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Summary of Differential Equations

§2.1 First-order diff. eq.: Method of Integrating Factors

$y' + p(t)y = g(t)$ standard form

$\mu(t) = e^{\int p(t)dt}$ integrating factor

$\frac{d}{dt}(\mu(t)y) = \mu(t)g(t)$ multiplification

$\mu(t)y = \int \mu(t)g(t)\,dt$ integration

$y = \frac{1}{\mu(t)} \int \mu(t)g(t)\,dt$ final answer

§3.4 Second-order diff. eq.: Reduction of order

$ay'' + by' + cy = 0$ standard form

$ar^2 + br + c = 0$ characteristic polynominal

$D = b_2 - 4ac \rightarrow$ ABC-formula determinant

if: ∘ $(r_1 \neq r_2)$; $D > 0$ real, not equal

$\rightarrow y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

∘ $(r_1 \neq r_2)$; $D < 0$ imaginary, not equal

$\rightarrow y = c_1 e^{\lambda t} \cos \mu t + c_2 e^{\lambda t} \sin \mu t$

with $r_{1,2} = \lambda \pm \mu i$

∘ $r_1 = r_2$ equal

$\rightarrow y = c_1 e^{r_1 t} + c_2 t e^{r_1 t}$

For: $y'' + p(t)y' + q(t)y = 0$ and we know $y_1(t)$

Assume: $y_2(t) = v(t)y_1(t)$

$y_2'(t) = v'(t)y_1(t) + v(t)y_1'(t)$

$y_2''(t) = v''(t)y_1(t) + 2v'(t)y_1'(t) + v(t)y_1''(t)$

Substitute these into basic eq. and choose the constants $c_1, c_2, \ldots$ you get after integrating such that $y_1(t) \neq y_2(t)$.

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§3.6 second-order diff. eq: Variation of Parameters

$ay'' + by' + cy = g(t)$ standard form

Solve the homogeneous eq. with reduction of order and substitute c_1 and c_2 with $u_1(t)$ and $u_2(t)$. Derive y' and y'' from the equation and set $u_1'(t)$ and $u_2'(t)$ equal to zero in the eq. for y'.

Substitute the y, y' and y'' equation in the given equation, and express $u_2'(t)$ in $u_1'(t)$ using the equation gotten from y'.

Integrate $u_1'(t)$ and $u_2'(t)$ to find $u_1(t)$ and $u_2(t)$. Substitute these in the equation for y to find the particular solution.

Add the homogeneous and particular solution together to find the general solution.

$y(t) = y_h(t) + y_p(t)$ general solution

§6.1 Laplace Transform: Definition

$$\mathcal{L}(f(t)) = \int_0^\infty f(t)e^{-st}\,dt = \lim_{A\to\infty}\left[\int_0^A f(t)e^{-st}\,dt\right]$$

§6.2 Laplace Transform: $\mathcal{L}(y) = F(s) + I.C.$

$ay'' + by' + cy = g(t)$

$ay'' \rightarrow a\cdot\mathcal{L}(y'') = a\cdot(s^2F(s) - sy(0) - y'(0))$

$by' \rightarrow b\cdot\mathcal{L}(y') = b\cdot(sF(s) - y(0))$

$cy \rightarrow c\cdot\mathcal{L}(y) = c\cdot F(s)$

Express $F(s)$ in terms of s. Eventually use the method of partial fractions.

