the solution  $y(t) = 0$ .

16. (a) Let  $M(t)$  be the total amount of the drug (*in milligrams*) in the patient's body at any given time  $t$  (*hrs*). The drug is administered into the body at a *constant* rate of 500 *mg/hr*.

The rate at which the drug *leaves* the bloodstream is given by  $0.4M(t)$ . Hence the accumulation rate of the drug is described by the differential equation

$$\frac{dM}{dt} = 500 - 0.4M \quad (\text{mg/hr}).$$

(b)



Based on the direction field, the amount of drug in the bloodstream approaches the equilibrium level of 1250 *mg* (*within a few hours*).

18. (a) Following the discussion in the text, the differential equation is

$$m \frac{dv}{dt} = mg - \gamma v^2$$

or equivalently,

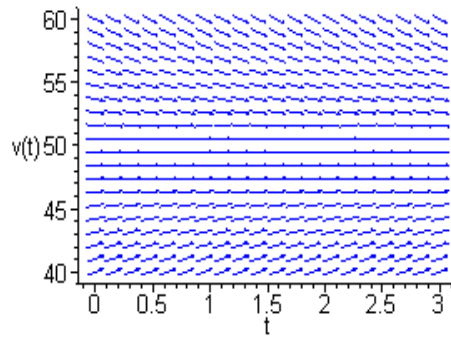
$$\frac{dv}{dt} = g - \frac{\gamma}{m} v^2.$$

(b) After a long time,  $\frac{dv}{dt} \approx 0$ . Hence the object attains a *terminal velocity* given by

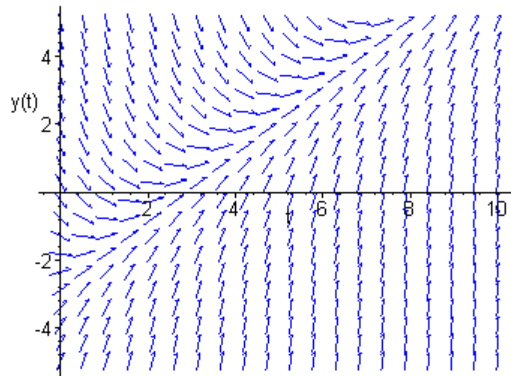
$$v_{\infty} = \sqrt{\frac{mg}{\gamma}}.$$

(c) Using the relation  $\gamma v_{\infty}^2 = mg$ , the required *drag coefficient* is  $\gamma = 0.0408 \text{ kg/sec}$ .

(d)

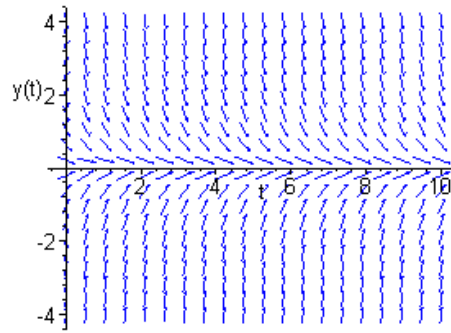


19.



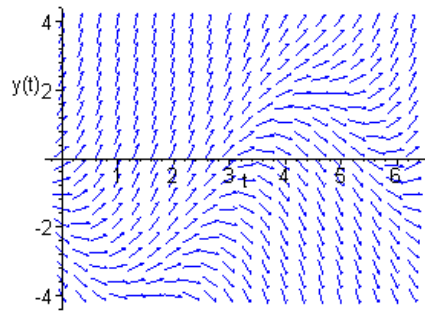
All solutions appear to approach a linear asymptote (*with slope equal to 1*). It is easy to verify that  $y(t) = t - 3$  is a solution.

20.



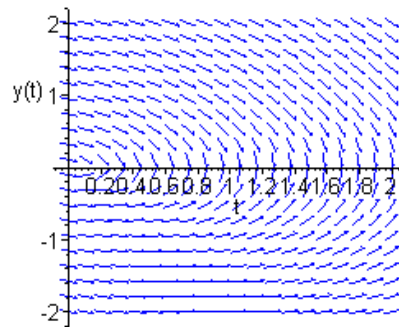
All solutions approach the equilibrium solution  $y(t) = 0$ .

23.



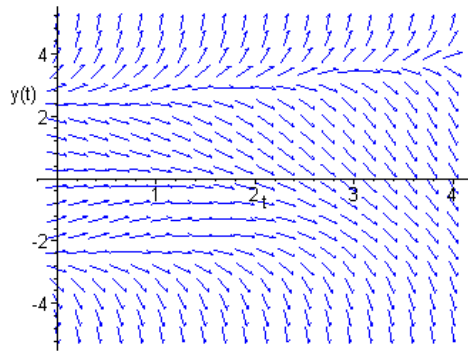
All solutions appear to *diverge* from the sinusoid  $y(t) = -\frac{3}{\sqrt{2}}\sin(t + \frac{\pi}{4}) - 1$ , which is also a solution corresponding to the initial value  $y(0) = -5/2$ .

