

# Root-locus plots

It's time we start designing systems ourselves. But how can we make sure our system gets the properties we want? In this chapter we'll investigate one method for that: the root-locus method.

## 1 General drawing rules

First we will examine the basic rules of the root-locus method. In the next part we will examine how they work.

### 1.1 The basic ideas

Suppose we have a (rather familiar) closed-loop system, as is shown in figure 1. We know that the transfer function is  $F(s) = G(s)/(1 + G(s)H(s))$ . We also know that the properties of the system strongly depend on the poles of  $F(s)$ . We would like to have some influence on these poles.

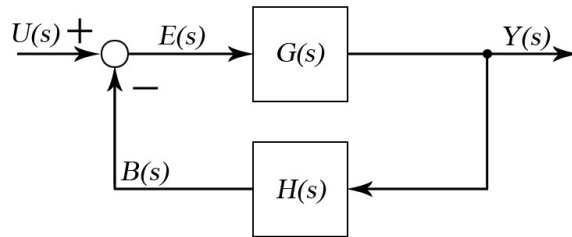


Figure 1: Again, a basic closed loop system.

Well, if we want to find the poles of  $F(s)$ , we should set its denominator to zero. So that gives us the equation

$$G(s)H(s) = K \frac{(s - z_1)(s - z_2) \dots (s - z_m)}{(s - p_1)(s - p_2) \dots (s - p_n)} = -1. \quad (1.1)$$

This equation is called the **characteristic equation**. Every  $s$  satisfying this equation is a pole of  $F(s)$ . By the way, the variables  $z_1, \dots, z_m$  and  $p_1, \dots, p_n$  in the above equation are the zeroes and poles, respectively, of  $G(s)H(s)$ . (Not to be confused with the zeroes and poles of  $F(s)$ .)

We assume that all these zeroes and poles of  $G(s)H(s)$  are already defined. There is, however, one variable left for us to define, being  $K$ . Luckily it turns out that the poles of  $F(s)$  (which we want to define) strongly depend on  $K$ . We therefore would like to do plot these poles as a function of  $K$ . In other words, we want to plot all  $s$  satisfying the above equation.

This is, however, more difficult than it seems. Especially because the above equation is a complex equation. So how do we know which  $s$  satisfy it? Well, we know that if two complex numbers  $z_1$  and  $z_2$  are equal, then both their arguments (angles) are equal ( $\arg(z_1) = \arg(z_2)$ ) and their lengths are equal ( $|z_1| = |z_2|$ ). Let's apply this to equation (1.1). If we examine lengths, we find the **length-equation**

$$|G(s)H(s)| = |K| \frac{|s - z_1| \dots |s - z_m|}{|s - p_1| \dots |s - p_n|} = |-1| = 1. \quad (1.2)$$

This isn't a very useful equation yet. So we also examine the arguments. This time we get the **argument-equation**

$$\arg(K) + \arg(s - z_1) + \dots + \arg(s - z_m) - \arg(s - p_1) - \dots - \arg(s - p_n) = \arg(-1) = (2k + 1) \cdot 180^\circ, \quad (1.3)$$

where  $k \in \mathbb{Z}$  can be any constant. (Note that multiplying complex numbers means adding their angle, while dividing complex numbers means subtracting their angle.)

We can even simplify the above equation a bit. We know  $K$  must be a real number. And usually it's also positive. This means that  $\arg(K) = 0$ . So the above equation doesn't even depend on our variable  $K$ ! And there's even more. If we can find an  $s$  satisfying this equation, then we can simply adjust our  $K$ , such that it also satisfies the length-equation. So all we need to do is find an  $s$  that satisfies the argument-equation. Most of the rules in the upcoming paragraph use this fact.

## 1.2 Root-locus drawing rules

There is no set of rules which will always enable you to draw a perfect root-locus diagram. However, there are rules which can give some info about its shape. So in the following plan of approach, we will mention most of these rules.

1. First we should draw a complex plane. In it, we should draw all the poles and zeroes of  $G(s)H(s)$ . We denote the  $n$  poles by a cross ( $x$ ) and the  $m$  (finite) zeroes by a circle ( $o$ ). (We will examine the infinite zeroes at point 4.)
2. Next we should examine which points  $s$  on the real line are part of the root-locus plot. Just pick a point on this axis. If the total number of poles and zeroes to the right of this point is odd, then this point is part of the root-locus plot. Otherwise, it is not. (By the way, this follows from the argument-equation.) In this way we can find all segments of the real axis which are part of the root-locus plot.
3. We will now find the **breakaway points**. These are the points where the root-locus lines break away from the real axis. To find them, we first have to solve equation (1.1) for  $K$ . We will find

$$K = -\frac{(s - p_1)(s - p_2) \dots (s - p_n)}{(s - z_1)(s - z_2) \dots (s - z_m)}. \quad (1.4)$$

We then use the condition  $dK/ds = 0$ . We will find several solutions  $s_i$  for this. However, not all of these solutions represent breakaway points. In fact, to find the actual breakaway points, we need to substitute the values of  $s$  back into the above equation. Only if a certain value  $s_i$  gives a positive  $K$ , then  $s_i$  is a valid breakaway point.

