Preliminary knowledge

Ever since mankind existed, they have tried to control things. During the past decennia, people have actually gotten quite good at it. So now it's our turn to take a look at how we can control systems.

1 Introduction to systems

To examine systems, we first need to know a few things: What are systems? And what kinds of systems are there? That's what we will look at first.

1.1 What is a system?

What is a system? A **system** is a combination of components that (together) perform a certain objective. Systems usually vary in time. This happens during a so-called **process**. A process is a set of controlled actions leading towards a certain end.

Another way to look at a system, is by considering it as some 'magic box'. You simply insert a certain input, denoted as $u(t)$ and out comes a certain output, denoted as $y(t)$. So a system merely takes an input and gives an output, according to a specified way.

1.2 System variables

Systems usually have variables. There are two important kinds of variables.

- First, there are the **manipulated variables**. They are controlled (set) by the controller. They are thus the input of the system. (Think of the deflection you give to the steering wheel of your car.)
- Second, there are the **controlled variables**. These variables are influenced by the manipulated variables. They are usually (but not always) the output of the system. (Think of the heading/position of the car.)

1.3 System types

There are many ways in which to distinguish systems. Let's have a look at a few important ones.

- Open/closed loop systems: In **closed-loop systems** (also called **feedback control systems**), the output is compared with the input. The result of this comparison is then used to change the output. In other words, in these systems the output effects the control action. Although such systems are usually relatively complicated, they handle disturbances rather well. However, in **open-loop system** the output does not effect the control action. These systems are usually quite simple. They aren't very good at dealing with disturbances though.
- Scalar/multivariable systems: In **scalar** systems there is only one input $u(t)$ and one output $y(t)$. **Multivariable systems** have multiple inputs $u_i(t)$ or multiple outputs $y_i(t)$ (or both).
- Linear/nonlinear systems: Let's suppose $y_1(t)$ is the respone of a system to input $u_1(t)$. Similarly, $y_2(t)$ is the respone to $u_2(t)$. Now examine the input signal $u(t) = c_1u_1(t) + c_2u_2(t)$. If the output of the system then is $y(t) = c_1y_1(t) + c_2y_2(t)$ (for all combinations of c_1, c_2, $u_1(t)$ and $u_2(t)$), then the system is **linear**. Otherwise, it is **nonlinear**.

2 The Laplace transform

A very important tool we need to evaluate systems, is the Laplace transform. This transform has been discussed in the differential equations course already. Although we will quickly go over it, it is advised that you have some preliminary knowledge on the Laplace transform already.

2.1 Definition of the Laplace transform

Let's suppose we have a function $f(t)$. The Laplace transform of $f(t)$, written as $\mathscr{L}(f(t))$, is defined as

$$\mathscr{L}(f(t)) = \int_0^\infty f(t)e^{-st}dt. \tag{2.1}$$

For example, the Laplace transform of the function $f(t) = at$ (with a constant) is

$$\mathscr{L}(at) = \int_0^\infty ate^{-st}dt = \left[-\frac{at}{s}e^{-st}\right]_0^\infty + \int_0^\infty \frac{a}{s}e^{-st}dt = 0 + \left[-\frac{a}{s^2}e^{-st}\right]_0^\infty = \frac{a}{s^2}. \tag{2.2}$$

Here s is a complex variable, which we will take a closer look at later. We denote the **inverse Laplace transform** as $\mathscr{L}^{-1}(F(s))$. For example, we have $\mathscr{L}^{-1}(a/s^2) = at$.

We can find Laplace transforms of a lot of functions with the above mentioned integral. This is, however, rather boring. And it has been done a lot of times already. Therefore so-called Laplace tables exist. These tables contain lists of function $f(t)$ and their corresponding Laplace transform $\mathscr{L}(f(t))$. If you want to take the Laplace transform of a function, I suggest you use such a table. (One can be found in the book on Control Theory, on pages 17 and 18. You can also find one on the page of the differential equations course.)