25.

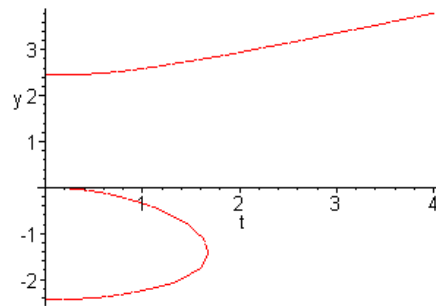


All solutions appear to converge to  $y(t) = 0$ . First, the rate of change is small. The slopes eventually increase very rapidly in *magnitude*.

26.



The direction field is rather complicated. Nevertheless, the collection of points at which the slope field is *zero*, is given by the implicit equation  $y^3 - 6y = 2t^2$ . The graph of these points is shown below:



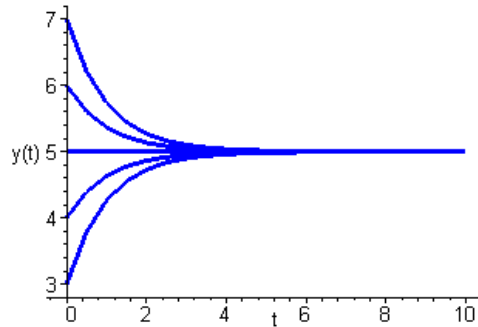
The *y*-intercepts of these curves are at  $y = 0, \pm\sqrt{6}$ . It follows that for solutions with initial values  $y > \sqrt{6}$ , all solutions increase without bound. For solutions with initial values in the range  $y < -\sqrt{6}$  and  $0 < y < \sqrt{6}$ , the slopes remain *negative*, and hence these solutions decrease without bound. Solutions with initial conditions in the range  $-\sqrt{6} < y < 0$  initially increase. Once the solutions reach the critical value, given by the equation  $y^3 - 6y = 2t^2$ , the slopes become negative and *remain* negative. These solutions eventually decrease without bound.

## Section 1.2

1(a) The differential equation can be rewritten as

$$\frac{dy}{5-y} = dt.$$

Integrating both sides of this equation results in  $-\ln|5-y| = t + c_1$ , or equivalently,  $5-y = ce^{-t}$ . Applying the initial condition  $y(0) = y_0$  results in the specification of the constant as  $c = 5 - y_0$ . Hence the solution is  $y(t) = 5 + (y_0 - 5)e^{-t}$ .

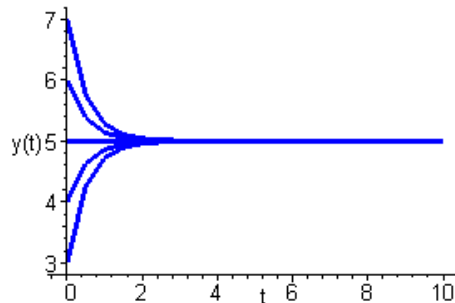


All solutions appear to converge to the equilibrium solution  $y(t) = 5$ .

1(c). Rewrite the differential equation as

$$\frac{dy}{10-2y} = dt.$$

Integrating both sides of this equation results in  $-\frac{1}{2}\ln|10-2y| = t + c_1$ , or equivalently,  $5-y = ce^{-2t}$ . Applying the initial condition  $y(0) = y_0$  results in the specification of the constant as  $c = 5 - y_0$ . Hence the solution is  $y(t) = 5 + (y_0 - 5)e^{-2t}$ .

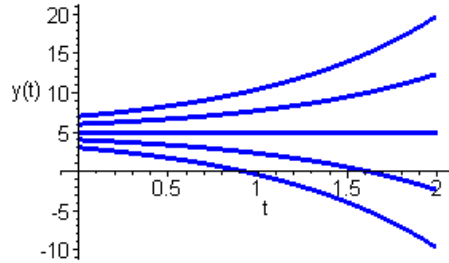


All solutions appear to converge to the equilibrium solution  $y(t) = 5$ , but at a *faster* rate than in Problem 1a.

