Fourier Series

1 Function types

1.1 Periodic functions

In the last chapter we have already mentioned periodic functions, but we will briefly repeat that here. A function f is **periodic** is

$$f(x+T) = f(x) \tag{1.1}$$

for every x. Here $T > 0$ is the period. The smallest value of T is called the **fundamental period** of f.

If f and g are two functions with equal period T, then their product fg and any linear combination $c_1 f + c_2 g$ also have period T.

1.2 Orthogonal functions

The inner product of two functions u and v on the interval $I : \alpha \leq x \leq \beta$ is defined as

$$(u, v) = \int_\alpha^\beta u(x)v(x)dx. \tag{1.2}$$

The function u and v are said to be **orthogonal** on I if $(u, v) = 0$. A set of functions is said to be **mutually orthogonal** if each distinct pair of functions is orthogonal.

Now consider the functions

$$u_m(x) = \cos\frac{m\pi x}{L} \qquad \text{and} \qquad v_n(x) = \sin\frac{n\pi x}{L}. \tag{1.3}$$

It can now be shown that $(u_m, v_n) = 0$ for every m, n on an interval $-L \leq x \leq L$. Also, if $m \neq n$, then $(u_m, u_n) = (v_m, v_n) = 0$ on the same interval. On the contrary, if $m = n$, then $(u_m, u_n) = (v_m, v_n) = L$ (also on $-L \leq x \leq L$).

1.3 Even and odd functions

A function f is said to be an **even function** if

$$f(-x) = f(x) \tag{1.4}$$

for every x. An example is $f(x) = x^2$. Let's take a graph of a function and mirror it along the y-axis. If we get back the same graph is we put in, then it is an even function.

A function is even if Similarly, a function f is said to be an **odd function** is

$$f(-x) = -f(x) \tag{1.5}$$

for every x. So odd functions always have $f(0) = 0$. An example is $f(x) = x$ or $f(x) = x^3$. Let's take a graph of a function and rotate it 180° about the origin. If we get back the same graph as we put in, then it is an odd function.

If f and g are even functions and p and q are odd functions, then

- $c_1 f + c_2 g$ and fg are even.
- $c_1 p + c_2 q$ is odd. However pq is even.
- fp is odd. $f + p$ is neither even nor odd.
- $\int_{-L}^{L} f(x)dx = 2\int_{-L}^{0} f(x)dx = 2\int_0^L f(x)dx$.
- $\int_{-L}^{L} p(x)dx = 0$.

1.4 Eigenfunctions

The difference between initial value problems and boundary value problems was previously discussed. Initial value problems concerned differential equations where y and y' were given at a certain point, while boundary problems have y given at two different points. While there usually is a unique solution to initial value problems, there is often not a unique solution to boundary problems. For boundary value problems here are either 0, 1 or infinitely many solutions.

Let's take a look at a boundary value problem concerning a homogeneous differential equations with a certain unknown constant. For example, let's consider

$$y'' + \lambda y = 0, \qquad y(0) = 0, \qquad y(\pi) = 0. \tag{1.6}$$

Here λ is an unknown constant. The above differential equation has solution $y = 0$ for all λ. This is the **trivial solution** in which we are not interested.

Instead, it turns out that for some values of λ there are infinitely many solutions. These values of λ for which nontrivial solutions occur are called **eigenvalues**. The nontrivial solutions are called **eigenfunctions**. For the above example, the eigenvalues turn out to be

$$\lambda_1 = 1, \qquad \lambda_2 = 4, \qquad \lambda_3 = 9, \qquad \ldots, \qquad \lambda_n = n^2, \tag{1.7}$$

where the corresponding eigenfunctions are

$$y_1(x) = \sin x, \qquad y_2(x) = \sin 2x, \qquad y_3(x) = \sin 3x, \qquad \ldots, \qquad y_(x) = \sin nx. \tag{1.8}$$

Just like in linear algebra, any linear combination of solutions (eigenfunctions) is also a solution to the differential equation.

