Problem Solving Guide

1 Introduction

Having trouble with differential equations? Then this problem solving guide can help you out. Follow it, and you can solve **almost** all differential equation exam problems.

Yes, almost. This strategy guide doesn't work for all problems. If you really want a 10, then don't read this file, but study the differential equations book for the coming (at least) two weeks. If you're content with a 9, then try to follow the steps below.

You don't have to read everything from this file. First of all, the addendums are optional. Read them if you have problems with certain parts, or want to go more into detail. Second, only the bold parts are important. The rest of the text is present to clarify the bold parts. If you understand the bold parts right away, you won't have to read anything else.

By the way, you might want to take a look at the examination solutions on the Aerostudents website as well. They are solved using the plans of approach described below. Sometimes they can help you understand one of the plans of approach.

2 Laplace Transform

2.1 When to use?

When it is asked. In the question is a line saying "Use the Laplace Transform to solve..."

2.2 Plan of Approach

1. **Take the Laplace Transform.** Do this for both sides of the given differential equation. Use the table of Laplace Transforms for this. When transforming the left side of the equation, use table item 18 and the relation $L\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 L\{f_1(t)\} + c_2 L\{f_2(t)\}$. For the right side of the equation, you can use table items 2, 3, 4, 6, 7, 9, 10, 14 and 17.
2. **Solve for $F(s)$.**
3. **Split the equation up in parts you can transform back.** Sometimes you need to split up fractions for this. Having trouble with this? See the addendum below.
4. **Transform every part back independently.** By doing this, you prevent yourself from mixing up things. And you keep a better overview.
5. **Add all the parts up.** Simply put all the results from the previous step together to find the solution to the question.

2.3 Addendum: Dealing with Fractions

Do you have trouble dealing with fractions? If you don't, then skip this addendum. The above plan of approach will assist you enough. If you do have problems, this addendum might help you out. It answers questions you might be having. First we consider when/how to split up fractions.

- **In what ways can we split up fractions?** There are two ways to do this. If we have a numerator consisting of parts with a + in between (like $\frac{a+b}{c}$) we can split it up quite easily (it becomes $\frac{a}{c} + \frac{b}{c}$). It is always wise to split fractions up in this way as much as possible.

  There is, however, another way to split up fractions. This is a bit more difficult. So we could now ask ourself:
- **When should we split up fractions in the difficult way?** To answer this, we look at the denominator of the fraction. The general rule is: Split up the fraction if the denominator consists of multiple factors. With a denominator of, for example, $(s^2 + 4)(s + 4)$ we split the fraction up. Now comes a following question:
- **How do we split a fraction up in the difficult way?** We do this by assuming a form of the split-up fraction. For every factor in the denominator, we create a separate fraction. So in this case we assume that

$$\frac{\text{Something}}{(s^2+4)(s+4)} = \frac{as+b}{s^2+4} + \frac{c}{s+4}.$$

  We could ask ourselves, why do we use $as+b$ at one fraction and only c at the other? Or in general, we could ask ourselves the following question:
- **When do we use $as + b$ above a fraction (in the assumed form) and when do we use only a constant c?** Once more we need to look at the denominator. More specifically, we need to look at the highest power of it. If the highest power is 1 (for example, in $s + 4$), we put a constant in the numerator. If the highest power is 2 (for example, in $s^2 + 4$), we assume $as + b$. (This can be continued for higher order denominators, but that never occurs in the Laplace Transform.)

  So we know which form to assume. The next problem arises.

- **How do we solve for the coefficients?** For this, we need to make the denominators equal. In our example we will then get

$$\frac{\text{Something}}{(s^2+4)(s+4)} = \frac{(as+b)(s+4)}{(s^2+4)(s+4)} + \frac{c(s^2+4)}{(s^2+4)(s+4)}.$$

  We now have equal denominators, so we can remove them. We also need to work out brackets. For our example, we will find

$$\text{Something} = as^2 + bs + 4as + 4b + cs^2 + 4c.$$

  Now we need to equate coefficients. First look at all the terms without any s. This gives you one equation. Then look at all the terms with s^1. This gives you another equation. If, in our example, we have Something $= 2$, we will find $2 = 4b + 4c$ and $0 = b + 4a$. We will also find a third equation when looking at all the terms with s^2. Can you find which equation it is?