2(a). The differential equation can be rewritten as

$$\frac{dy}{y-5} = dt.$$

Integrating both sides of this equation results in  $\ln|y-5| = t + c_1$ , or equivalently,  $y-5 = ce^t$ . Applying the initial condition  $y(0) = y_0$  results in the specification of the constant as  $c = y_0 - 5$ . Hence the solution is  $y(t) = 5 + (y_0 - 5)e^t$ .

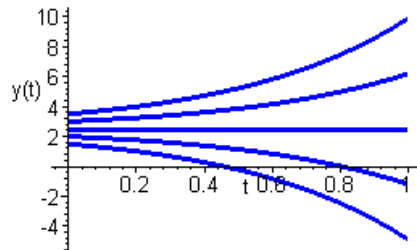


All solutions appear to diverge from the equilibrium solution  $y(t) = 5$ .

2(b). Rewrite the differential equation as

$$\frac{dy}{2y-5} = dt.$$

Integrating both sides of this equation results in  $\frac{1}{2}\ln|2y-5| = t + c_1$ , or equivalently,  $2y-5 = ce^{2t}$ . Applying the initial condition  $y(0) = y_0$  results in the specification of the constant as  $c = 2y_0 - 5$ . Hence the solution is  $y(t) = 2.5 + (y_0 - 2.5)e^{2t}$ .



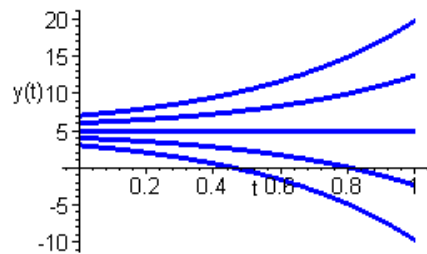
All solutions appear to diverge from the equilibrium solution  $y(t) = 2.5$ .

2(c). The differential equation can be rewritten as

$$\frac{dy}{2y-10} = dt.$$

Integrating both sides of this equation results in  $\frac{1}{2}\ln|2y-10| = t + c_1$ , or equivalently,  $y-5 = ce^{2t}$ . Applying the initial condition  $y(0) = y_0$  results in the specification of the constant as  $c = y_0 - 5$ . Hence the solution is  $y(t) = 5 + (y_0 - 5)e^{2t}$ .





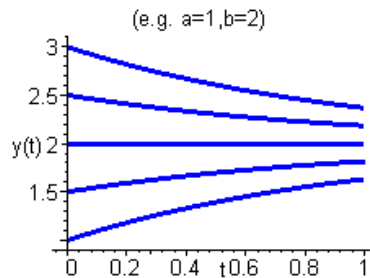
All solutions appear to diverge from the equilibrium solution  $y(t) = 5$ .

3(a). Rewrite the differential equation as

$$\frac{dy}{b - ay} = dt,$$

which is valid for  $y \neq b/a$ . Integrating both sides results in  $-\frac{1}{a} \ln|b - ay| = t + c_1$ , or equivalently,  $b - ay = c e^{-at}$ . Hence the general solution is  $y(t) = (b - c e^{-at})/a$ . Note that if  $y = b/a$ , then  $dy/dt = 0$ , and  $y(t) = b/a$  is an equilibrium solution.

(b)



(i) As  $a$  increases, the equilibrium solution gets closer to  $y(t) = 0$ , from above. Furthermore, the *convergence rate* of all solutions, that is,  $a$ , also increases.

(ii) As  $b$  increases, then the equilibrium solution  $y(t) = b/a$  also becomes larger. In this case, the convergence rate remains the same.

(iii) If  $a$  and  $b$  both increase (*but*  $b/a = \text{constant}$ ), then the equilibrium solution  $y(t) = b/a$  remains the same, but the *convergence rate* of all solutions increases.

5(a). Consider the simpler equation  $dy_1/dt = -ay_1$ . As in the previous solutions, rewrite the equation as

$$\frac{dy_1}{y_1} = -a dt.$$

Integrating both sides results in  $y_1(t) = c e^{-at}$ .

(b). Now set  $y(t) = y_1(t) + k$ , and substitute into the original differential equation. We find that

$$-ay_1 + 0 = -a(y_1 + k) + b.$$

That is,  $-ak + b = 0$ , and hence  $k = b/a$ .

(c). The general solution of the differential equation is  $y(t) = ce^{-at} + b/a$ . This is exactly the form given by Eq. (17) in the text. Invoking an initial condition  $y(0) = y_0$ , the solution may also be expressed as  $y(t) = b/a + (y_0 - b/a)e^{-at}$ .