§6.3 Step Functions

$f(t) = \begin{cases} g(t), & t < c \\ h(t), & t \geq c \end{cases}$ then: $f(t) = u_c(t)f(t-c)$

$\mathcal{L}\{u_c f(t-c)\} = e^{-cs}F(s)$ Tabel Laplace nr. 17

§6.5 Impulse Functions

$$\mathcal{L}\{\delta(t-c)\} = \mathcal{L}\{\delta_c(t)\} = e^{-cs}$$

Remark: $\mathcal{L}\{a\delta(t-c)\} = ae^{-cs}$ where a is a constant

§6.6 Convolution Integral

$H(s) = F(s)G(s) = \mathcal{L}\{h(t)\}$ with the convolution integral:

$$h(t) = \int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t f(\tau)g(t-\tau)d\tau$$

HS6 Remark: Partial Fractions

$$\frac{b}{x(x+a)} = \frac{A}{x} + \frac{B}{x+a} = \frac{A(x+a)}{x(x+a)} + \frac{Bx}{x(x+a)} = \frac{(A+B)x+aA}{x(x+a)}$$

So: $A+B=0$ & $aA=b \longrightarrow A=\frac{b}{a}$ & $B=-\frac{b}{a}$

If the denominator has a higher function of x in it, write a polynomial that is one degree less in the numerator. E.g.: $\frac{ax}{(x^n+b)(x+c)} = \frac{Ax^{n-1}+B}{x^n+b} + \frac{C}{x+c}$.

For repeated roots: If the root is repeated n-times, write out the partial factor n times, starting from 1 and increasing by 1 each time.

$$\frac{ax}{(x-b)^n} = \frac{A}{x-b} + \frac{B}{(x-b)^2} + \dots + \frac{Z}{(x-b)^n}$$

§7.5 Homogeneous Systems

Substitute $x = \xi e^{rt}$ and $x' = r\xi e^{rt}$

Divide by e^{rt}: $(A-rI)\xi = 0$

Find $\det(A-rI)=0$ and find all r's.

Fill in the r's found into $(A-rI)\xi=0$ to find $\xi^{(n)}$

Construct answer: $x = c_1\xi^{(1)}e^{r_1 t} + c_2\xi^{(2)}e^{r_2 t} + \dots$

For large matrices:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \rightarrow \det A = a_{11}\cdot\det A_{11} - a_{12}\det A_{12} + \dots$$

A_{11} is the free part of the matrix

§7.6 Complex Eigenvalues

Determine $r_{1,2}$ using $\det(A-rI)=0$

Implement one eigenvalue in $A-rI$ and reduce the matrix using the complex conjugate

For $z=\lambda+i\mu \rightarrow z\cdot\overline{z}=\lambda^2+\mu^2$

Then $\overline{\xi^{(1)}}=\xi^{(2)}$. Use $x^{(1)}=\xi^{(1)}e^{rt}$ and $e^{-i\mu t}=\cos\mu t-i\sin\mu t$

Write out $x^{(1)}$ in a real and an imaginairy part.

Then: General Solution: $x=\overline{x^{(1)}}$ with c_n for every term.

§7.8 Repeated Eigenvalues

Determine r using $\det(A-rI)=0$

Find ξ using the standard procedure.

From there you get: $x^{(1)}=\xi e^{rt}$

Now construct: $(A-rI)\eta=\xi$ and fill in $\eta_1=k$ to find η_2. Write $\eta=a+kb$ so easy fill in:

$x^{(2)}=\xi te^{rt}+\eta e^{rt}$

should equal ξ so remove from answer

General Solution: $x(t)=c_1x^{(1)}+c_2x^{(2)}$

§7.9 Nonhomogeneous Linear Systems

$x'=Ax+\underline{g}(t)$ basic form

First solve the homogeneous part using §7.5, 7.6, 7.8 and $\underline{g}(t)=\underline{0}$ as assumption.

General Solution: $\underline{x}_h=\psi(t)\cdot\underline{c}$

For particular part, say $\underline{c}$ is $\underline{u}(t)$. Then:

$\underline{x}_p=\psi(t)\underline{u}(t)$ and $x'=\psi(t)\underline{u}'(t)+\psi'(t)\underline{u}(t)$

since $\psi'(t)=\psi(t)\cdot A \rightarrow \psi(t)\underline{u}'(t)=\underline{g}(t)$

$\rightarrow \underline{u}'(t)=\psi^{-1}(t)g(t) \rightarrow \int\underline{u}'(t)=\int\psi^{-1}(t)\underline{g}(t)+\underline{c}$

Fill this back in $\underline{x}_p$ and add up with x_h to find gen. sol.