2.2 Special functions

There are several special functions, which have a rather basic Laplace transform. Let's have a look at them. First, there is the **ramp function** $r(t)$, defined as

$$r(t) = \begin{cases} 0 & \text{if } t < 0, \\ t & \text{if } t \geq 0. \end{cases} \tag{2.3}$$

The Laplace transform of this function is $\mathscr{L}(r(t)) = 1/s^2$. Second, there is the **unit step function** $u(t)$. This function is, in fact, the derivative of the ramp function. It is defined as

$$u(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } t \geq 0. \end{cases} \tag{2.4}$$

Its Laplace transform is given by $\mathscr{L}(u(t)) = 1/s$. And third, there is the **unit impulse function** $\delta(t)$. This, in turn, is the derivative of the unit step function. It is defined such that

$$\delta(t) = \begin{cases} 0 & \text{if } t \neq 0, \\ \infty & \text{if } t = 0, \end{cases} \qquad \text{but also} \qquad \int_{-\infty}^\infty \delta(t)\, dt = 1. \tag{2.5}$$

Its Laplace transform is given by $\mathscr{L}(\delta(t)) = 1$.

2.3 Special rules for derivatives

When using the Laplace transform, we can also apply several rules. Some of these rules are actually quite handy. For example, the rules for integrals and derivatives. Let's suppose we have a function $f(x)$ and know its Laplace transform $F(s) = \mathscr{L}(f(t))$. What are the Laplace transforms of the derivate and integral of $f(x)$? Well, they are given by

$$\mathscr{L}\left(\frac{df}{dt}\right) = sF(s) - f(0) \qquad \text{and} \qquad \mathscr{L}\left(\int f(t)\,dt\right) = \frac{F(s) + f(0)}{s}. \tag{2.6}$$

In this course, we usually assume that $f(0) = 0$. In that case the relations above simplify to

$$\mathscr{L}\left(\frac{df}{dt}\right) = sF(s) \qquad \text{and} \qquad \mathscr{L}\left(\int f(t)\,dt\right) = \frac{F(s)}{s}. \tag{2.7}$$

So we see that, if we take a derivative of $f(t)$, we should multiply its Laplace transform by s. Similarly, if we integrate $f(t)$, we should divide its Laplace transform by s. This is an interesting rule, which you should remember. By the way, if you look at the previous paragraph (about the special functions), you see that this rule holds.

2.4 Change-of-variable rules

But there are more special rules for the Laplace transform. We can, for example, apply a change-of-variable. Let's once more suppose that we know $F(s) = \mathscr{L}(f(t))$. Then what is the Laplace transform of $f(at + b)$? Well, it is

$$\mathscr{L}(f(at + b)) = \frac{e^{\frac{b}{a}}}{a} F\left(\frac{s}{a}\right). \tag{2.8}$$

We can also apply a change-of-variable the other way around. Again we have $F(s) = \mathscr{L}(f(t))$. But what is the inverse Laplace transform of $F(as + b)$? This then is

$$\mathscr{L}^{-1}(F(as + b)) = \frac{e^{-\frac{b}{a}t}}{a} f\left(\frac{t}{a}\right). \tag{2.9}$$

By the way, it is also important to know that the Laplace transform is a **linear operator**. This means that

$$\mathscr{L}(c_1 f_1(t) + c_2 f_2(t)) = c_1 \mathscr{L}(f_1(t)) + c_2 \mathscr{L}(f_2(t)). \tag{2.10}$$

2.5 Roots and poles of a function

Let's suppose we have a function $F(s)$. For example, let's examine

$$F(s) = \frac{K(s + 2)(s + 10)}{s(s + 1)(s + 5)^2(s + 10)(s^2 + 9)}. \tag{2.11}$$

We know that s is a complex variable. So we can examine this function on all points in the complex plane. The points where this function is analytic (where a unique derivative exists) are called **ordinary points**. Other points are called **singular points**. Singular points can be classified as either **zeroes**, **poles** or neither.

To find the zeroes, we need to look at the numerator. This should be zero. For $F(s)$, the value $s = -2$ is a zero. To find the poles, we need to look at the denominator. Now this should be zero. So, for $F(s)$, the poles are $s = 0$, $s = -1$, $s = -5$, $s = 3i$ and $s = -3i$. One may initially think that $s = -10$ is both a pole and a zero. However, it is neither. $s = -10$ is just a singular point.

Poles and zeroes also have an **order**. This is the amount of times which they occur. In our example, the pole $s = -1$ occurred just once. (There was only one factor $(s + 1)$.) Its order is thus 1. The order of $s = -5$, however, is two. This is because the factor $(s + 5)$ occurred twice.