6(a). The general solution is  $p(t) = 900 + ce^{t/2}$ , that is,  $p(t) = 900 + (p_0 - 900)e^{t/2}$ . With  $p_0 = 850$ , the specific solution becomes  $p(t) = 900 - 50e^{t/2}$ . This solution is a *decreasing* exponential, and hence the time of extinction is equal to the number of months

it takes, say  $t_f$ , for the population to reach *zero*. Solving  $900 - 50e^{t_f/2} = 0$ , we find that  $t_f = 2 \ln(900/50) = 5.78$  months.

(b) The solution,  $p(t) = 900 + (p_0 - 900)e^{t/2}$ , is a *decreasing* exponential as long as  $p_0 < 900$ . Hence  $900 + (p_0 - 900)e^{t_f/2} = 0$  has only *one* root, given by

$$t_f = 2 \ln\left(\frac{900}{900 - p_0}\right).$$

(c). The answer in part (b) is a general equation relating time of extinction to the value of

the initial population. Setting  $t_f = 12$  months, the equation may be written as

$$\frac{900}{900 - p_0} = e^6,$$

which has solution  $p_0 = 897.7691$ . Since  $p_0$  is the initial population, the appropriate answer is  $p_0 = 898$  mice.

7(a). The general solution is  $p(t) = p_0 e^{rt}$ . Based on the discussion in the text, time  $t$  is measured in *months*. Assuming 1 month = 30 days, the hypothesis can be expressed as  $p_0 e^{r \cdot 1} = 2p_0$ . Solving for the rate constant,  $r = \ln(2)$ , with units of *per month*.

(b).  $N$  days =  $N/30$  months. The hypothesis is stated mathematically as  $p_0 e^{rN/30} = 2p_0$ .

It follows that  $rN/30 = \ln(2)$ , and hence the rate constant is given by  $r = 30 \ln(2)/N$ . The units are understood to be *per month*.

9(a). Assuming *no air resistance*, with the positive direction taken as *downward*, Newton's Second Law can be expressed as

$$m \frac{dv}{dt} = mg$$

in which  $g$  is the *gravitational constant* measured in appropriate units. The equation can be

written as  $dv/dt = g$ , with solution  $v(t) = gt + v_0$ . The object is released with an initial velocity  $v_0$ .

(b). Suppose that the object is released from a height of  $h$  units above the ground. Using the fact that  $v = dx/dt$ , in which  $x$  is the *downward displacement* of the object, we obtain the differential equation for the displacement as  $dx/dt = gt + v_0$ . With the origin placed at the point of release, direct integration results in  $x(t) = gt^2/2 + v_0t$ . Based on the chosen coordinate system, the object reaches the ground when  $x(t) = h$ . Let  $t = T$  be the time that it takes the object to reach the ground. Then  $gT^2/2 + v_0T = h$ . Using the quadratic formula to solve for  $T$ ,

$$T = \frac{-v_0 \pm \sqrt{v_0^2 + 2gh}}{g}.$$

The *positive* answer corresponds to the time it takes for the object to fall to the ground. The *negative* answer represents a previous instant at which the object could have been launched upward (*with the same impact speed*), only to ultimately fall downward with speed  $v_0$ , from a height of  $h$  units above the ground.

(c). The impact speed is calculated by substituting  $t = T$  into  $v(t)$  in part (a). That is,  $v(T) = \sqrt{v_0^2 + 2gh}$ .

10(a,b). The general solution of the differential equation is  $Q(t) = ce^{-rt}$ . Given that  $Q(0) = 100$  mg, the value of the constant is given by  $c = 100$ . Hence the amount of thorium-234 present at any time is given by  $Q(t) = 100e^{-rt}$ . Furthermore, based on the hypothesis, setting  $t = 1$  results in  $82.04 = 100e^{-r}$ . Solving for the rate constant, we find that  $r = -\ln(82.04/100) = .19796/\text{week}$  or  $r = .02828/\text{day}$ .

(c). Let  $T$  be the time that it takes the isotope to decay to *one-half* of its original amount.

From part (a), it follows that  $50 = 100e^{-rT}$ , in which  $r = .19796/\text{week}$ . Taking the natural logarithm of both sides, we find that  $T = 3.5014$  weeks or  $T = 24.51$  days.

11. The general solution of the differential equation  $dQ/dt = -rQ$  is  $Q(t) = Q_0e^{-rt}$ , in which  $Q_0 = Q(0)$  is the initial amount of the substance. Let  $\tau$  be the time that it takes the substance to decay to *one-half* of its original amount,  $Q_0$ . Setting  $t = \tau$  in the solution,

we have  $0.5Q_0 = Q_0e^{-r\tau}$ . Taking the natural logarithm of both sides, it follows that  $-r\tau = \ln(0.5)$  or  $r\tau = \ln 2$ .

