Analyzing systems subject to basic inputs

We now know how we can display systems. The next step is to analyze the systems. Let's subject systems to basic input functions. What can we then say about their behaviour?

1 Analyzing basic systems

This is where we start our journey into the wondrous world of stability. We begin by examining the definitions of stability, after which we examine some simple systems.

1.1 Definition of stability

Let's suppose we have a system. This system has zero input (so $u(t) = 0$). It may occur that the position/state of the system doesn't change in time. If this is the case, then the system is in a so-called **equilibrium position**.

Now let's give the system certain initial conditions. (We put it in a position other than the equilibrium position.) What will happen? This, in fact, depends on the system. Some systems converge back to an equilibrium position. These systems are called **stable**. Other systems oscillate about an equilibrium position, but don't converge to it. Such systems are **critically stable**. If the system diverges, then it is **unstable**.

1.2 First order systems

Let's examine a first order system. The transfer function of such a system can be written as

$$F(s) = \frac{1}{Ts+1}. \tag{1.1}$$

We want to investigate the stability of this system. There is, however, a slight problem. Every system behaves differently to different input functions. So we can't give any general rules about the behaviour of a system.

However, we want to get at least some indication of the system behaviour. So we examine its response to some basic functions: the unit impulse function, the unit step function and the unit ramp function. Another basic function is the sine function. However, we will not examine the response of a system to periodic inputs now. We leave that for the chapter on frequency response analysis.

Let's suppose the input of the system is a unit impulse function. In this case $U(s) = 1$ and thus $Y(s) = F(s)$. We then find that

$$y(t) = \mathscr{L}^{-1}\left(F(s)\right) = \mathscr{L}^{-1}\left(\frac{1}{Ts+1}\right) = \frac{1}{T}e^{-t/T}. \tag{1.2}$$

But what if the input is a unit step function? We then have $U(s) = 1/s$ and thus $Y(s) = F(s)/s$. The output of the system then becomes

$$y(t) = \mathscr{L}^{-1}\left(F(s)/s\right) = \mathscr{L}^{-1}\left(\frac{1}{Ts^2+s}\right). \tag{1.3}$$

To find the inverse Laplace transform, we can apply a partial-fraction expansion. This would give us

$$y(t) = \mathscr{L}^{-1}\left(\frac{1}{Ts^2+s}\right) = \mathscr{L}^{-1}\left(\frac{1}{s} - \frac{T}{Ts+1}\right) = 1 - e^{-t/T}. \tag{1.4}$$

Finally, we examine the unit ramp function. We now have $U(s) = 1/s^2$ and thus $Y(s) = F(s)/s^2$. The output of the system now is

$$y(t) = \mathscr{L}^{-1}\left(F(s)/s^2\right) = \mathscr{L}^{-1}\left(\frac{1}{Ts^3+s^2}\right) = \mathscr{L}^{-1}\left(\frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{Ts+1}\right) = t - T + Te^{-t/T}. \qquad (1.5)$$

1.3 The steady-state error

There are a couple of important parameters describing the behaviour of a system. The **error** $e(t)$ is the difference between the output and the input. So, $e(t) = u(t) - y(t)$. The **steady-state error** error e_{ss} is now defined as

$$e_{ss} = \lim_{t\to\infty} e(t). \qquad (1.6)$$

This limit may seem rather annoying, because we first have to find $e(t)$. But luckily, we don't always have to! This is because we can use **initial/final value theorem**. This theorem states that

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s) - f(0) \qquad \text{and} \qquad \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) - f(0). \qquad (1.7)$$

We can use the second part of this theorem. Our initial conditions are zero, so $e(0) = 0$. We also have $E(s) = U(s) - Y(s)$. This gives

$$e_{ss} = \lim_{t\to\infty} s(U(s) - Y(s)). \qquad (1.8)$$

Now let's try to find the steady state error for the first order system of the previous paragraph. If the input is a unit impulse function or a unit step function, the steady-state error is zero. However, for the unit ramp function, this is not the case. This steady-state error is equal to

$$e_{ss} = \lim_{t\to\infty} u(t) - y(t) = \lim_{t\to\infty} t - (t - T - e^{-t/T}) = \lim_{t\to\infty} T + e^{-t/T} = T. \qquad (1.9)$$

(Note that we don't have to apply the theorem we just showed. This is because we already have an expression for $e(t)$.) You may wonder, why does the system have an error when it is subject to a unit ramp function? We will investigate the reasons behind this later.