  Now we know when to split up the fraction. If the denominator doesn't consist of factors (like for example $s^2 + 2s + 4$) we can't split the fraction up. What do we do then?

- **What do we do with fractions we can't split up?** This time we need to write the denominator differently. To be more precise, we need to set it in the form $(s+a)^2 \pm b^2$, for some constants a and b. In our example with denominator $s^2 + 2s + 4$, we will find $a = 1$ and $b = \sqrt{3}$. Once you have the denominator in this form, you can use the Laplace Transform table to find the inverse Laplace Transform. If $b = 0$ you need to use either table item 3, 6, 7 or 8. If $b \neq 0$ you need to combine this with table item 14.

By now you're an expert in splitting up fractions. There's one remaining thing we could discuss. Often we have $F(s)$ written as

$$F(s) = \ldots + \ldots + \frac{\text{Some numerator}}{\text{Some denominator}} e^{-as}.$$

Students often find this difficult, but it is quite simple, once you know the trick. We first need to define

$$H(s) = \frac{\text{Some numerator}}{\text{Some denominator}}.$$

If we use table item 13 (from the given Laplace Transform table) we will get

$$L^{-1}\left\{\frac{\text{Some numerator}}{\text{Some denominator}} e^{-as}\right\} = L^{-1}\{e^{-as}H(s)\} = u_a(t)h(t-a), \quad \text{where} \quad h(t) = L^{-1}\{H(s)\}.$$

So the only thing we need to do is find the inverse laplace transform of some fraction $H(s)$. And you know how to do that!

3 Systems of First Order Differential Equations

3.1 When to use?

When it is asked. The question will start with "Find the general solution to the system of differential equations..."

3.2 Plan of Approach

To find the general solution, just follow the following steps.

- **Solve the homogeneous system**

$$\mathbf{x}'(t) = A\mathbf{x}(t) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mathbf{x}(t).$$

  We do this by finding two independent solutions $\mathbf{x_1}(t)$ and $\mathbf{x_2}(t)$. Of course that's easier said than done. So we go into a little bit more detail.

  - **Assume $\mathbf{x} = \xi e^{\lambda t}$.** Here ξ is a constant vector. Do not forget to write this assumption down on your exam!
  - **Find the eigenvalues λ_1 and λ_2.** We do this by solving $(a-\lambda)(d-\lambda) - bc = 0$.
  - **Find the corresponding eigenvectors ξ_1 and ξ_2.** We do this by solving $(A - \lambda_1 I)\xi_1 = \mathbf{0}$ and $(A - \lambda_2 I)\xi_2 = \mathbf{0}$. It can be shown that the solutions are

$$\xi_1 = \begin{bmatrix} -b \\ a - \lambda_1 \end{bmatrix} \qquad \text{and} \qquad \xi_2 = \begin{bmatrix} -b \\ a - \lambda_2 \end{bmatrix}.$$

  - **Look at the eigenvalues.**
    - **Are they real and different? Then**

$$\mathbf{x_1}(t) = \xi_1 e^{\lambda_1 t} \qquad \text{and} \qquad \mathbf{x_2}(t) = \xi_2 e^{\lambda_2 t}.$$

    - **Are they complex? Then write**

$$\xi_1 = \mathbf{u} + \mathbf{v}i \qquad \text{and} \qquad \lambda_1 = \alpha + \beta i.$$

      Now we have

$$\begin{aligned} \mathbf{x_1}(t) &= e^{\alpha t} \left(\mathbf{u} \cos \beta t - \mathbf{v} \sin \beta t \right), \\ \mathbf{x_2}(t) &= e^{\alpha t} \left(\mathbf{u} \sin \beta t + \mathbf{v} \cos \beta t \right). \end{aligned}$$

      You may be wondering, why we only use one eigenvalue and one eigenvector. That is because the two eigenvalues, and also the two eigenvectors, are complex conjugates (so $\lambda_2 = \alpha - \beta i$ and $\xi_2 = \mathbf{u} - \mathbf{v}i$). This plan of approach explicitly uses that fact. (An exception occurs if A is complex, but that situation isn't part of this course.)
    - **Are they equal? Then solve**

$$(A - \lambda_1 I)\, \eta = \xi_1$$

      for the constant vector η. Now we have

$$\begin{aligned} \mathbf{x_1}(t) &= \xi_1 e^{\lambda_1 t}, \\ \mathbf{x_2}(t) &= \xi_1 t e^{\lambda_1 t} + \eta e^{\lambda_1 t}. \end{aligned}$$

So that's how we find the solution to the homogeneous solution.