REMARK: Stating $\Psi(t)\underline{u}'(t) = \underline{g}(t)$ and using row reduction to find $\underline{u}'(t)$ might be sufficient. If this vector $\underline{u}'(t)$ is then integrated, $\underline{u}(t)$ is found.

Otherwise use: $\Psi^{-1}(t) = \frac{1}{\det(\Psi(t))}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

§9.1 Linear Systems: stability

- Homogeneous case: $\underline{x}' = A\underline{x}$

  Find $\det(A - rI) = 0 \rightarrow$ determine r's (eigenvalues)

  Fill in those r's in $(A - rI)$ to find eigenvectors.

  Look in the table to find stability for the point x_{cr}

- Nonhomogeneous case: $x' = Ax + g(t)$

  Set $x' = 0 \rightarrow Ax = -g(t) \rightarrow$ Row reduce $[A \mid g(t)]$

  From there determine $\underline{x}^0 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and set $\underline{x} = \underline{x}^0 + \underline{u}$

  So that: $\underline{u}' = A\underline{u}$ which is solved homogeneous.

§9.2 Autonomous systems & stability

Given: $x' = f_1(x,y)$ and $y' = f_2(x,y)$

Set $x' = y' = 0$ and find all possible values for x and y in either f_1 or f_2 that make it zero. Substitute these values one at a time in the other function to find the corresponding x or y.

The combinations of x and y gotten here are called the critical points.

Find the Jacobian: $J(x,y) = \begin{bmatrix} \frac{f_1(x,y)}{dx} & \frac{f_1(x,y)}{dy} \\ \frac{f_2(x,y)}{dx} & \frac{f_2(x,y)}{dy} \end{bmatrix}$

Fill in the critical points and set $\det(J(x_{cr}, y_{cr})) = 0$ to find the eigenvalues. Look in table for stability.

§9.7 Limit Cycles

Given: $r(r-a)(r-b) = r'$ Set $r' = 0$ and find all r's

Check the situation of r' between every found value of r, by filling in a value between 2 r's in the equation

If $r' > 0$ lines go away, if $r' < 0$ lines approach.

When:
- 2 lines approach the limit cycle: stable
- 2 lines go away from limit cycle: unstable
- 1 line approaches, 1 line goes away from limit cycle: semi-stable

§10.1 Boundary Value Problems

Given: $y'' + ay = g(x)$ → find $y(x)$ using §3.4 or 3.6

Differentiate $y \to y'$ and use boundary conditions given to find c_1 and c_2.

Given: $y'' + \lambda y = g(x)$ and asked for eigenvalues and eigenfunctions

3 Cases:
- $\lambda < 0 \to \lambda = -\mu^2$
- $\lambda = 0 \to$ straight into BVP: $y'' = g(x)$
- $\lambda > 0 \to \lambda = \mu^2$

Set the assumption in the BVP and find $y(x)$, the rest is the same procedure as in the beginning of §10.1 but now also μ needs to be found besides c_1 and c_2.

From μ_n determine λ_n and $y_n(x)$.

§10.2 Fourier Series

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \quad ; \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right) \text{ and } f(x+2L) = f(x)$$

§10.4 Odd & Even functions

$$\left.\begin{array}{l}\text{Odd functions: } f(x) = x,\ \sin x,\ x^3 \\ \quad \to \text{Fourier Sine Series}\end{array}\right\} a_n = 0$$

$$\left.\begin{array}{l}\text{Even functions: } f(x) = x^2,\ \cos x,\ 1,\ |x| \\ \quad \to \text{Fourier Cosine Series}\end{array}\right\} b_n = 0$$

§10.5 Heat Equation

Given: $u_{xx} = \frac{1}{\alpha^2} u_t$

Say: $u(x,t) = X(x)T(t) \rightarrow u_{xx}(x,t) = X''(x)T(t)$
$\rightarrow u_t(x,t) = X(x)T'(t)$