1.4 Second order systems

Now let's examine second order systems. Let's suppose we have a transfer function of the form

$$F(s) = \frac{c}{as^2 + bs + c}. \qquad (1.10)$$

We can rewrite this to the so-called **standard form** of the second order system, being

$$F(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \qquad (1.11)$$

where $\omega_n^2 = c/a$ and $2\zeta\omega_n = b/a$. By the way, ω_n is the **undamped natural frequency** and ζ is the **damping ratio**. If you have attended the vibrations course, both terms will be very familiar to you.

Let's suppose that the input is a unit step function. We thus have $Y(s) = F(s)/s$. What will the output $y(t)$ of the system be? If $0 < \zeta < 1$, then we are dealing with the **underdamped case**. In this case the solution will have oscillation. In fact, we can find that

$$y(t) = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right). \qquad (1.12)$$

By the way, the variable $\omega_d = \omega_n\sqrt{1-\zeta^2}$ is the **damped natural frequency**. If $\zeta = 1$, then we are dealing with the **critically damped case**. This time no oscillations occur. Instead, we have

$$y(t) = 1 - e^{-\omega_n t}(1 + \omega_n t). \tag{1.13}$$

Finally, there is the case $\zeta > 1$. We now have an **overdamped system**. The solution for $y(t)$ is

$$y(t) = 1 - \frac{1}{2\sqrt{\zeta^2-1}}\left(\frac{e^{-(\zeta-\sqrt{\zeta^2-1})\omega_n t}}{\zeta-\sqrt{\zeta^2-1}} - \frac{e^{-(\zeta+\sqrt{\zeta^2-1})\omega_n t}}{\zeta+\sqrt{\zeta^2-1}}\right). \tag{1.14}$$

This time no oscillations occur either. Instead, the system slowly converges to the equilibrium position.

1.5 Other important parameters

Next to the steady-state error, there are several other important parameters describing the response of a system. Their definition is most simple if the input of a system is a unit step function. (Thus $U(s) = 1/s$.) So we examine that case.

- The **delay time** t_d is the time required for the response to reach half of the final value. For a unit step function, we thus have $y(t_d) = 1/2$.
- The **rise time** t_r is the time required for the response to rise from either 10% to 90%, from 5% to 95% or from 0% to 100%. (Which one to use depends on the circumstances.)
- The **peak time** t_p is the time required for the response to reach the first peak. So, t_p is the smallest positive value for which $dy(t_p)/dt = 0$.
- The **maximum percent overshoot** is the percentage of overshoot at the first peak. So this is

$$\frac{y(t_p) - y(\infty)}{y(\infty)} \times 100\%. \tag{1.15}$$

- The **settling time** t_s is the time required for the response to reach and stay within a certain percentage of the final value. This percentage is usually 2% or 5%. We thus have, for the 2% case,

$$\frac{|y(t) - y(\infty)|}{|y(\infty)|} \leq 0.02 \qquad \text{for} \qquad t \geq t_s. \tag{1.16}$$

2 General rules for stability

We have previously examined first order and second order systems. Now let's examine systems in general. What can we then say about the stability of a system?