- **Find a particular solution $\mathbf{x_p}(t)$ for the nonhomogeneous system**

$$\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{g}(t).$$

  - **Look at $\mathbf{g}(t)$. Is it an exponential or a polynomial?** $\mathbf{g}(t)$ is an exponential if it has the form $\alpha e^{\beta t}$. It is a polynomial if its form is $\alpha t + \beta$ (or incidentally with higher powers of t).
  - **Yes? Then use method of undetermined coefficients.** We do this because it is much simpler than the other methods. Only if we can't use this method, we take our refuge to a different method.
    * **Assume the general form of $\mathbf{x_p}(t)$.** If $g(t) = \alpha e^{\beta t}$ assume $\mathbf{x_p}(t) = \mathbf{c_1} e^{\beta t}$. If $g(t) = \alpha t + \beta$, assume $\mathbf{x_p}(t) = \mathbf{c_1} t + \mathbf{c_2}$.
    * **Find $\mathbf{x_p}'(t)$.**
    * **Insert $\mathbf{x_p}(t)$ and $\mathbf{x_p}'(t)$ in the differential equation $\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{g}(t)$.**
    * **Solve for $\mathbf{c_1}$ (and $\mathbf{c_2}$).** If we are dealing with a polynomial, it will seem like we will have 2 equations and 4 unknowns. To solve this problem, we need to apply a small trick. We need to look at all the coefficients without a factor t (this gives us two equations) and at all the coefficients with a factor t (giving us another two equations). This is similar to what was discussed in the addendum about fractions.
  - **No? Use variation of parameters.**
    * **Assume $\mathbf{x_p}(t) = \Phi(t)\mathbf{u}(t)$.**
    * **Assemble the fundamental matrix**

$$\Phi(t) = \begin{bmatrix} \mathbf{x_1}(t) & \mathbf{x_2}(t) \end{bmatrix}.$$

    * **Find $\Phi^{-1}(t)$.** This is usually the part requiring most work. One way to find Φ^{-1} is by using the relation $\Phi\Phi^{-1} = I$. But there are a lot more methods to find the inverse matrix.
    * **Find $\mathbf{u}(t)$ using**

$$\mathbf{u}(t) = \int \Phi^{-1}(t)\mathbf{g}(t)\,dt.$$

    * **Find $\mathbf{x_p}(t) = \Phi(t)\mathbf{u}(t)$.**

  And now we've found the particular solution.

- **Write down the general solution set**

$$\mathbf{x}(t) = c_1\mathbf{x_1}(t) + c_2\mathbf{x_2}(t) + \mathbf{x_p}(t).$$

And we're done! That's all there is to it.

4 Stability of Systems of Differential Equations

4.1 When to use?

When it is asked. The question starts with "Determine the type and (in-)stability..."

4.2 Plan of Approach

- **Find the critical points.** Of course you won't have to do this when they are given.
  - **Set** $dx/dt = 0$ **and** $dy/dt = 0$**.**
  - **Find all solutions (**x,y**).**

    What often goes wrong is that students do not find all solutions. It goes wrong when they remove a term on both sides of the equation. Be careful with this! For example, consider the equation $(x+1)y^2 = (x+1)y$. We can't say right away that this is equivalent to $y^2 = y$. If $(x+1) = 0$ this doesn't have to be true! So if you stripe away some term, separately consider the case where that term is 0.

    By the way, we are only looking for real critical points. If you find $x^2 = -2$, you can simply say "no solutions" for that case.
- **For every critical point, examine the type and stability.**
  - **Define** $dx/dt = F(x,y)$ **and** $dy/dt = G(x,y)$**.**
  - **Find the Jacobian matrix**

$$J = \begin{bmatrix} F_x(x_{cr}, y_{cr}) & F_y(x_{cr}, y_{cr}) \\ G_x(x_{cr}, y_{cr}) & G_y(x_{cr}, y_{cr}) \end{bmatrix}.$$

  - **Find the eigenvalues** λ_1 **and** λ_2 **of** J**.** Don't remember how to do this? See the previous chapter. (Although we ignored complex critical points, we don't ignore complex eigenvalues. So if $\lambda^2 + 1 = 0$, we have $\lambda_1 = i$ and $\lambda_2 = -i$.)
  - **If** $\lambda_1 = \lambda_2$**, also find the eigenvectors.** In other cases you don't need to know the eigenvectors to determine the stability.
  - **Use the tables to find the critical point type.** (The tables are on the next page.) You won't get these tables on your exam. Learn them by heart! Note that you need to write down the type and stability, both for the linear system and the almost linear system. So for every critical point, you need to find 4 pieces of data.