We have actually missed a zero of $F(s)$. Let's ask ourselves, what happens if $s \to \infty$? In this case the constant coefficients within the brackets don't really matter anymore. (For example, as $s \to \infty$, then $s + 5 \approx s$.) So $F(s) \to Ks^2/s^7 = K/s^5$. Thus $s = \infty$ is a zero of fifth order.

It is important to note that zeroes and poles aren't always real numbers. They can also be complex numbers, as we have already seen. An interesting fact, however, is that complex zeroes/poles always come in pairs. If $s = a + bi$ is a pole, then its complex conjugate $\bar{s} = a - bi$ is also a pole. The same goes for zeroes.

2.6 Finding the inverse Laplace transform using partial-fraction expansion

Let's suppose that we have a function $F(s)$, and want to find the inverse Laplace transform $\mathscr{L}^{-1}(F(s))$. We can get a Laplace transform table, and hope that $F(s)$ is in there. But if it is not, what do we do then? Panic? Rather not. Instead, we can apply a partial-fraction expansion to $F(s)$. For that, we first have to rewrite $F(s)$ in the form

$$F(s) = \frac{B(s)}{A(s)} = K\frac{(s-z_1)^{r_{z1}}(s-z_2)^{r_{z2}}\dots(s-z_m)^{r_{zm}}}{(s-p_1)^{r_{p1}}(s-p_2)^{r_{p2}}\dots(s-p_n)^{r_{pn}}}. \quad (2.12)$$

Here $z_1, z_2, \dots, z_m$ are the zeroes of $F(s)$, having orders $r_{z1}, r_{z2}, \dots, r_{zn}$, respectively. The poles p_i are defined similarly. Now we want to rewrite $F(s)$. How can we do that? Well, to simplify matters, we first examine the case where all poles have order 1 (so $r_{p1} = r_{p2} = \dots = r_{pn} = 1$). We can then write $F(s)$ as

$$F(s) = \frac{a_1}{(s-p_1)^{r_{p1}}} + \frac{a_2}{(s-p_2)^{r_{p2}}} + \dots + \frac{a_n}{(s-p_n)^{r_{pn}}}. \quad (2.13)$$

The coefficients $a_1, a_2, \dots, a_n$ are called the **residues** at the poles. The question now is, how can we find these residues? Well, to find a residue a_k, we simply multiply both sides by the factor $(s - p_k)$. We then insert $s = p_k$ into the equation. Due to this, a lot of terms will vanish. We remain with

$$a_k = \left((s-p_k)\frac{B(s)}{A(s)}\right)_{s=p_k}. \quad (2.14)$$

It may seem like the right side of the equation will be zero. This is, however, not the case, because $A(s)$ has a factor $(s - p_k)$ as well. These factors will thus cancel each other out. So then we can simply find the residue a_k.

But what do we do if there is a pole p_k with order greater than 1? Let's, for example, say that this order is $r_{pk} = 3$. In this case we have to include more terms in $F(s)$. 3 more terms, to be exact. Let's call the part of $F(s)$ representing this pole $F_k(s)$. We then have

$$F_k(s) = \frac{a_{k_1}}{(s-p_k)} + \frac{a_{k_2}}{(s-p_k)^2} + \frac{a_{k_3}}{(s-p_k)^3}. \quad (2.15)$$

How do we find the residues a_{k_1}, a_{k_2} and a_{k_3} now? We can find a_{k_3} in a rather familiar way. We multiply both sides of the equation by $(s - p_k)^3$. We then get

$$(s-p_k)^3 F_k(s) = a_{k_1}(s-p_k)^2 + a_{k_2}(s-p_k) + a_{k_3}. \quad (2.16)$$

By inserting $s = p_k$ we can find a_{k_3}. But how do we find the other coefficients? To find them, we differentiate with respect to s. Differentiating once gives

$$\frac{d}{ds}\left((s-p_k)^3 F_k(s)\right) = 2a_{k_1}(s-p_k) + a_{k_2}. \quad (2.17)$$

By inserting $s = p_k$, we can find a_{k_2}. To find a_{k_1}, we have to differentiate once more. (And if the order of the pole is even higher, you have to differentiate even more times to find the other coefficients.)

In the end you should have the partial-fraction expansion of $F(s)$. And then it is quite easy to find the inverse Laplace transform $f(t)$.