2 Fourier Series

2.1 Introduction to Fourier series

Let's suppose we have a continuous periodic function f with period $T = 2L$. In that case, it can be expressed as a **Fourier series**, being an infinite sum of sines and cosines that converges to $f(x)$. This goes according to

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \right). \tag{2.1}$$

Here the coefficients $a_0, a_1, \ldots$ and $b_1, b_2, \ldots$ need to be determined. It can be shown that

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx \qquad \text{and} \qquad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx. \tag{2.2}$$

If f is not a periodic function, it can not entirely be expressed as a Fourier series. However, the part of f on interval $-L \leq x \leq L$ can be expressed as a Fourier series, according to the above procedure.

2.2 Discontinuous functions

If the periodic function f is not a continuous function but a piecewise continuous function, it is still possible to express the function using a Fourier series. However, at positions of discontinuity (where the graph makes a "jump") the Fourier series never really converges to $f(x)$. This behavior is known as the **Gibbs phenomenon**.

Another interesting phenomenon always occurs. If the value of f at a certain point x jumps from y_1 to y_2, then the Fourier series at point x always returns a value of $\frac{y_1+y_2}{2}$.

For functions that are not even piecewise continuous (such as for example $\frac{1}{x}$), Fourier series often do not converge. Therefore Fourier series are hardly ever applicable for such functions.

2.3 Sine and cosine series

Let's suppose we have an even function f and want to find the corresponding Fourier series. When we are trying to find the b-coefficients, we will be integrating over $f(x)\sin\frac{n\pi x}{L}$. Since $\sin(x)$ is an odd function, this product is also odd. We know that an integral from $-L$ to L over an odd function will give 0 as result. Therefore $b_n = 0$ for every n.

Since b_n is always zero in the Fourier series of even functions, all the terms with sines disappear. Such a series thus only consists of cosines and is therefore called a **Fourier cosine series**.

Now let's suppose f is odd. If we make use of the fact that $\cos(x)$ is an even function, we will find that $a_n = 0$ for every n. Therefore the Fourier series for an odd function consists only of sines. and is thus called a **Fourier sine series**.

3 Heat Conduction Problems

3.1 Heat conduction in a rod

Let's consider a thin rod, ranging from $x = 0$ to $x = L$ of which the sides are insulated. Heat can only enter via the two edges. The temperature u is now only a function of x and t. To solve this problem, we need to use the **heat conduction equation**

$$\alpha^2 u_{xx} = u_t, \tag{3.1}$$

where α^2 is the **thermal diffusivity** (a material property). There are several boundary values for this problem. First there is the initial state of the rod $u(x, 0)$. This is simply equal to some known function $f(x)$, so

$$u(x, 0) = f(x). \tag{3.2}$$

3.2 Rod with open ends at $u = 0$

If heat can pass in/out of the rod at the edges, then the edges will always have constant temperature. For simplicity's sake we will assume that this temperature is 0 for both edges. Later we will consider the case in which this is not true. So the other boundary conditions are

$$u(0, t) = 0, \qquad u(L, t) = 0. \tag{3.3}$$

This differential equation is hard to solve. So to solve it, we make an assumption. We assume that the function $u(x, t)$ can be written as

$$u(x, t) = X(x)T(t). \tag{3.4}$$

So we assume it is a product of a function of x and a function of t. Using this assumption we can separate the problem in two differential equations

$$X'' + \lambda X = 0, \qquad T' + \alpha^2\lambda T = 0, \tag{3.5}$$

where λ is an unknown separation constant. Now let's look at the first equation and combine it with the second boundary equation. Ignoring the trivial solution $X = 0$, we will find that the solutions are the eigenfunctions

$$X_n(x) = \sin \frac{n\pi x}{L}, \qquad n = 1, 2, 3, \dots, \tag{3.6}$$

associated with the eigenvalues $\lambda_n = \frac{n^2\pi^2}{L^2}$. Inserting these values in the second differential equation gives

$$T_n(x) = e^{-\frac{n^2\pi^2\alpha^2 t}{L^2}}. \tag{3.7}$$

We can now find a solution $u_n = X_n T_n$. The general solution is then any linear combination of the specific solutions, so