  What often goes wrong here is that people forget critical points. When the stability is determined for one point, people think they are done. Don't forget the second (and incidentally third) critical point!

4.3 Stability Tables

Eigenvalues	**Type of Critical Point**	**Stability**
$r_1 > r_2 > 0$	Nodal Source (Node)	Unstable
$r_1 < r_2 < 0$	Nodal Sink (Node)	Asymptotically Stable
$r_2 < 0 < r_1$	Saddle Point	Unstable
$r_1 = r_2 > 0$, independent eigenvectors	Proper node	Unstable
$r_1 = r_2 < 0$, independent eigenvectors	Proper node	Asymptotically Stable
$r_1 = r_2 > 0$, missing eigenvector	Improper node	Unstable
$r_1 = r_2 < 0$, missing eigenvector	Improper node	Asymptotically Stable
$r_1 = \lambda + \mu i$, $r_2 = \lambda - \mu i$, $\lambda > 0$	Spiral point	Unstable
$r_1 = \lambda + \mu i$, $r_2 = \lambda - \mu i$, $\lambda < 0$	Spiral point	Asymptotically Stable
$r_1 = \lambda + \mu i$, $r_2 = \lambda - \mu i$, $\lambda = 0$	Center	Stable

Stability for the linear system.

Eigenvalues of linear system	**Type of Critical Point**	**Stability**
$r_1 > r_2 > 0$	Nodal Source (Node)	Unstable
$r_1 < r_2 < 0$	Nodal Sink (Node)	Asymptotically Stable
$r_2 < 0 < r_1$	Saddle Point	Unstable
$r_1 = r_2 > 0$, independent eigenvectors	Node or Spiral Point	Unstable
$r_1 = r_2 < 0$, independent eigenvectors	Node or Spiral Point	Asymptotically Stable
$r_1 = r_2 > 0$, missing eigenvector	Node or Spiral Point	Unstable
$r_1 = r_2 < 0$, missing eigenvector	Node or Spiral Point	Asymptotically Stable
$r_1 = \lambda + \mu i$, $r_2 = \lambda - \mu i$, $\lambda > 0$	Spiral point	Unstable
$r_1 = \lambda + \mu i$, $r_2 = \lambda - \mu i$, $\lambda < 0$	Spiral point	Asymptotically Stable
$r_1 = \lambda + \mu i$, $r_2 = \lambda - \mu i$, $\lambda = 0$	Center or Spiral Point	Indeterminate

Stability for the almost linear system.

5 Power Series

5.1 When to use?

When it is asked. The question starts with "Find the general solution of the following differential equation by means of a power series expansion..."

5.2 Plan of Approach

- **Write down**

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n \left(x - x_0\right)^n , \\
y' &= \sum_{n=0}^{\infty} a_{n+1}(n+1) \left(x - x_0\right)^n , \\
y'' &= \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1) \left(x - x_0\right)^n .
\end{aligned}
$$

  You can fill in the value for x_0 here right away. By the way, x_0 is always given in the question.
- **Fill y, y' and y'' in into the differential equation.**
- **Pull all factors $(x - x_0)$ within the sum.** This will only cause the power above the $(x - x_0)$ to change. For example, $(x-2)^2 \sum_{n=0}^{\infty} a_n (x-2)^n$ becomes $\sum_{n=0}^{\infty} a_n (x-2)^{n+2}$.
- **Set the power in the sum back to $(x - x_0)^n$.** Do this by changing the starting number. Also the other occurrences of n will now change. For example, $\sum_{n=0}^{\infty} a_n (x-2)^{n+2}$ becomes $\sum_{n=2}^{\infty} a_{n-2} (x-2)^n$.
- **Make the starting number of the sums equal by pulling out terms.** Just look for the highest starting number there is. Then make the starting numbers of all sums equal to that number. For example, if the highest starting number is 2, then $\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)(x-2)^n$ becomes $2a_2 + 6a_3(x-2) + \sum_{n=2}^{\infty} a_{n+2}(n+2)(n+1)(x-2)^n$.
- **Join the sums together.** That was the entire goal. Now that the powers and the starting numbers are equal, we're finally allowed to do this.
- **Equate the coefficients.** First look at all terms without a factor $(x - x_0)$. That gives you one equation. If present, look at all terms with a factor $(x - x_0)^1$. This might give you an equation too. Finally look at the general case: all terms with a factor $(x - x_0)^n$. This gives you another equation. Solve this equation for the highest indexed coefficient. For example, if there is a_{n+2} and a_n in the equation, solve for a_{n+2}. Now you have found the recurrence relation.
- **Use the recurrence relation to find the first few coefficients.** Sometimes you can find exact values. Sometimes you need to express the coefficients in a_0 (and incidentally in a_1 too).
- **Write down the first few terms.** Do this in the form $y = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \ldots$.