12. The differential equation governing the amount of radium-226 is  $dQ/dt = -rQ$ , with solution  $Q(t) = Q(0)e^{-rt}$ . Using the result in Problem 11, and the fact that the half-life  $\tau = 1620$  years, the decay rate is given by  $r = \ln(2)/1620$  per year. The amount of radium-226, after  $t$  years, is therefore  $Q(t) = Q(0)e^{-0.00042786t}$ . Let  $T$  be the time that it takes the isotope to decay to  $3/4$  of its original amount. Then setting  $t = T$ , and  $Q(T) = \frac{3}{4}Q(0)$ , we obtain  $\frac{3}{4}Q(0) = Q(0)e^{-0.00042786T}$ . Solving for the decay time, it follows that  $-0.00042786T = \ln(3/4)$  or  $T = 672.36$  years.

13. The solution of the differential equation, with  $Q(0) = 0$ , is  $Q(t) = CV(1 - e^{-t/CR})$ . As  $t \rightarrow \infty$ , the exponential term vanishes, and hence the limiting value is  $Q_L = CV$ .

14(a). The *accumulation* rate of the chemical is  $(0.01)(300)$  grams per hour. At any given time  $t$ , the *concentration* of the chemical in the pond is  $Q(t)/10^6$  grams per gallon. Consequently, the chemical *leaves* the pond at a rate of  $(3 \times 10^{-4})Q(t)$  grams per hour. Hence, the rate of change of the chemical is given by

$$\frac{dQ}{dt} = 3 - 0.0003Q(t) \text{ gm/hr.}$$

Since the pond is initially free of the chemical,  $Q(0) = 0$ .

(b). The differential equation can be rewritten as

$$\frac{dQ}{10000 - Q} = 0.0003 dt.$$

Integrating both sides of the equation results in  $-\ln|10000 - Q| = 0.0003t + C$ .

Taking

the natural logarithm of both sides gives  $10000 - Q = ce^{-0.0003t}$ . Since  $Q(0) = 0$ , the value of the constant is  $c = 10000$ . Hence the amount of chemical in the pond at any time

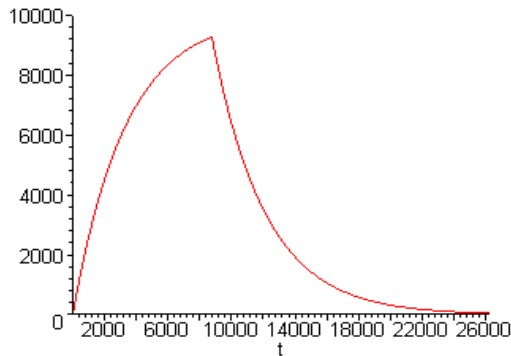
is  $Q(t) = 10000(1 - e^{-0.0003t})$  grams. Note that 1 year = 8760 hours. Setting  $t = 8760$ , the amount of chemical present after *one year* is  $Q(8760) = 9277.77$  grams, that is, 9.27777 kilograms.

(c). With the *accumulation* rate now equal to zero, the governing equation becomes  $dQ/dt = -0.0003Q(t)$  gm/hr. Resetting the time variable, we now assign the new initial value as  $Q(0) = 9277.77$  grams.

(d). The solution of the differential equation in Part (c) is  $Q(t) = 9277.77e^{-0.0003t}$ . Hence, one year *after* the source is removed, the amount of chemical in the pond is  $Q(8760) = 670.1$  grams.

(e). Letting  $t$  be the amount of time after the source is removed, we obtain the equation  $10 = 9277.77 e^{-0.0003t}$ . Taking the natural logarithm of both sides,  $-0.0003t = \ln(10/9277.77)$  or  $t = 22,776 \text{ hours} = 2.6 \text{ years}$ .

(f)



15(a). It is assumed that dye is no longer entering the pool. In fact, the rate at which the dye leaves the pool is  $200 \cdot [q(t)/60000] \text{ kg/min} = 200(60/1000)[q(t)/60] \text{ gm per hour}$ .

Hence the equation that governs the amount of dye in the pool is

$$\frac{dq}{dt} = -0.2q \quad (\text{gm/hr}).$$

The initial amount of dye in the pool is  $q(0) = 5000 \text{ grams}$ .

(b). The solution of the governing differential equation, with the specified initial value, is  $q(t) = 5000 e^{-0.2t}$ .

(c). The amount of dye in the pool after four hours is obtained by setting  $t = 4$ . That is,  $q(4) = 5000 e^{-0.8} = 2246.64 \text{ grams}$ . Since size of the pool is 60,000 gallons, the concentration of the dye is 0.0374 grams/gallon.

