Basic Differential Equations

1 First order equations

1.1 Basic form

Equations containing derivatives are **differential equations**. Differential equations of the **first order** (meaning only first derivatives can occur, but no second or higher derivatives) can be written as

$$\frac{dy}{dt} = y' = f(t, y). \tag{1.1}$$

Note that to keep equations short, we write $\frac{dy}{dt} = y'$. A function $y = \phi(t)$ is called a **solution** if it satisfies the above equation. No simple solution method exists that can solve all differential equations of this form.

1.2 Linear equations

However, for some forms there are methods to find solutions. For example, if the equation is **linear in** y, it can be written as

$$y' + p(t)y = g(t). \tag{1.2}$$

Note that sometimes differential equations have to be rewritten to bring them to the right form. To find a solution on a particular interval (α, β), $p(t)$ must be **continuous on** (α, β), that is, $p(t)$ exists for every t in the interval (α, β).

The **technique of integrating factor** can be applied to solve this form of differential equation. First find any integral of $p(t)$. Then define the **integrating factor** $\mu(t)$ as

$$\mu(t) = e^{\int p(t)\,dt}. \tag{1.3}$$

Now multiply equation 1.2 by $\mu(t)$. Using the chain rule, it can be rewritten as

$$\frac{d(\mu(t)y)}{dt} = \int \mu(t)\,g(t)\,dt. \tag{1.4}$$

The solution then becomes

$$y(t) = \frac{1}{\mu(t)} \int_{t_0}^{t} \mu(s)\,g(s)\,ds + \frac{c}{\mu(t)}. \tag{1.5}$$

Of this equation, the part $\frac{1}{\mu(t)} \int_{t_0}^{t} \mu(s)\,g(s)\,ds$ is called the **particular solution** and $\frac{c}{\mu(t)}$ is called the **general solution**. In differential equations the complete set of solutions is usually formed by the general solution, plus any linear combination of the particular solution(s).

1.3 Separable differential equations

A differential equation is called a **separable differential equation**, if it can be written as

$$\frac{dy}{dx} = \frac{M(x)}{N(y)}. \tag{1.6}$$

We can rewrite it as

$$N(y)\,dy = M(x)\,dx. \tag{1.7}$$

The solution of this differential equation is acquired by simple integration of the equation

$$\int N(y)\,dy = \int M(x)\,dx. \tag{1.8}$$

2 Second order linear differential equations

2.1 Basic form

The basic form of a **second order differential equations** is

$$\frac{d^2y}{dt^2} = y'' = f(t, y, y'). \tag{2.1}$$

Such equations are hard to solve. So we will be looking at **second order linear differential equations**, which have the form

$$y'' + p(t)y' + q(t)y = g(t). \tag{2.2}$$

If a second order equation can be written in the form of the above equation, it is called **linear**, and otherwise **nonlinear**. For such differential equations solving methods exist. However, we do assume that the function $p(t)$, $q(t)$ and $g(t)$ are continuous functions.

A second order linear differential equation is said to be **homogeneous** if the term $g(t)$ in equation 2.2 is 0 for all t. Otherwise it is called **nonhomogeneous**.

2.2 Homogeneous equations with constant coefficients

Suppose our differential equation has the form

$$ay'' + by' + cy = 0, \tag{2.3}$$

with a, b and c constants. Let's define the **characteristic equation** to be

$$ar^2 + br + c = 0. \tag{2.4}$$

If we can find an r that satisfies the characteristic equation, then we know that $y = e^{rt}$ is a solution. In fact all linear combinations $y = ce^{rt}$ are solutions. So let's look at three specific cases.

- $b^2 - 4ac > 0$

  There are two real solutions r_1 and r_2 to equation 2.4. Both $y_1 = e^{r_1 t}$ and $y_2 = e^{r_2 t}$ and all linear combinations of them are solutions. So the general solution of the differential equation is:

  $$y = c_1 y_1 + c_2 y_2 = c_1 e^{r_1 t} + c_2 e^{r_2 t} \tag{2.5}$$

- $b^2 - 4ac = 0$

  There is only one solution $r = -\frac{b}{2a}$ to the characteristic equation. We know that $y_1 = e^{rt}$ is a solution. However, also $y_2 = te^{rt}$ is a solution. So the general solution of the differential equation is:

  $$y = c_1 y_1 + c_2 y_2 = c_1 e^{rt} + c_2 t e^{rt} \tag{2.6}$$

- $b^2 - 4ac < 0$

  There are no real solutions now, only complex ones. So if $\alpha = -\frac{b}{2a}$ and $\beta = \frac{\sqrt{4ac-b^2}}{2a}$, and also $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$, then $y_1 = e^{r_1 t}$ and $y_2 = e^{r_2 t}$ are solutions. Working out the complex numbers in all linear combinations of the two solutions gives as general solution:

  $$y = c_1 y_1 + c_2 y_2 = e^{\alpha t}\left(c_1 \cos \beta t + c_2 \sin \beta t\right) \tag{2.7}$$

The solutions given by the methods above are all possible solutions of the differential equation.