4. We can sometimes also find the **asymptotes** of the graph. (However, this only works if  $H(s)G(s)$  has enough infinite zeroes. With enough, I mean at least 2. The number of infinite zeroes of  $G(s)H(s)$  is equal to  $n - m$ .) To find the asymptotes, we first have to find the **asymptote center**. It is located at the point

$$s = \frac{p_1 + p_2 + \dots + p_n - z_1 - z_2 - \dots - z_m}{n - m}. \quad (1.5)$$

From this point, we now draw asymptote lines in  $n - m$  directions. These directions have angles

$$\text{asymptote angles} = \frac{\pm(2k + 1) \cdot 180^\circ}{n - m}. \quad (1.6)$$

When  $|s| \rightarrow \infty$  the root-locus plots will converge to these asymptotes.

5. Sometimes we can also find the points where the root-locus plot crosses the imaginary axis. (In fact, this usually only works if  $n - m \geq 2$ .) On these points, we can write  $s = \omega i$ . So we can insert this condition into the characteristic equation (1.1). We then find a complex equation. If we equate both the real and complex parts of this equation, we can find  $\omega$ .

6. Finally, we can determine the **angle of departure/angle of arrival** of the complex poles/zeros. (This, of course, only works if  $G(s)H(s)$  has complex poles or zeros.) We can use the argument-equation for this. Let's suppose we want to find the angle of departure/arrival of the pole  $p_1$ . We examine some point  $s_a$ , infinitely close to  $p_1$ , but at an angle  $\theta = \arg(s_a - p_1)$ . If we insert this into the argument-equation, we find that

$$\arg(s_a - z_1) + \dots + \arg(s_a - z_m) - \theta - \arg(s - p_2) - \dots - \arg(s_a - p_n) = (2k + 1) \cdot 180^\circ. \quad (1.7)$$

Since  $s_a$  is infinitely close to  $p_1$ , we can simply substitute  $s_a = p_1$  in the above equation. We can then solve for the angle of departure  $\theta$  at this pole.

Once we have gone through all of these rules, we should be able to draw a more or less accurate graph. In fact, we only have to draw one half of the graph, since root-locus plots are always symmetric about the real axis.

When drawing the root-locus plot, you do have to keep in mind that every line should go from a pole to a (possibly infinite) zero. If you have a zero or a pole without a line, then you probably haven't completed your root-locus plot yet.

## 2 Example problems

The root-locus drawing rules might seem rather vague. So let's clarify them with some examples.

### 2.1 An example problem without infinite zeroes

Let's consider the basic feedback system with  $G(s)$  and  $H(s)$  given by

$$G(s) = K \frac{(s+2)(s+3)}{s(s+1)} \quad \text{and} \quad H(s) = 1. \quad (2.1)$$

We want to plot the root-locus graph for this system. In other words, we want the possible range of poles of  $F(s)$ , if we can vary  $K$ . To find this, let's simply apply the rules.

1. The function  $G(s)H(s)$  has  $n = 2$  poles, being  $s = 0$  and  $s = -1$ . It also has  $m = 2$  zeroes, being  $s = -2$  and  $s = -3$ . We draw a complex plane, and insert these four points. (For the final solution, in which these four points are also drawn, see figure 2 at the end of this section.)
2. Let's examine a point  $s$  on the real axis and count the number of poles/zeros to the right of it. This number is only odd if  $-1 < s < 0$  or  $-3 < s < -2$ . So only these parts of the real axis are part of our root-locus plot. We mark these sections in our graph. (Again, see figure 2.)
3. Let's find the breakaway points. We know that

$$K = -\frac{s(s+1)}{(s+2)(s+3)}. \quad (2.2)$$

We can take the derivative with respect to  $s$ . This gives us the condition

$$\frac{dK}{ds} = -\frac{(s+2)(s+3)(2s+1) - s(s+1)(2s+5)}{(s+2)^2(s+3)^2} = 0. \quad (2.3)$$

Solving this for  $s$  gives  $s = -\frac{1}{2}(3 \pm \sqrt{3})$ . So we have two solutions for  $s$ . We do have to check whether they are actual breakaway points. The solution  $s = -\frac{1}{2}(3 - \sqrt{3})$  implies that  $K = 0.0718$ , while  $s = -\frac{1}{2}(3 + \sqrt{3})$  gives  $K = 13.93$ . Both values of  $K$  are positive, so both values of  $s$  are breakaway points. We can thus draw breakaway points at these positions.