(d). Let  $T$  be the time that it takes to reduce the concentration level of the dye to 0.02 grams/gallon. At that time, the amount of dye in the pool is 1,200 grams. Using the answer in part (b), we have  $5000 e^{-0.2T} = 1200$ . Taking the natural logarithm of both sides of the equation results in the required time  $T = 7.14 \text{ hours}$ .

(e). Note that  $0.2 = 200/1000$ . Consider the differential equation

$$\frac{dq}{dt} = -\frac{r}{1000}q.$$

Here the parameter  $r$  corresponds to the flow rate, measured in gallons per minute. Using the same initial value, the solution is given by  $q(t) = 5000 e^{-rt/1000}$ . In order to determine the appropriate flow rate, set  $t = 4$  and  $q = 1200$ . (Recall that 1200 gm of

dye has a concentration of  $0.02 \text{ gm/gal}$ ). We obtain the equation  $1200 = 5000 e^{-r/250}$ . Taking the natural logarithm of both sides of the equation results in the required flow rate  $r = 357 \text{ gallons per minute}$ .

## Section 1.3

1. The differential equation is *second order*, since the highest derivative in the equation is of order *two*. The equation is *linear*, since the left hand side is a linear function of  $y$  and its derivatives.

3. The differential equation is *fourth order*, since the highest derivative of the function  $y$  is of order *four*. The equation is also *linear*, since the terms containing the dependent variable is linear in  $y$  and its derivatives.

4. The differential equation is *first order*, since the only derivative is of order *one*. The dependent variable is *squared*, hence the equation is *nonlinear*.

5. The differential equation is *second order*. Furthermore, the equation is *nonlinear*, since the dependent variable  $y$  is an argument of the *sine function*, which is *not* a linear function.

7.  $y_1(t) = e^t \Rightarrow y_1'(t) = y_1''(t) = e^t$ . Hence  $y_1'' - y_1 = 0$ .

Also,  $y_2(t) = \cosh t \Rightarrow y_2'(t) = \sinh t$  and  $y_2''(t) = \cosh t$ . Thus  $y_2'' - y_2 = 0$ .

9.  $y(t) = 3t + t^2 \Rightarrow y'(t) = 3 + 2t$ . Substituting into the differential equation, we have  $t(3 + 2t) - (3t + t^2) = 3t + 2t^2 - 3t - t^2 = t^2$ . Hence the given function is a solution.

10.  $y_1(t) = t/3 \Rightarrow y_1'(t) = 1/3$  and  $y_1''(t) = y_1'''(t) = y_1''''(t) = 0$ . Clearly,  $y_1(t)$  is a solution. Likewise,  $y_2(t) = e^{-t} + t/3 \Rightarrow y_2'(t) = -e^{-t} + 1/3$ ,  $y_2''(t) = e^{-t}$ ,  $y_2'''(t) = -e^{-t}$ ,  $y_2''''(t) = e^{-t}$ . Substituting into the left hand side of the equation, we find that  $e^{-t} + 4(-e^{-t}) + 3(e^{-t} + t/3) = e^{-t} - 4e^{-t} + 3e^{-t} + t = t$ . Hence both functions are solutions of the differential equation.

11.  $y_1(t) = t^{1/2} \Rightarrow y_1'(t) = t^{-1/2}/2$  and  $y_1''(t) = -t^{-3/2}/4$ . Substituting into the left hand side of the equation, we have

$$\begin{aligned} 2t^2(-t^{-3/2}/4) + 3t(t^{-1/2}/2) - t^{1/2} &= -t^{1/2}/2 + 3t^{1/2}/2 - t^{1/2} \\ &= 0 \end{aligned}$$

Likewise,  $y_2(t) = t^{-1} \Rightarrow y_2'(t) = -t^{-2}$  and  $y_2''(t) = 2t^{-3}$ . Substituting into the left hand side of the differential equation, we have  $2t^2(2t^{-3}) + 3t(-t^{-2}) - t^{-1} = 4t^{-1} - 3t^{-1} - t^{-1} = 0$ . Hence both functions are solutions of the differential equation.

