Series Solutions of Second Order Linear Equations

1 Power Series Solutions

1.1 Power series

A power series $f(x)$ is a series given by

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = \lim_{m\to\infty} \sum_{n=0}^{m} a_n(x-x_0)^n. \tag{1.1}$$

It is said to converge if this limit exists. Otherwise it diverges. There is a **radius of convergence** ρ such that the series converges for $|x-x_0|<\rho$ and diverges for $|x-x_0|>\rho$. For $|x-x_0|=\rho$ the series may either converge or diverge. This needs to be tested separately.

The radius of convergence can often be found using the **ratio test**. Consider the limit of the absolute value of the ratio of two subsequent terms in the summation, being

$$\lim_{n\to\infty}\left|\frac{a_{n+1}(x-x_0)^{n+1}}{a_n(x-x_0)^n}\right| = |x-x_0| \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x-x_0|L. \tag{1.2}$$

The series converges if $|x-x_0|L<1$ and diverges if $|x-x_0|L>1$. In other words, the radius of convergence here is $\rho = 1/L$.

Series can also be tested for equality. If we have two series a and b such that

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = \sum_{n=0}^{\infty} b_n(x-x_0)^n, \tag{1.3}$$

then all coefficients must be equal. So $a_n = b_n$ for every n.

Let $f^{(n)}(x)$ denote the n^{th} derivative of $f(x)$. If the coefficients a_n are such that

$$a_n = \frac{f^{(n)}(x_0)}{n!}, \tag{1.4}$$

then the series is called a **Taylor series** for the function f about $x = x_0$.

1.2 Ordinary and singular points

Let's consider second order linear differential equations, where the coefficients are functions of the independent variable (which is usually x). The general form of the homogeneous equation is

$$P(x)y'' + Q(x)y' + R(x) = 0. \tag{1.5}$$

For simplicity we will assume that $P(x)$, $Q(x)$ and $R(x)$ are all polynomials. Let's suppose we want to solve this equation in the neighborhood of a point x_0. Such a point x_0 is called a **ordinary point** if $P(x_0) \neq 0$. If, however, $P(x_0) = 0$, then the point is called a **singular point**.

1.3 Series solutions near an ordinary point

It's often hard to find a normal solution for equation 1.5. But let's suppose that we look for solutions of the form

$$y = a_0 + a_1(x - x_0) + \ldots + a_n(x - x_0)^n + \ldots = \sum_{n=0}^{\infty} a_n(x - x_0)^n. \tag{1.6}$$

We assume that this series converges in an interval $|x - x_0| < \rho$ for some $\rho > 0$. For example, if we want to solve the differential equation $y'' + y = 0$, we can first find that

$$y'' = 2a_2 + 6a_3(x - x_0) + \ldots + n(n-1)a_n(x - x_0)^{n-2} + \ldots = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x - x_0)^n. \tag{1.7}$$

The differential equation thus becomes

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x - x_0)^n + a_n(x - x_0)^2 = y'' + y = 0 = \sum_{n=0}^{\infty} 0(x - x_0)^n. \tag{1.8}$$

We now have an equation with two sums. The two sums are only equal if all the coefficients are equal. This results in

$$(n+2)(n+1)a_{n+2} + a_n = 0 \qquad \Rightarrow \qquad a_{n+2} = -\frac{a_n}{(n+2)(n+1)}. \tag{1.9}$$

This relation is a **recurrence relation**, expressing a coefficient as a function of its predecessors. For arbitrary coefficients a_0 and a_1 we can find all the coefficients, and thus find the solution to the differential equation.

1.4 Convergence of the solution

The solution found in the last paragraph converges around x_0. But what is the radius of convergence? It turns out that this depends on the roots of $P(x)$ (being the values x such that $P(x) = 0$). Let's consider all the roots of $P(x)$ and draw them in the complex plane. Now let's also draw x_0. The radius of convergence is the minimum distance between x_0 and any root of $P(x)$.

For example, if $P(x) = x^2 - 2x + 2 = (x-1)^2 + 1$, then the roots are $1 \pm i$. If also $x_0 = 0$, then the radius of convergence is simply $\sqrt{2}$.

2 Singular Points

2.1 Regular singular points

Let's define $p(x) = \frac{Q(x)}{P(x)}$ and $q(x) = \frac{R(x)}{P(x)}$. Normally we could rewrite the differential equation to

$$y'' + p(x)y' + q(x)y = 0. \tag{2.1}$$

For singular points this isn't possible since $P(x_0) = 0$. In this case using power series gives problems. That's why we need to find other ways to solve these problems. Solving this problem can be split up in two separate cases, depending on whether x_0 is a **regular singular point** or an **irregular singular point**. To determine this, we need to examine the limits

$$p_0 = \lim_{x \to x_0}(x - x_0)p(x) = \lim_{x \to x_0}(x - x_0)\frac{Q(x)}{P(x)}, \qquad \text{and} \qquad q_0 = \lim_{x \to x_0}(x - x_0)^2 = q(x) = \lim_{x \to x_0}(x - x_0)^2\frac{R(x)}{P(x)}. \tag{2.2}$$

If both these limits exist (they are finite), then the point x_0 is a regular singular point. If either of these limits (or both) do not exist, then x_0 is an irregular singular point.

