Phase Portraits and Stability

1 Phase Portraits for Linear Systems

1.1 Phase Portraits

Many differential equations can't be solved analytically. If we have a system, described by a differential equation, we still want to get an idea of how that system behaves. Let's consider the system

$$\mathbf{x}' = A\mathbf{x}. \tag{1.1}$$

If at some given $\mathbf{x}$ the value of $\mathbf{x}' = 0$, the system doesn't change. In that case the vector $\mathbf{x}$ is an **equilibrium solution**, also called a **critical point**. These points are often of special importance. However, for a consistent matrix A ($\det A = 0$) only the point $\mathbf{0}$ is a critical point. In the rest of this chapter we assume A is consistent.

Let's suppose we have found a vector function $\mathbf{x}(t)$ that satisfies equation 1.1. In case A is a 2×2 matrix, such a function can be viewed as a parametric representation for a curve in the x_1x_2-plane. Such a curve is called a **trajectory**, the x_1x_2-plane is called the **phase plane** and a representative set of trajectories is called a **phase portrait**.

Phase portraits can have many shapes. To get a general idea of them, we examine phase portraits of first-order linear differential equations, which we have already studied in detail. In the following paragraphs we will only be looking at a 2×2 matrix A.

1.2 Real unequal eigenvalues of the same sign

If the matrix A has two real unequal eigenvalues of the same sign, then the solution of system 1.1 is

$$\mathbf{x} = c_1\xi_1 e^{r_1 t} + c_2\xi_2 e^{r_2 t}. \tag{1.2}$$

If r_1 and r_2 are both negative, then as $t \to \infty$, $\mathbf{x} \to \mathbf{0}$. In this case the point $\mathbf{0}$ is called a **nodal sink**. All trajectories go to this sink.

If, however, r_1 and r_2 are both positive, then as $t \to \infty$, $\mathbf{x}$ diverges away from $\mathbf{0}$. Now the point $\mathbf{0}$ is called a **nodal source**. All trajectories go away from this source.

Another thing can be noted for these kinds of solutions. If $r_1 > r_2 > 0$ or $r_1 < r_2 < 0$, then ξ_1 has the most influence on the trajectory of $\mathbf{x}$. Therefore the trajectory will be mostly tangent to ξ_1.

1.3 Real eigenvalues of opposite sign

If the matrix A has two eigenvalues of opposite sign, then the solution still has the form of equation 1.2. However, there won't be a sink or a source, but a **saddle point**. Let's suppose $r_1 > 0 > r_2$. As $t \to \infty$ the part of the solution $\xi_2 e^{r_2 t}$ disappears and $\mathbf{x}$ will be (approximately) a multiple of ξ_1. If, however, $c_1 = 0$ (which is the case if $\mathbf{x}_0$ is a multiple of ξ_2), then the trajectory of $\mathbf{x}$ does converge to $\mathbf{0}$.

1.4 Equal eigenvalues with independent eigenvectors

If A has two equal eigenvalues (so an eigenvalue with multiplicity 2) with independent eigenvectors, the solution will still be of the form of equation 1.2. In this case $r_1 = r_2 = r$. If $r < 0$, then all trajectories will directly converge to $\mathbf{0}$ in a straight line. If $r > 0$ all trajectories will diverge away from $\mathbf{0}$ in a straight line. As the phase portrait therefore looks like a star, the point $\mathbf{0}$ is called a **star point**. It's also called a **proper node**.

1.5 Equal eigenvalues with a missing eigenvector

If A has only one eigenvalue with one eigenvector, then the solution will be of the form

$$\mathbf{x} = c_1 \xi e^{rt} + c_2 (\xi t e^{rt} + \eta e^{rt}). \tag{1.3}$$

This can also be written as

$$\mathbf{x} = ((c_1 \xi + c_2 \eta) + c_2 \xi t)\, e^{rt} = \mathbf{y} e^{rt}. \tag{1.4}$$

Here the vector $\mathbf{y}$ largely determines the direction of the vector, while e^{rt} determines the magnitude. As $t \to \infty$ the part $c_2 \xi t$ will increase, so the direction of $\mathbf{y}$ will be in the direction of ξ. It is also interesting to note that at $t = 0$ always $\mathbf{x} = c_1 \xi + c_2 \eta$.

The trajectories will always converge to $\mathbf{0}$ if $r < 0$ and diverge from it if $r > 0$. This critical point is in this case called an **improper** or **degenerate node**.