12.  $y_1(t) = t^{-2} \Rightarrow y_1'(t) = -2t^{-3}$  and  $y_1''(t) = 6t^{-4}$ . Substituting into the left hand side of the differential equation, we have  $t^2(6t^{-4}) + 5t(-2t^{-3}) + 4t^{-2} = 6t^{-2} - 10t^{-2} + 4t^{-2} = 0$ . Likewise,  $y_2(t) = t^{-2} \ln t \Rightarrow y_2'(t) = t^{-3} - 2t^{-3} \ln t$  and  $y_2''(t) = -5t^{-4} + 6t^{-4} \ln t$ . Substituting into the left hand side of the equation, we have  $t^2(-5t^{-4} + 6t^{-4} \ln t) + 5t(t^{-3} - 2t^{-3} \ln t) + 4(t^{-2} \ln t) = -5t^{-2} + 6t^{-2} \ln t +$

$+ 5t^{-2} - 10t^{-2} \ln t + 4t^{-2} \ln t = 0$ . Hence both functions are solutions of the differential equation.

13.  $y(t) = (\cos t) \ln \cos t + t \sin t \Rightarrow y'(t) = -(\sin t) \ln \cos t + t \cos t$  and  $y''(t) = -(\cos t) \ln \cos t - t \sin t + \sec t$ . Substituting into the left hand side of the differential equation, we have  $(-(\cos t) \ln \cos t - t \sin t + \sec t) + (\cos t) \ln \cos t + t \sin t = -(\cos t) \ln \cos t - t \sin t + \sec t + (\cos t) \ln \cos t + t \sin t = \sec t$ . Hence the function  $y(t)$  is a solution of the differential equation.

15. Let  $y(t) = e^{rt}$ . Then  $y''(t) = r^2 e^{rt}$ , and substitution into the differential equation results in  $r^2 e^{rt} + 2 e^{rt} = 0$ . Since  $e^{rt} \neq 0$ , we obtain the algebraic equation  $r^2 + 2 = 0$ . The roots of this equation are  $r_{1,2} = \pm i\sqrt{2}$ .

17.  $y(t) = e^{rt} \Rightarrow y'(t) = r e^{rt}$  and  $y''(t) = r^2 e^{rt}$ . Substituting into the differential equation, we have  $r^2 e^{rt} + r e^{rt} - 6 e^{rt} = 0$ . Since  $e^{rt} \neq 0$ , we obtain the algebraic equation  $r^2 + r - 6 = 0$ , that is,  $(r - 2)(r + 3) = 0$ . The roots are  $r_{1,2} = -3, 2$ .

18. Let  $y(t) = e^{rt}$ . Then  $y'(t) = r e^{rt}$ ,  $y''(t) = r^2 e^{rt}$  and  $y'''(t) = r^3 e^{rt}$ . Substituting the derivatives into the differential equation, we have  $r^3 e^{rt} - 3r^2 e^{rt} + 2r e^{rt} = 0$ . Since  $e^{rt} \neq 0$ , we obtain the algebraic equation  $r^3 - 3r^2 + 2r = 0$ . By inspection, it follows that  $r(r - 1)(r - 2) = 0$ . Clearly, the roots are  $r_1 = 0$ ,  $r_2 = 1$  and  $r_3 = 2$ .

20.  $y(t) = t^r \Rightarrow y'(t) = r t^{r-1}$  and  $y''(t) = r(r - 1)t^{r-2}$ . Substituting the derivatives into the differential equation, we have  $t^2[r(r - 1)t^{r-2}] - 4t(r t^{r-1}) + 4t^r = 0$ . After some algebra, it follows that  $r(r - 1)t^r - 4r t^r + 4t^r = 0$ . For  $t \neq 0$ , we obtain the algebraic equation  $r^2 - 5r + 4 = 0$ . The roots of this equation are  $r_1 = 1$  and  $r_2 = 4$ .

21. The order of the partial differential equation is *two*, since the highest derivative, in fact each one of the derivatives, is of *second order*. The equation is *linear*, since the left hand side is a linear function of the partial derivatives.

23. The partial differential equation is *fourth order*, since the highest derivative, and in fact each of the derivatives, is of order *four*. The equation is *linear*, since the left hand side is a linear function of the partial derivatives.

24. The partial differential equation is *second order*, since the highest derivative of the function  $u(x, y)$  is of order *two*. The equation is *nonlinear*, due to the product  $u \cdot u_x$  on the left hand side of the equation.

25.  $u_1(x, y) = \cos x \cosh y \Rightarrow \frac{\partial^2 u_1}{\partial x^2} = -\cos x \cosh y$  and  $\frac{\partial^2 u_1}{\partial y^2} = \cos x \cosh y$ .