This plan of approach can initially seem vague. If that is the case, I would advice you to look at some example solutions. The ones on the Aerostudents website follow this plan of approach. See if you can identify the individual steps in it.

6 Eigenfunctions

6.1 When to use?

When it is asked. Once every few years, they ask directly for eigenfunctions of a certain differential equation. Then you need to use this.

As part of a Fourier Series Application question. We'll address those later.

6.2 Plan of Approach

- **Find the eigenfunctions y_n and eigenvalues λ_n.** The equation you need to determine the eigenfunctions from is virtually always $y'' + \lambda y = 0$. Its solutions are the eigenfunctions y_n. The corresponding values of λ are the eigenvalues λ_n.
  - **Consider three cases: $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$.**
  - **For every case, do the following.**
    - **Find the general solution set.**
      For $\lambda < 0$ this is $y = c_1 e^{\mu x} + c_2 e^{-\mu x}$, with $\mu = \sqrt{-\lambda}$. You usually won't find any non-trivial solutions for this case.
      For $\lambda = 0$ this is $y = c_1 x + c_2$. In 50% of the cases you find a solution for this case. It is customary to call this solution y_0.
      For $\lambda > 0$ this is $y = c_1 \cos \mu x + c_2 \sin \mu x$, with now $\mu = \sqrt{\lambda}$. You usually find infinitely many solutions for this, each depending on some constant n. It is customary to call these solutions y_n.
    - **Apply the boundary conditions.** Use these conditions to find the two constants.
    - **Write down all non-trivial solutions.** The trivial solution is $y = 0$. We're not looking for that solution. So if you only find $y = 0$, you don't have to write down anything. (Note that $y = c$ is not trivial, so we do want such a solution!) By the way, if you have any undetermined constants in your solution, just give them some value.

  So now we've got the eigenfunctions! The above three steps might seem easy, but they are a lot of work with many opportunities to make mistakes. Do not underestimate them.
- **Normalize the eigenfunctions.** You only have to do this when they specifically ask for the normalized eigenfunctions. Otherwise, just skip this step. So how do we normalize eigenfunctions?
  - **Write down**
    $$\int_0^1 (c_n y_n(t))^2 \, dt = 1.$$
  - **Solve this equation for the constant c_n.** (If there is an eigenfunction y_0, evaluate the above integral separately for that case.)
  - **Write down the normalized eigenfunctions $c_n y_n(t)$.**

7 Fourier Series

7.1 When to use?

When it is asked. The question then starts with "Find the Fourier Series of the function..."

As a part of Fourier Series Applications. We'll address those later.

7.2 Plan of Approach

- **First assume**
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_m \sin\left(\frac{n\pi x}{L}\right)\right).$$
- **Find the period T and the value $L = T/2$.** The period T is explicitly given in the question, by $f(x+T) = f(x)$.
- **Find a_0 using**
$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx.$$
- **Find a_n using**
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx.$$
- **Find b_n using**
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx.$$
- **Fill a_n and b_n in, into the expression for $f(x)$.**

This isn't very difficult, so usually very few points are rewarded for such a question. The most difficult part is the integration. Usually integration by parts is needed.

8 Fourier Series Applications

8.1 When to use?

When it is asked. The question starts with "Find a formal solution (...) $u(x,t)$ of the..."