1.6 Complex eigenvalues

Let's suppose A has only complex eigenvalues $\lambda \pm \mu i$ (with $\lambda \neq 0$ and $\mu > 0$). The system is typified by

$$\mathbf{x}' = \begin{pmatrix} \lambda & \mu \\ -\mu & \lambda \end{pmatrix} \mathbf{x}. \tag{1.5}$$

We can transfer this system to polar coordinates, such that $r = \sqrt{x_1^2 + x_2^2}$ and $\theta = \tan^{-1} x_2/x_1$. Solving the system will give

$$r = c e^{\lambda t} \qquad \text{and} \qquad \theta = -\mu t + \theta_0. \tag{1.6}$$

As t increases, the trajectory will spiral around the origin, which is thus called a **spiral point**. If $r < 0$ it will spiral inward, so then the origin is a **spiral sink**. If $r > 0$ it will spiral outward, so then the origin is a **spiral source**.

Let's now look at the same situation, except we assume that $\lambda = 0$. In this case r is constant. So the trajectories are circles, with center at the origin. The origin is therefore called a **center**.

1.7 Intermediate summary

Eigenvalues	**Type of Critical Point**	**Stability**
$r_1 > r_2 > 0$	Nodal Source (Node)	Unstable
$r_1 < r_2 < 0$	Nodal Sink (Node)	Asymptotically Stable
$r_2 < 0 < r_1$	Saddle Point	Unstable
$r_1 = r_2 > 0$, independent eigenvectors	Proper node/Star point	Unstable
$r_1 = r_2 < 0$, independent eigenvectors	Proper node/Star point	Asymptotically Stable
$r_1 = r_2 > 0$, missing eigenvector	Improper node	Unstable
$r_1 = r_2 < 0$, missing eigenvector	Improper node	Asymptotically Stable
$r_1 = \lambda + \mu i$, $r_2 = \lambda - \mu i$, $\lambda > 0$	Spiral point	Unstable
$r_1 = \lambda + \mu i$, $r_2 = \lambda - \mu i$, $\lambda < 0$	Spiral point	Asymptotically Stable
$r_1 = \lambda + \mu i$, $r_2 = \lambda - \mu i$, $\lambda = 0$	Center	Stable

Table 1: Overview of behavior of linear systems.

All that we have discussed in this part can be summarized in a table. This is done in table 1. In this table is also a column concerning stability. This topic will be discussed in the next part.

2 Stability

2.1 Autonomous systems

Previously we have looked at systems of linear first order differential equations. Linear meant that only x_1, x_2 and such appeared in the equation, and not something like x_1^2 or $\ln x_1$. First order meant that only x' and not x'' or x''' appeared.

Now let's widen our view a bit more. Let's also consider systems of nonlinear first order differential equation. But we won't consider all nonlinear systems. We only consider systems that can be written as

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}). \tag{2.1}$$

Here the function $\mathbf{f}(\mathbf{x})$ does not depend on t. So given any position vector $\mathbf{x}$, the velocity $\mathbf{x}'$ will always be the same. In other words, the phase portrait of the system is constant in time. Such a system is said to be **autonomous**. An example of such a system is the linear system $\mathbf{x}' = A\mathbf{x}'$ (with A a constant matrix).

2.2 Stability Definitions

A point for which $\mathbf{x}' = 0$ is called a **critical point**. Now imagine a circle with radius ϵ around a critical point $\mathbf{x}_{cr}$. Also imagine a second smaller circle with radius δ. Let's take a point $\mathbf{x}_0$ in the δ-circle. If the trajectory of that point leaves the ϵ-circle, then the critical point is called **unstable**. If, however, the trajectory of every starting point $\mathbf{x}_0$ in the δ-circle remains entirely within the ϵ-circle, the critical point is called **stable**.

If a point is stable, it can also be **asymptotically stable**. This is the case if also

$$\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}_{cr}, \tag{2.2}$$

meaning that the trajectory of the starting point $\mathbf{x}_0$ goes to $\mathbf{x}_{cr}$. If a trajectory forms, for example, a circle around the critical point, then it is stable but not asymptotically stable.

For asymptotically stable points, certain trajectories approach the origin. If all trajectories approach the origin, then the critical point is said to be **globally asymptotically stable**. Linear systems with $\det A = 0$ always have only 1 critical point $\mathbf{x}_{cr} = \mathbf{0}$. If $\mathbf{0}$ is then stable, it is also globally asymptotically stable.

2.3 Almost linear systems

Let's now consider an **isolated** critical point $\mathbf{x}_{cr}$. A critical point is isolated if there are no other critical points very close next to it. For simplicity, let's assume $\mathbf{x}_{cr} = \mathbf{0}$.