Substitute these into heat equation. Separate x & t:

$X''(x)T(t) = \frac{1}{\alpha^2} X(x)T'(t) \longrightarrow \frac{X''(x)}{X(x)} = \frac{1}{\alpha^2}\frac{T'(t)}{T(t)} = -\lambda$

So: $\begin{cases} X''(x) + \lambda X(x) = 0 \\ T'(t) + \alpha^2 \lambda T(t) = 0 \end{cases}$

Use the Boundary Conditions in $u(x,t) = X(x)T(t)$ where $T(t) \neq 0$. Use the 3 cases from §10.1 to solve.

Solve for T using: $T_n = ^{-\lambda_n \alpha^2 t}$

Use X_n and T_n to find $u_n = X_n T_n$ and determine u_0

So: $u(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \underbrace{e^{-\lambda_n \alpha^2 t}}_{T(t)} \underbrace{\cos/\sin(\lambda_n x)}_{\text{either sin or cos} \rightarrow X(x)}$

$f(x) = \sum_{n=1} c_n \sin\left(\frac{n\pi x}{L}\right)$ to find c_n and apply it in $u(x,t)$

§10.6 Other heat problems

Steady State: $T''(t) = T'(t) = 0$

§10.7 Wave Equation

$u_{xx} = \frac{1}{\alpha^2} u_{tt}$ same procedure as §10.5, but mark $T''(t)$!

§11.1 Two Point BVP

Same procedure as the $y'' + \lambda y = g(t)$ case in §10.1

Only μ sometimes needs to be found using a calculator.

§11.2 Sturm-Liouville BVP and normalisation

Sturm-Liouville: use $f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x) \rightarrow c_n = \int_0^L f(x)\phi_n(x)\,dx$

Normalisation: use $\int_0^1 y_n^2(x) r(x)\,dx = 1$ to find c_n

most of the time is 1

§5.1 Ratio Test

To check for absolute convergence:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}(x-x_0)^{n+1}}{a_n(x-x_0)^n}\right| = |x-x_0|\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x-x_0|L$$

To check a limit: Apply the formula in your calculator with a X where the n should be. Then save a value very near the lim-point in the slot for X, and use that to solve the formula.

§5.2 Power Series

Say:
$$\begin{cases} y = \sum_{n=0}^{\infty} a_n (x-x_0)^n \\ y' = \sum_{n=0}^{\infty} a_{n+1}(n+1)(x-x_0)^n \\ y'' = \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)(x-x_0)^n \end{cases}$$

Substitute these in the given equation. Get every x-term inside the series by raising the power. Equal all powers by changing the count. Make every serie start the count at the same level of the highest count, by pulling out terms. Add up all the series, and also all the pulled-out terms. Solve by stating that the pulled-out terms are zero as well as the series. Answer in a_0 or a_1 terms.

§5.4 Euler and Regularity

Given: $ax^2y + bxy + cy = 0$

Set: $y = x^r$; $y' = rx^{r-1}$; $y'' = r(r-1)x^{r-2}$

So: $x^r(ar(r-1) + br + c) = 0$ where $x^r \neq 0$

Find r's:
$$\begin{cases} r_1 \neq r_2 ;\ D>0 : & y = c_1 x^{r_1} + c_2 x^{r_2} \\ r_1 = r_2 ;\ D=0 : & y = (c_1 + c_2 \ln|x|)x^r \\ r_1 \neq r_2 ;\ D<0 : & y = c_1 x^\lambda \cos(\mu \ln|x|) + c_2 x^\lambda \sin(\mu \ln|x|) \end{cases}$$

$$\left.\begin{aligned} &\lim_{x\to x_0}(x-x_0)\frac{Q(x)}{P(x)} \\ &\lim_{x\to x_0}(x-x_0)^2\frac{R(x)}{P(x)} \end{aligned}\right\}$$

If both these limits are finite, then x_0 is regular singular, otherwise it is irregular singular.