8.2 Plan of Approach

There are three types of questions for this: Heat conduction problems (HE), vibrating string problems (VS) and the Laplace equation (LE). We will discuss the first two here. The Laplace equation is a bit too difficult to discuss right away, so we will only discuss it in an addendum. I'd advise you to only spend time on it when you have sufficient time left.

- **Look at the form of the differential equation to determine its type.** If the form is $\alpha^2 u_{xx} = u_t$, then we have a heat conduction problem. If the form is $a^2 u_{xx} = u_{tt}$, we have a vibrating string. If we have $u_{xx} + u_{yy} = 0$, we are dealing with the Laplace equation. We will treat this case later.
- **Assume that**
$$u(x,t) = X(x)T(t).$$
- **Derive that**

Heat conduction problem	Vibrating string problem	
$\alpha^2 X''T = XT'$	$a^2 X''T = XT''$	
$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T} = -\lambda$	$\frac{X''}{X} = \frac{1}{a^2}\frac{T''}{T} = -\lambda$	
$X'' + \lambda X = 0$	$X'' + \lambda X = 0$	(1)
$T' + \alpha^2 \lambda T = 0$	$T'' + a^2 \lambda T = 0$	(2)

- **Transform the boundary conditions.** Although we have a differential equation for X, we have boundary conditions like for example $u(l,t) = 0$. We first need to transform them to $X(x)$. We do this as follows. We know that $u(l,t) = X(l)T(t) = 0$ for every t. So $X(l) = 0$ or $T(t) = 0$. Since $T(t) = 0$ for every t would give the trivial solution, we have $X(l) = 0$. All boundary conditions can be transformed in this way. (Note that the initial condition $u(x,0) = f(x)$ can not be transformed like this. We will use this one later.)
- **Use equation (1) and the boundary conditions to find the eigenfunctions $X_n(x)$ and eigenvalues λ_n.** So now we need to find eigenvalues. We already know how to do this. We do not have to normalize those. (In fact, we may not even do this.)
- **Substitute λ_n in (2) to find $T_n(t)$.** In case of a vibrating string problem, we also have to use an initial boundary condition (one for T) to find the solution.
- **Calculate $u_n(x,t) = X_n(x)T_n(t)$.**
- **Write down the form of the solution**
$$u(x,t) = c_0 \frac{u_0(x,t)}{2} + \sum_{m=1}^{\infty} c_n u_n(x,t).$$
If $u_0(x,t)$ doesn't exist (which is the case if $X_0(x)$ doesn't exist), then just ignore that term.
- **Use the initial condition $f(x) = u(x,0)$.** Just fill in $t = 0$ in the earlier found expression for $u(x,0)$.

- **If**

$$f(x) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos\left(\frac{\alpha x}{l}\right)$$

for some value α, then write down

$$c_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{\alpha x}{l}\right) dx.$$

But if

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{\alpha x}{l}\right)$$

then write down

$$c_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{\alpha x}{l}\right) dx.$$

Note that you only have to write down the equation for c_n. You do not have to solve it. So in the end your solution consists of two parts: The relation for $u(x,t)$ with unknown coefficients c_n, and an equation showing how c_n can be derived from $f(x)$.

8.3 Addendum: The Laplace Equation

The Laplace equation is very similar to the vibrating string problem. So we will simply note the differences, and for the rest refer to the plan of approach for vibrating strings.

When dealing with the Laplace Equation, you will have an equation $u_{xx} + u_{yy} = 0$ and some boundary conditions. Let's take a close look at those boundary conditions.

Two boundary conditions are given for any x at constant y. (For example, $u(x,0) = 0$ and $u(x,h) = 0$.) Two other conditions are given for any y at constant x. (For example $u(0,y) = 0$ and $u(w,y) = f(y)$.) In these boundary conditions, there is always a function $f(x)$ or $f(y)$.

If we have a boundary condition equal to $f(x)$, then consider this as a vibrating string (VS) problem where

- t (in VS) has become y (in LE).
- The differential equations have become $X'' + \lambda X = 0$ (1) and $Y'' - \lambda Y = 0$ (2).

If we have a boundary condition equal to $f(y)$ (as in our example conditions), then we also consider this as a vibrating string problem. But now

- t (in VS) has become x (in LE).
- x (in VS) has become y (in LE). (So we solve for $Y_n(y)$ first and then look for $X_n(x)$.)
- The differential equations have become $Y'' + \lambda Y = 0$ (1) and $X'' - \lambda X = 0$ (2).