Differential Equations Problems

1 The Laplace Transform

1.1 Question 1 - January 19, 2007 (4 points)

Use the Laplace transform to solve the initial value problem

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 4y = \delta\left(t - \pi\right), \qquad y(0) = 1, \qquad \frac{dy}{dt}(0) = 0. \tag{1.1}$$

1.2 Question 1 - April 2, 2007 (6 points)

Use the Laplace transform to solve the initial value problem

$$y' + 4y = \sin 2t + \delta\left(t - \pi\right), \qquad y(0) = 1. \tag{1.2}$$

1.3 Question 1 - January 21, 2005 (7 points)

Use the Laplace Transform to solve

$$\frac{d^2y}{dt^2} + y = g(t), \qquad g(t) = \begin{cases} \frac{1}{2}t & \text{if } 0 \leq t < 6 \\ 3 & \text{if } 6 \leq t \end{cases}, \qquad y(0) = 0, \qquad y'(0) = 1. \tag{1.3}$$

2 Second Order Linear Differential Equations

2.1 Question 2 - January 19, 2007 (4 points)

Find (using the method of variation of parameters) the general solution of

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = \frac{e^{-2t}}{t^2}. \tag{2.1}$$

3 Systems of First Order Linear Differential Equations

3.1 Question 3 - January 19, 2007 (7 points)

Find the general solution of the system of differential equations

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}. \tag{3.1}$$

3.2 Question 3 - April 2, 2007 (8 points)

Find the general solution of the system of differential equations (thereby expressing the General solution of the corresponding homogeneous system in terms of real-valued functions)

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} -3 & 5 \\ -1 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-t}. \tag{3.2}$$

3.3 Question 2 - January 21, 2005 (7 points)

Find the general solution of the system of equations (thereby expressing the general solution of the corresponding homogeneous system in terms of real-valued functions)

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-2t}. \tag{3.3}$$

4 Stability of Systems of Differential Equations

4.1 Question 4 - January 19, 2007 (5 points)

Consider the system of nonlinear equations

$$\frac{dx}{dt} = x - xy, \tag{4.1}$$

$$\frac{dy}{dt} = x^2 + y^2 + y. \tag{4.2}$$

Determine type and (in-)stability of each critical point of this almost linear system (for the linearised case as well as for the nonlinear case).

4.2 Question 4 - April 2, 2007 (5 points)

Consider the system of nonlinear equations

$$\frac{dx}{dt} = (1+x)\sin y, \tag{4.3}$$

$$\frac{dy}{dt} = 1 - x - \cos y. \tag{4.4}$$

Points $(0,0)$ en $(2,\pi)$ are critical points. Determine type and (in-)stability of these two points of the given almost linear system (for the linearized case as well as for the nonlinear case).

4.3 Question 3 - April 4, 2005 (6 points)

Consider the system of nonlinear equations

$$\frac{dx}{dt} = x + x^2 + y^2, \tag{4.5}$$

$$\frac{dy}{dt} = y - xy. \tag{4.6}$$

Determine type and (in-)stability of each critical point of this almost linear system (linearized case and nonlinear case).

5 Eigenfunctions

5.1 Question 4 - April 2, 2007 (5 points)

Determine the normalised eigenfunctions (assume that all eigenvalues are real) of the following problem:

$$y'' + \lambda y = 0, \qquad y'(0) = 0, \qquad y'(1) = 0. \tag{5.1}$$

6 Power Series

6.1 Question 6 - January 21, 2005 (5 points)

Solve the following initial value problem by means of a power series expansion near $x_0 = 0$

$$\frac{dy}{dx} + xy = 1, \qquad y(0) = 0. \tag{6.1}$$

(You may stop after three non-zero terms.)

6.2 Question 6 - January 10, 2003 (6 points) (adjusted)

Find the general solution of the following differential equation by means of a power series expansion about the point $x_0 = 0$

$$x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + xy = 0, \qquad x > 0. \tag{6.2}$$

[You may stop after some non-zero terms].

6.3 Question 6 - April 2, 2007 (6 points)

Find the general solution of the following differential equation by means of a power series expansion about the point $x_0 = 1$

$$\frac{d^2y}{dx^2} - (x-1)^2y = 0. \tag{6.3}$$

[You may stop after some non-zero terms].

7 Fourier Series

7.1 Question 5b - January 19, 2007 (3 points)

Find the Fourier series of the function (first, sketch its graph!) given by

$$g(x+2) = g(x), \qquad g(x) = x \text{ for } 0 \leq x < 1, \qquad g(x) = 0 \text{ for } 1 \leq x < 2. \tag{7.1}$$

7.2 Question 5 - April 4, 2005 (2 points)

Find the Fourier series of the function (at first, sketch its graph!) given by

$$f(x) = |x|, \qquad -1 \leq x < 1, \qquad f(x+2) = f(x). \tag{7.2}$$

8 Fourier Series Applications

8.1 Question 5a - January 19, 2007 (5 points)

Find a formal solution (using the method of separation of variables) $u(x,t)$ of the initial-boundary value heat conduction problem

$$\frac{1}{\alpha^2}u_t = u_{xx}, \qquad u(0,t) = 0, \qquad u_x(l,t) = 0, \qquad u(x,0) = f(x). \tag{8.1}$$

8.2 Question 4 - January 9, 2004 (7 points)

Find a formal solution (using the method of separation of variables) $u(x,t)$ to the initial-boundary value heat conduction problem

$$u_t = u_{xx}, \qquad u_x(0,t) = 0, \qquad u_x(l,t) = 0, \qquad u(x,0) = f(x). \tag{8.2}$$

8.3 Question 4 - April 4, 2005 (7 points)

Find a formal solution $u(x,y)$ of the potential equation

$$u_{xx} + u_{yy} = 0 \qquad \text{in the rectangle } 0 < x < a, 0 < y < b, \tag{8.3}$$

that satisfies the boundary conditions

$$u(0,y) = 0, \qquad u(a,y) = 0, \qquad 0 < y < b, \tag{8.4}$$

$$u_y(x,0) = 0, \qquad u(x,b) = g(x), \qquad 0 < x < a. \tag{8.5}$$