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t) = \sum_{n=1}^{\infty} c_n X_n(t) T_n(t) = \sum_{n=1}^{\infty} c_n e^{-\frac{n^2\pi^2\alpha^2 t}{L^2}} \sin \frac{n\pi x}{L}. \tag{3.8}$$

But we haven't satisfied the first boundary conditions yet. Using $u(x,0) = f(x)$ we can find the coefficients c_n. The procedure for this is identical to finding a sine series for $f(x)$. From this follows that

$$c_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx = \frac{2}{L}\int_{0}^{L} f(x) \sin \frac{n\pi x}{L} dx. \tag{3.9}$$

3.3 Rod with open ends not at $u = 0$

But what if the edges don't have $u = 0$? Let's suppose $u(0,t) = T_1$ and $u(L,t) = T_2$. Now the problem is not homogeneous anymore. So we will first make it homogeneous. We can see that

$$g(x) = T_1 + (T_2 - T_1)\frac{x}{L} \tag{3.10}$$

is a solution. In fact, this is the limit solution as $t \to \infty$. If we now not use the initial condition $u(x,0) = f(x)$, but instead use $u(x,0) = f(x) - g(x)$, then we once more have a homogeneous problem. Then the coefficients can be found using

$$c_n = \frac{2}{L}\int_{0}^{L} (f(x) - g(x)) \sin \frac{n\pi x}{L} dx = \frac{2}{L}\int_{0}^{L} \left(f(x) - T_1 - (T_2 - T_1)\frac{x}{L}\right) \sin \frac{n\pi x}{L} dx. \tag{3.11}$$

Note that this is equal to what we saw last chapter, except we replaced $f(x)$ by $f(x) - g(x)$. The corresponding solution then becomes

$$u(x,t) = g(x) + \sum_{n=1}^{\infty} e^{-\frac{n^2\pi^2\alpha^2 t}{L^2}} \sin \frac{n\pi x}{L} = T_1 + (T_2 - T_1)\frac{x}{L} + \sum_{n=1}^{\infty} e^{-\frac{n^2\pi^2\alpha^2 t}{L^2}} \sin \frac{n\pi x}{L}. \tag{3.12}$$

This is also equal to the solution of the last paragraph, except that we put the part $g(x)$ in front of it.

3.4 Rod with insulated ends

What happens if the ends of the rod are insulated? In that case they are no longer a constant temperature. Instead, in that case $X'(0) = 0$ and $X'(L) = 0$. The solution process is more or less similar to that of a rod without insulated ends. But instead of finding a sine series, the result now is a cosine series, given by

$$u(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n e^{\frac{-n^2\pi^2\alpha^2 t}{L^2}} \cos \frac{n\pi x}{L}. \tag{3.13}$$

The coefficients are given by the equation

$$c_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx. \tag{3.14}$$

A funny thing to note is that as $t \to \infty$, the temperature in the entire bar becomes equal to $c_0/2$. It can be shown that this just happens to be the average temperature of the bar.

4 The Wave Equation

4.1 Vibrations of an elastic string

Let's examine an elastic string, connected at $x = 0$ and $x = L$. Every point x at a time t has a deflection $u(x, t)$. If the string is given an initial deflection, it will vibrate. If damping effects are neglected, the governing equation is

$$a^2 u_{xx} = u_{tt}, \tag{4.1}$$

where a^2 is a constant. This equation is called the **wave equation**. One of the boundary conditions of this problem is rather trivial. As the ends of the string are fixed, we know that

$$u(0, t) = 0, \qquad u(L, t) = 0. \tag{4.2}$$

To solve the problem, we also need to know the initial position $u(x, 0)$. But this won't suffice to solve the problem. Also the initial velocity $u_t(x, 0)$ needs to be known. These boundary conditions can be expressed as

$$u(x, 0) = f(x), \qquad u_t(x, 0) = g(x). \tag{4.3}$$

4.2 String with initial displacement

Suppose the string has been given an initial displacement, but no initial velocity. So $u_t(x, 0) = 0$. This implies that $T'(0) = 0$. Solving the wave equation is very similar to solving the heat conduction equation. The solution for $X(x)$ will be exactly the same. The solution for $T(t)$ will be

$$T_n(t) = \cos\frac{n\pi at}{L}. \tag{4.4}$$

The final solution will then have the form

$$u(x, t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L}\cos\frac{n\pi at}{L}. \tag{4.5}$$

The constants c_n can be found using

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx. \tag{4.6}$$

For a fixed value of n the expression $\sin\frac{n\pi x}{L}\cos\frac{n\pi at}{L}$ is periodic with period $T = \frac{2L}{na}$ or equivalently having the frequency $\frac{na}{2L}$. This frequency is called the **natural frequency** of the string - being the frequency at which it will freely vibrate.