4. Since  $n = m$ , there are no infinite zeroes. So there are no asymptotes either. This rule therefore isn't of much use.
5. Since there are no asymptotes, there aren't any points where the root-locus plot crosses the imaginary axis.
6. There are no complex poles/zeroes, so we can't find any angles of departure either.

We have got some nice info on the shape of our root-locus diagram already. Experienced root-locus-drawers can already predict its shape. If you are, however, new to root-locus drawing, you might want to find some extra points. We could use the argument-equation for that. We could also simply try to find some point in a brute force kind of fashion. We'll use that second method now, since it's a bit more exact.

We know that root-locus lines always run from a pole to a zero. So there must be some point on the root-locus plot with real part  $\text{Re}(s) = -3/2$ . What point would that be? To find that out, we can insert  $s = -3/2 + \omega i$  into the characteristic equation. If we do this, we can solve for  $\omega$ . We will find that  $\omega = \pm\sqrt{3}/2$ . So we just found another point of our root-locus plot!

If you want, you can continue to do this for other points. You will find some more points of the root-locus diagram. When you feel you're sure what the shape of the plot will be, you can draw it. And then you should hopefully have found the plot of figure 2.

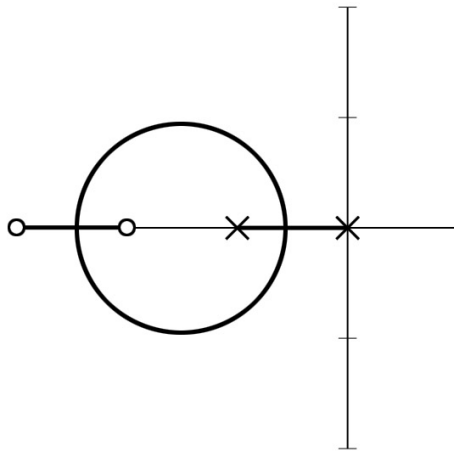


Figure 2: The solution of our first example problem.

## 2.2 An example problem with multiple infinite zeroes

This time, let's consider a system with

$$G(s) = \frac{K}{s(s+1)(s^2+4s+13)} \quad \text{and} \quad H(s) = 1. \quad (2.4)$$

Let's use the drawing rules to draw a root-locus plot.

1. We find that  $G(s)H(s)$  has no (finite) zeroes. (So  $m = 0$ .) It does have four poles. (So  $n = 4$ .) They are  $s = 0$ ,  $s = -1$  and  $s = -2 \pm 3i$ . We draw those poles in the complex plane.
2. We examine all points on the real axis which have an odd number of poles/zeroes to the right of it. We find that this condition is only valid for  $-1 < s < 0$ . We mark this part in our plot.

3. Let's find the breakaway points. We have  $K = -s(s+1)(s^2+4s+13)$ . Taking the derivative will give

$$\frac{dK}{ds} = 4s^3 + 15s^2 + 34s + 13 = 0. \quad (2.5)$$

This is a cubic equation. So we let a computer program solve it. We get as solutions  $s = -0.467$  and  $s = -1.642 \pm 2.067i$ . Inserting the first solution in the characteristic equation gives  $K = 2.825$ , which is positive. So  $s = -0.467$  is a breakaway point. We mark this one in our plot. However, inserting the other solutions results in a complex  $K$ . They are therefore not breakaway points.

4. We have  $n - m = 4$ . So there are four infinite zeroes, and thus also four asymptotes. The asymptote center is at

$$s = \frac{z_1 + z_2 + \dots + z_m - p_1 - p_2 - \dots - p_n}{n - m} = \frac{-0 - 1 - (2 + \sqrt{3}) - (2 - \sqrt{3})}{4} = -\frac{5}{4}. \quad (2.6)$$

The angle of the asymptotes can be found using

$$\text{asymptote angles} = \frac{\pm(2k+1) \cdot 180^\circ}{4}. \quad (2.7)$$

If we let  $k$  vary, we will find angles  $-135^\circ$ ,  $-45^\circ$ ,  $45^\circ$  and  $135^\circ$ . We lightly mark these asymptotic lines in our plot.