An autonomous nonlinear system can be written like

$$\mathbf{x}' = A\mathbf{x} + \mathbf{g}(\mathbf{x}). \tag{2.3}$$

If $\mathbf{g}(\mathbf{x})$ is small, then this system is close to the linear system $\mathbf{x}' = A\mathbf{x}$. More precisely, the system is said to be an **almost linear system** if $\mathbf{g}$ has continuous partial derivatives and

$$\frac{|\mathbf{g}(\mathbf{x})|}{|\mathbf{x}|} \to 0 \qquad \text{as} \qquad \mathbf{x} \to \mathbf{0}. \tag{2.4}$$

If we define $r = |\mathbf{x}|$, then this can be written in scalar form as

$$\frac{g_1(\mathbf{x})}{r} \to 0, \ldots, \frac{g_2(\mathbf{x})}{r} \to 0 \qquad \text{as} \qquad r \to 0. \tag{2.5}$$

It can be shown that if $\mathbf{g}(\mathbf{x})$ is twice differentiable, then the system is always an almost linear system.

Previously we have treated stability for linear systems. An overview was shown in table 1. The stability for an almost linear system is shown in table 2. It is important to note the difference. For most eigenvalues the stability and the type of critical point stay the same. There are a few exceptions.

Let's consider the case when $r_1 = \lambda + \mu i$ and $r_2 = \lambda - \mu i$ with $\lambda = 0$. If small deviations occur, it is likely that $\lambda \neq 0$. So the critical point has become a spiral point. The other difference occurs when $r_1 = r_2$. But now there are several more types to which the critical point can change.

Eigenvalues of linear system	**Type of Critical Point**	**Stability**
$r_1 > r_2 > 0$	Nodal Source (Node)	Unstable
$r_1 < r_2 < 0$	Nodal Sink (Node)	Asymptotically Stable
$r_2 < 0 < r_1$	Saddle Point	Unstable
$r_1 = r_2 > 0$, independent eigenvectors	Node or Spiral Point	Unstable
$r_1 = r_2 < 0$, independent eigenvectors	Node or Spiral Point	Asymptotically Stable
$r_1 = r_2 > 0$, missing eigenvector	Node or Spiral Point	Unstable
$r_1 = r_2 < 0$, missing eigenvector	Node or Spiral Point	Asymptotically Stable
$r_1 = \lambda + \mu i$, $r_2 = \lambda - \mu i$, $\lambda > 0$	Spiral point	Unstable
$r_1 = \lambda + \mu i$, $r_2 = \lambda - \mu i$, $\lambda < 0$	Spiral point	Asymptotically Stable
$r_1 = \lambda + \mu i$, $r_2 = \lambda - \mu i$, $\lambda = 0$	Center or Spiral Point	Indeterminate

Table 2: Overview of behavior of almost linear systems.

2.4 Periodic Solutions

It may occur that autonomous systems $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ have periodic solutions. Such solutions satisfy

$$\mathbf{x}(t+T) = \mathbf{x}(t), \tag{2.6}$$

where $T > 0$ is called the period. This period is usually the smallest value such that the above relation is satisfied. The corresponding trajectories form closed curves. If other non-closed curves spiral towards this curve, then it is called a **limit cycle**.

If all trajectories that start near the limit cycle (both inside and outside) spiral towards it, then it is called **asymptotically stable**. If all trajectories spiral outward, then the limit cycle is called **unstable**. If trajectories on one side spiral inward and on the other side spiral outward, it is called **semistable**. It may also occur that other trajectories neither spiral to nor away from a limit cycle. In that case the limit cycle is called **stable**.

It is usually difficult to determine whether limit cycles exist in a system. However, there are a few rules that may help. A closed trajectory always encloses at least one critical point. If it encloses only one critical point, then that critical point can not be a saddle point.

We can also consider the value

$$\frac{df_1(\mathbf{x})}{dx_1} + \frac{df_2(\mathbf{x})}{dx_2} + \ldots + \frac{df_n(\mathbf{x})}{dx_n}. \tag{2.7}$$

If this has the same sign throughout a simply connected region D (meaning that D has no holes), then there is no closed trajectory lying entirely in D.

Suppose a region R contains no critical points. If a certain trajectory lies entirely in R, then this trajectory either is a closed trajectory or spirals towards one. In either case, there is a closed trajectory present. This last rule is called the **Poincaré-Bendixson Theorem**.