While vibrating, certain displacement patterns appear. Each displacement pattern is called a **natural mode of vibration** and is periodic in space. The corresponding spacial period $\frac{2L}{n}$ is called the **wavelength** of the mode.

4.3 String with initial velocity

Now let's examine a string without initial displacement, but with initial velocity. So this time $u(x,0) = 0$, implying that $T(0) = 0$. Now we will find that

$$T_n(t) = \sin \frac{n\pi at}{L}. \tag{4.7}$$

Working out the results will give

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L} \sin \frac{n\pi at}{L}. \tag{4.8}$$

To find the coefficients c_n, we have to do a little bit more work than previously, as now we haven't been given an initial value but an initial velocity. Differentiating $u(x,t)$ and solving for c_n using Fourier series will give

$$c_n = \frac{2}{n\pi a} \int_0^L g(x) \sin \frac{n\pi x}{L} dx. \tag{4.9}$$

4.4 String with both initial position and initial velocity

The two cases above have a disadvantage. To use the first case, we have to have a string of which all points have no velocity at $t = 0$. For the second all points on the string have to have zero displacement at that moment. This doesn't always occur.

If $f(x) \neq 0$ and $g(x) \neq 0$ it is possible to solve the problem using separation of variables, as we have previously performed. This is difficult though, and there is an easier way to solve the problem.

Let $v(x,t)$ be the solution for the vibrating string with no initial velocity ($g(x) = 0$). Also let $w(x,t)$ be the solution for the string with no initial displacement ($f(x) = 0$). If we add the two solutions up, we get

$$u(x,t) = v(x,t) + w(x,t). \tag{4.10}$$

It can now be shown that this solution satisfies all the boundary conditions. So if you have a string with both initial displacement and initial velocity, simply split the problem up and then add up the results.

5 Problem Variations

5.1 Expanding to multiple dimensions

The heat conduction problem and the wave problem described in the previous parts are only one-dimensional. They can be made two-dimensional or three-dimensional rather easily. We can replace the term u_{xx} by $u_{xx} + u_{yy}$ for a two-dimensional case or $u_{xx} + u_{yy} + u_{zz}$ for a three-dimensional case. This would make the heat conduction equation

$$\alpha^2 \left(u_{xx} + u_{yy} + u_{zz}\right) = u_t. \tag{5.1}$$

The wave equation would then be

$$a^2 \left(u_{xx} + u_{yy} + u_{zz}\right) = u_{tt}. \tag{5.2}$$

5.2 Steady-State problems

In for example heat conduction problems, the variable u usually converges to a constant value in time. But to what value does it occur? It stops changing if $u_t = 0$ or equivalently

$$u_{xx} + u_{yy} + u_{zz} = 0. \tag{5.3}$$

This equation is called **Laplace's Equation** for three dimensions. But can we solve it for a three-dimensional problem? What do we need to know before we can solve it?

In a one-dimensional problem we needed to know either the value of u or u_t at the edges of the rod. This can be expanded to three dimensions. To solve Laplace's equation in three dimensions, we need to know the value of u or u_t along the entire boundary of the three-dimensional space.

If u is given, the problem is slightly different than if u_t is given. It therefore also has a different name. If u is known along the edges, then the problem is called a **Dirichlet problem**. However, if we have been given u_t, then the problem is called a **Neumann problem**.

Both types of problems can be solved using the techniques demonstrated in this chapter. However, the equations for the solution and the corresponding coefficients need to be derived once more. As there are very many types of these problems, it is not possible to give the solution for every single type.