5. There are infinite zeroes. So the root-locus might cross the imaginary axis as well. To find out where it crosses it, we substitute  $s = \omega i$  into the characteristic equation. We then find that

$$K = -\omega^4 + 5\omega^3 i + 17\omega^2 - 13\omega i. \quad (2.8)$$

If we equate both the real and complex part, we will find that

$$K = -\omega^4 + 17\omega^2 \quad \text{and} \quad 0 = 5\omega^3 - 13\omega. \quad (2.9)$$

Solutions for  $\omega$  are  $\omega = 0$  and  $\omega = \pm\sqrt{13/5}$ . The corresponding solutions for  $K$  are  $K = 0$  and  $K = -(13/5)^2 + 17 \cdot 13/5 = 37.44$ , respectively. The solution  $\omega = 0$  isn't altogether very surprising. However, we find that the root-locus plot also crosses the imaginary axis at  $s = \pm\sqrt{13/5}i = \pm 1.612$ . We thus mark those points in our plot.

6. Finally we would like to know the angles of departure of the two complex poles. For that, we use the argument-equation. Let's find the angle of departure  $\theta$  for the pole  $p_1 = -2 + 3i$ . First we find the arguments we need to put into the argument-equation. They are

$$\arg((-2 + 3i) - (-2 - 3i)) = \arg(6i) = 90^\circ, \quad (2.10)$$

$$\arg((-2 + 3i) - 0) = \arg(-2 + 3i) = 123.69^\circ, \quad (2.11)$$

$$\arg((-2 + 3i) - (-1)) = \arg(-1 + 3i) = 108.43^\circ. \quad (2.12)$$

It now follows that the angle of departure of  $p_1$  is

$$\theta = (2k+1) \cdot 180^\circ - 90^\circ - 123.69^\circ - 108.43^\circ = -142.13^\circ. \quad (2.13)$$

From symmetry we can find that the angle of departure of  $p_2 = -2 - 3i$  is  $\theta = 142.13^\circ$ . We can mark these angles in our root-locus plot.

Now we should have sufficient data to sketch our root-locus plot. The result can be seen in figure 3.

### 3 Finding the appropriate points in the plot

It's nice that we know what poles we can give to our system. But how do we choose the right value of  $K$ ? That's what we'll look at now.

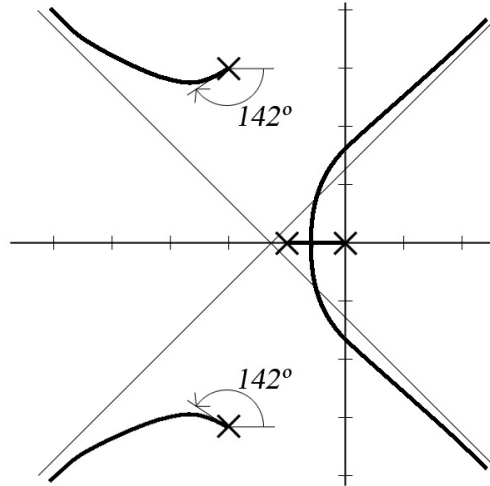


Figure 3: The solution of our second example problem.

### 3.1 Setting demands

Let's suppose we have drawn a root-locus plot. How can we use this plot to influence our system? For that, we will examine a pole. But first we have to make sure we examine the right pole of  $F(s)$ . But which pole is the right pole?

As you know, poles with a bigger real part are more influential. So actually, the pole closest to the origin is the only pole we should focus on. (The other poles are less significant.) Then, using certain **system demands**, we can find this pole.

But what kind of properties can we demand from our system? Let's make a list of that.

- We can set the dampening factor  $\zeta$ . (Often a value between 0.4 and 0.7 is preferred.) Let's suppose we use this criterion to find a pole  $p_i$ . We define the argument of this pole to be  $\phi = \arg(p_i)$ . It can then be shown that  $\zeta = \cos(180^\circ - \phi)$ .

So how do we find  $p_i$ ? Well, we draw a line from the origin with an angle  $\phi = 180^\circ - \arccos \zeta$ . The point where it intersects our root-locus plot is the position of our pole  $p_i$ .

- We can also set the damped natural frequency  $\omega_n$ . It can now be shown that the undamped natural frequency due to some pole  $p_i$  is  $|p_i| = \omega_n$ .

So how do we find  $p_i$ ? We simply draw a circle with radius  $\omega_n$  around the origin. The point where it intersects the root-locus plot is where our pole  $p_i$  is.

- Finally we can set the undamped natural frequency  $\omega_d$ . We can find that the damped natural frequency due to a pole  $p_i$  is equal to the real part of  $p_i$ . So,  $\text{Re}(p_i) = \pm\omega_d$ .

So how do we find  $p_i$ ? We simply draw a horizontal line at a height  $\pm\omega_d$ . The point where it intersects the root-locus plot is where our pole  $p_i$  is.

### 3.2 Finding the gain $K$

We now know how to find the right pole. But how do we use this to find  $K$ ? Well, we can simply use the characteristic equation for this. Insert the pole  $p_i$  we just found into the characteristic equation. From this, you can solve for  $K$  quite easily. And once you know  $K$ , all that is left to do is to find the other poles of  $F(s)$ .