It is evident that  $\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} = 0$ . Likewise, given  $u_2(x, y) = \ln(x^2 + y^2)$ , the second derivatives are



$$\frac{\partial^2 u_2}{\partial x^2} = \frac{2}{x^2 + y^2} - \frac{4x^2}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 u_2}{\partial y^2} = \frac{2}{x^2 + y^2} - \frac{4y^2}{(x^2 + y^2)^2}$$

Adding the partial derivatives,

$$\begin{aligned} \frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} &= \frac{2}{x^2 + y^2} - \frac{4x^2}{(x^2 + y^2)^2} + \frac{2}{x^2 + y^2} - \frac{4y^2}{(x^2 + y^2)^2} \\ &= \frac{4}{x^2 + y^2} - \frac{4(x^2 + y^2)}{(x^2 + y^2)^2} \\ &= 0. \end{aligned}$$

Hence  $u_2(x, y)$  is also a solution of the differential equation.

27. Let  $u_1(x, t) = \sin \lambda x \sin \lambda at$ . Then the second derivatives are

$$\frac{\partial^2 u_1}{\partial x^2} = -\lambda^2 \sin \lambda x \sin \lambda at$$

$$\frac{\partial^2 u_1}{\partial t^2} = -\lambda^2 a^2 \sin \lambda x \sin \lambda at$$

It is easy to see that  $a^2 \frac{\partial^2 u_1}{\partial x^2} = \frac{\partial^2 u_1}{\partial t^2}$ . Likewise, given  $u_2(x, t) = \sin(x - at)$ , we have

$$\frac{\partial^2 u_2}{\partial x^2} = -\sin(x - at)$$

$$\frac{\partial^2 u_2}{\partial t^2} = -a^2 \sin(x - at)$$

Clearly,  $u_2(x, t)$  is also a solution of the partial differential equation.

28. Given the function  $u(x, t) = \sqrt{\pi/t} e^{-x^2/4\alpha^2 t}$ , the partial derivatives are

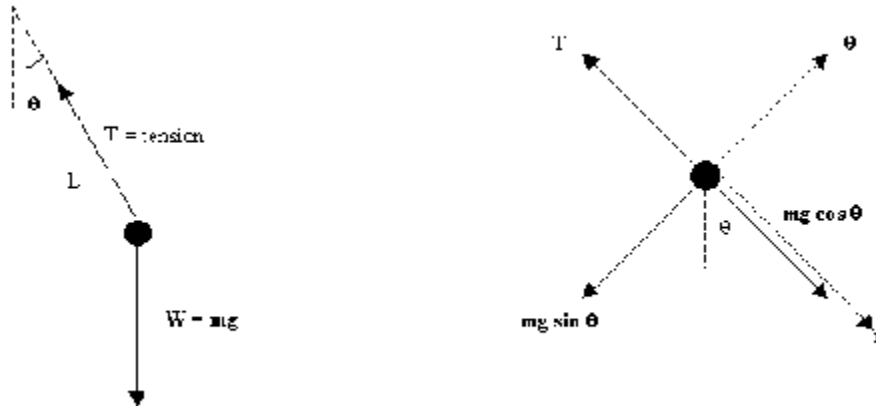
$$u_{xx} = -\frac{\sqrt{\pi/t} e^{-x^2/4\alpha^2 t}}{2\alpha^2 t} + \frac{\sqrt{\pi/t} x^2 e^{-x^2/4\alpha^2 t}}{4\alpha^4 t^2}$$

$$u_t = -\frac{\sqrt{\pi t} e^{-x^2/4\alpha^2 t}}{2t^2} + \frac{\sqrt{\pi} x^2 e^{-x^2/4\alpha^2 t}}{4\alpha^2 t^2 \sqrt{t}}$$

It follows that  $\alpha^2 u_{xx} = u_t = -\frac{\sqrt{\pi} (2\alpha^2 t - x^2) e^{-x^2/4\alpha^2 t}}{4\alpha^2 t^2 \sqrt{t}}$ .

Hence  $u(x, t)$  is a solution of the partial differential equation.

29(a).



(b). The path of the particle is a circle, therefore *polar coordinates* are intrinsic to the problem. The variable  $r$  is radial distance and the angle  $\theta$  is measured from the vertical. Newton's Second Law states that  $\sum \mathbf{F} = m\mathbf{a}$ . In the *tangential* direction, the equation of motion may be expressed as  $\sum F_\theta = m a_\theta$ , in which the *tangential acceleration*, that is, the linear acceleration *along* the path is  $a_\theta = L d^2\theta/dt^2$ . ( $a_\theta$  is *positive* in the direction of increasing  $\theta$ ). Since the only force acting in the tangential direction is the component of weight, the equation of motion is

$$-mg \sin \theta = mL \frac{d^2\theta}{dt^2}.$$

(Note that the equation of motion in the radial direction will include the tension in the rod).

(c). Rearranging the terms results in the differential equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{L} \sin \theta = 0.$$