2.3 Nonhomogeneous equations - Method of undetermined coefficients

Suppose our differential equation has the form

$$ay'' + by' + cy = g(t). \tag{2.8}$$

with a, b and c still constants. The function $g(t)$ here is called the **forcing function**. Suppose we find any particular solution $Y(t)$ that satisfies the above equation. We already know from the previous paragraph how to find the general solution set $c_1y_1 + c_2y_2$ for the homogeneous differential equation $ay'' + by' + c = 0$. If we add those two solutions up, we find all solutions for the above differential equation.

So the trick is to find a single $Y(t)$ that satisfies the differential equation. One way to do that is to use the **method of undetermined coefficients**. We make an initial assumption on the form of $Y(t)$ (called the **auxiliary equation**), with a couple of undetermined coefficients, and then try to find the coefficients. The downside of this method is that it only works on equations that contain terms at^n, $e^{\alpha}t$ and $\sin\beta t$, or combinations of those terms.

First take a look at $g(t)$. If it consists of multiple parts, separated by plus or minus signs (for example, $g(t) = t + \sin t - e^t$), then split the problem up in parts and find a particular solution $Y_i(t)$ for every part $g_i(t)$.

To find a particular solution for $g_i(t)$, use the auxiliary equation

$$t^s\left(\left(a_0 + a_1t + a_2t^2 + \ldots + a_nt^n\right)e^{\alpha t}\cos\beta t + \left(b_0 + b_1t + b_2t^2 + \ldots + b_nt^n\right)e^{\alpha t}\sin\beta t\right). \tag{2.9}$$

The variables α, β and n can be found in $g_i(t)$. (For example, for $g_i(t) = te^{2t}$ the auxiliary equation becomes $t^s\left((a_0 + a_1t)\,e^{2t}\right)$.) The variable s, however, is a different story. It's a matter of trial and error. Usually $s = 0$ works. If this doesn't work, try $s = 1$. If it still doesn't work (unlikely, but possible), try $s = 2$.

Now we have an auxiliary equation $Y_i(t)$ with undetermined coefficients $a_0, \ldots, a_n, b_0, \ldots b_n$. First find $Y_i'(t)$ and $Y_i''(t)$. Then write down the equation

$$aY_i''(t) + bY_i'(t) + cY_i(t) = g_i(t). \tag{2.10}$$

Use this equation to solve the undetermined coefficients and find the particular solution for $ay'' + by' + cy = g_i(t)$.

So having found all the particular solutions $Y_i(t)$ for $ay'' + by' + cy = g_i(t)$, add them all up to find the particular solution $Y(t) = Y_1(t) + \ldots + Y_n(t)$. Now add this up to the general solution $c_1y_1 + c_2y_2$ of the homogeneous equation $ay'' + by' + cy = 0$ to find the full solution set of the differential equation:

$$y = c_1y_1 + c_2y_2 + (Y_1(t) + \ldots + Y_n(t)). \tag{2.11}$$

2.4 Nonhomogeneous equations - Variation of parameters

The method **variation of parameters** is applied to differential equations of the form of equation 2.8 and goes as follows. First find the solution $y = c_1y_1 + c_2y_2$ of the differential equation $ay'' + by' + c = 0$. Now replace c_1 by $u_1(t)$ and c_2 by $u_2(t)$ to get $y = u_1(t)y_1 + u_2(t)y_2$. Now it is possible to find y' and y''. Let's first (for no specific reason but that the outcome will be convenient) assume that

$$u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0. \tag{2.12}$$

Working everything out, we eventually find that

$$u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t). \tag{2.13}$$

Now, let's define the **Wronskian determinant** (or simply **Wronskian**) $W(y_1, y_2)$ as

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_1'(t)y_2(t). \tag{2.14}$$

If we solve for u_1' and u_2' in equations 2.12 and 2.13, we find that

$$u_1'(t) = -\frac{y_2(t)g(t)}{W(y_1, y_2)(t)} \qquad \text{and} \qquad u_2'(t) = \frac{y_1(t)g(t)}{W(y_1, y_2)(t)}. \tag{2.15}$$

Solving this gives as a particular solution for the differential equation

$$Y(t) = -y_1(t) \int_{t_0}^{t} \frac{y_2(s)g(s)}{W(y_1, y_2)(s)} ds + y_2(t) \int_{t_0}^{t} \frac{y_1(s)g(s)}{W(y_1, y_2)(s)} ds, \tag{2.16}$$

for any convenient t_0 in the interval. So this makes the general solution for the differential equation:

$$y = c_1 y_1 + c_2 y_2 + Y(t). \tag{2.17}$$

3 Initial value problems and boundary value problems

3.1 Initial value problems

Sometimes, next to a differential equation, also certain conditions are given. For example, the values of y and y' at a given time t_0 are given:

$$y(t_0) = y_0, \qquad y'(t_0) = y_0'. \tag{3.1}$$

Such conditions are called **initial conditions**. If these conditions are given, the problem is called an **initial value problem**. Finding the general solution for the differential equation isn't sufficient to solve the problem. You have to find the values c_1 and c_2 such that the initial conditions are met.

3.2 Boundary value problems

Sometimes not the initial conditions at a time t_0 are given, but the conditions are two different times are given:

$$y(\alpha) = y_0, \qquad y(\beta) = y_1. \tag{3.2}$$

Such conditions are called **boundary conditions**. If these conditions are given, the problem is called a **(two-point) boundary value problem**. Once more the values c_1 and c_2 should be found such that the boundary conditions are met, to solve the problem.