2.2 Euler equation

A relatively simple differential equation with a regular singular point is the **Euler equation**, being

$$x^2y'' + \alpha xy' + \beta y = 0. \tag{2.3}$$

Let's assume a certain solution has the form $y = x^r$. The differential equation then becomes

$$x^2(x^r)'' + \alpha x(x^r)' + \beta x^r = x^r\left(r(r-1) + \alpha r + \beta\right) = 0. \tag{2.4}$$

So we need to solve $(r(r-1) + \alpha r + \beta)$ to find r. There are three given possibilities. If r has two real distinct roots r_1 and r_2, then the general solution is

$$y = c_1 x^{r_1} + c_2 x^{r_2}. \tag{2.5}$$

If the roots are real, but equal, then the general solution can be shown to be

$$y = (c_1 + c_2 \ln x)x^{r_1}. \tag{2.6}$$

If the roots are complex, such that $r = \lambda \pm \mu i$, then the general solution is

$$y = x^\lambda \left(c_1 \cos(\mu \ln x) + c_2 \sin(\mu \ln x)\right). \tag{2.7}$$

2.3 Negative x

The above solution are right for positive x. If $x < 0$ strange situation occur with possibly complex or undefined numbers. But if we define $\xi = -x$, then we find the same solutions (with ξ instead of x). So we can rewrite the equations of the last paragraph to

$$y = c_1|x|^{r_1} + c_2|x|^{r_2}, \tag{2.8}$$

$$y = (c_1 + c_2 \ln|x|)|x|^{r_1}, \tag{2.9}$$

$$y = |x|^\lambda \left(c_1 \cos(\mu \ln|x|) + c_2 \sin(\mu \ln|x|)\right). \tag{2.10}$$

2.4 Series solutions near a singular point

Let's consider a regular singular point. We assume this point is $x_0 = 0$. (If $x_0 \neq 0$ simply make the change of variable $t = x - x_0$.) We can rewrite our differential equation to

$$y'' + p(x)y' + q(x)y = x^2y'' + x(xp(x))y' + x^2q(x)y = 0. \tag{2.11}$$

Note that for a regular singular point the parts $xp(x)$ and $x^2q(x)$ have a value as $x \to 0$. Let's now assume that a solution has the form

$$y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{r+n}. \tag{2.12}$$

We only need to find the values of r and the coefficients a_n. If we calculate y' and y'' and put this back in the differential equation, we find that

$$F(r) = r(r-1) + p_0 r + q_0 = 0. \tag{2.13}$$

This equation is called the **indicial equation**. Its roots r_1 and r_2 (in which we suppose that $r_1 \geq r_2$) are called the **exponents of singularity**. We have now found r. The coefficients can be found using the **recurrence relation**

$$F(r+n)a_n + \sum_{k=0}^{n-1} a_k \left((r+k)p_{n-k} + q_{n-k}\right) = 0. \tag{2.14}$$

Note that the coefficients depend on the values of a_0 and r. a_0 is arbitrary, and for simplicity usually $a_0 = 1$ is chosen. r is not arbitrary though. To indicate which r has been used to calculate the coefficients, the coefficients are usually written as $a_n(r_1)$ or $a_n(r_2)$. Now that we have found the coefficients, we can write the solutions. Since we have two solutions r_1 and r_2, we have two solutions, being

$$y_1(x) = |x|^{r_1} \left(1 + \sum_{n=1}^{\infty} a_n(r_1)x^n\right) \qquad \text{and} \qquad y_2(x) = |x|^{r_2} \left(1 + \sum_{n=1}^{\infty} a_n(r_2)x^n\right). \tag{2.15}$$

Note that we have taken the absolute value of x again, according to the trick of the previous paragraph. The general set of solutions now consists of all linear combination $c_1y_1 + c_2y_2$ of these solutions.

2.5 Equal roots

There are, however, a few catches to the method described in the last paragraph. If $r_1 = r_2$ we will find the same solution twice. We want two solutions to find the general solution set, and we only have one, being y_1. So we need another method to find a new solution. Let's assume that the new solution has the form

$$y_2(x) = y_1(x)\ln|x| + |x|^{r_1} \sum_{n=1}^{\infty} b_n x^n. \tag{2.16}$$

All we need to do now is find the coefficients b_n. The procedure for this is simple. First calculate y_2' and y_2''. Then substitute these in the differential equation, and solve for the coefficients b_n.

2.6 Roots differing by an integer

Let's consider the term $F(r+n)$ of the recurrence relation. If this term is 0, it is impossible to find a_n. If $r = r_1$ we will always find a correct solution y_1. But if $r = r_2$ and $n = r_1 - r_2$ we find that $F(r+n) = F(r_2) = 0$. So there is a problem. Now let's assume the second solution has the form

$$y_2(x) = ay_1(x)\ln|x| + |x|^{r_2} \left(1 + \sum_{n=1}^{\infty} c_n x^n\right). \tag{2.17}$$

Just like in the previous paragraph, the values of the constant a and the coefficients c_n can be found by substituting y_2, y_2' and y_2'' in the differential equation.