2.1 Finding the general solution

Let's examine a system with transfer function $F(s)$. We can write this function as

$$F(s) = K\frac{(s-z_1)(s-z_2)\dots(s-z_m)}{(s-p_1)(s-p_2)\dots(s-p_n)}, \tag{2.1}$$

where $z_1, \dots, z_m$ are the zeroes and $p_1, \dots, p_n$ are the poles. Now what will happen if this system is subject to a unit step function as input? To find that out, we will try to find the solution for $y(t)$. We know we have $Y(s) = F(s)/s$. We can use a partial-fraction expansion to rewrite this to

$$Y(s) = \frac{F(s)}{s} = \frac{a}{s} + \sum_{i=1}^{n}\frac{a_i}{s-p_i}. \tag{2.2}$$

Let's take a closer look at these poles p_i. They are either real or complex. However, if they are complex, then they always come in pairs of complex conjugates. So we can assume that there are q real poles $p_1, \ldots, p_q$ and r complex pole pairs, being $(\lambda_1 \pm \eta_1 i), \ldots, (\lambda_r \pm \eta_r i)$. (Note that $q + 2r = n$.) We can then find that

$$Y(s) = \frac{F(s)}{s} = \frac{a}{s} + \sum_{j=1}^{q} \frac{a_j}{s - p_j} + \sum_{k=1}^{r} \frac{b_k(s - \lambda_k) + c_k \eta_k}{s^2 - 2\lambda_k s + (\lambda_k^2 + \eta_k^2)}. \tag{2.3}$$

Finding the inverse Laplace transform gives us

$$y(t) = a + \sum_{j=1}^{q} a_j e^{p_j t} + \sum_{k=1}^{r} b_k e^{\lambda_k t} \cos \eta_k t + \sum_{k=1}^{r} c_k e^{\lambda_k t} \sin \eta_k t. \tag{2.4}$$

And this is the solution to our problem.

2.2 The importance of poles

It's very nice that we found a solution in the previous paragraph. But what can we conclude from it? Well, something very important! We can find whether $y(t)$ converges.

Let's suppose there is some pole of $Y(s)$ with a real part bigger than zero. (So either $p_j > 0$ for a real pole or $\lambda_k > 0$ for a complex pole.) In this case $y(t) \to \infty$ as $t \to \infty$. In other words, the system is unstable! Therefore, to ensure stability, the real part of all poles must be smaller than zero.

However, we can conclude more from this. Let's examine two poles p_i and p_j, one with a very negative real part λ_i, and one with an (also) negative real part λ_j closer to zero. Which one is more important? Well, the term $e^{\lambda_i t}$ (for the very negative pole) goes to zero very quickly as t increases. However, the term $e^{\lambda_j t}$ (for the less negative pole) stays relatively big. Therefore the pole p_j is more important. And in general we can say: the higher the real part of the pole, the more influential this pole is.

You may wonder, what is the effect of the complex part of the poles? Well, for that we examine the part $\sin \eta_k t$. We see that η_k is the angular frequency of this part. In other words, complex poles with a high value for η_k cause high frequency oscillations.

And finally, you may be wondering, don't zeroes have any effect? Well, you are right that zeroes aren't very important. However, they do have some effect. It can be shown that if a zero z_i is close to some pole p_j, then the residue a_j of that pole is small. This reduces the effect of the pole. What can we conclude from this? A zero near a pole reduces the effect of the pole.

2.3 System types

Now we know whether $y(t)$ converges. But if it converges, does it converge to the actual input? Or will there be a steady-state error? That's what we will examine shortly.

But first, we examine a system. In fact, we examine a basic feedback system, with unity feedback. (This means that $H(s) = 1$.) The transfer function of the system thus becomes

$$F(s) = \frac{Y(s)}{U(s)} = \frac{G(s)}{1 + G(s)}. \tag{2.5}$$

Let's take a closer look at this control function $G(s)$. In fact, let's write it as

$$G(s) = K \frac{(1 - s/z_1)(1 - s/z_2) \ldots (1 - s/z_m)}{s^N (1 - s/p_1)(1 - s/p_2) \ldots (1 - s/p_{n-N})}. \tag{2.6}$$

Here the values p_i are all nonzero. In other words, we define N as the number of times which the pole $s = 0$ occurs for $G(s)$. We now say that our entire feedback system is a **type N system**. So if $N = 0$

(and $s = 0$ is thus not a pole of $G(s)$), then we have a type 0 system. If $N = 1$, we have a type 1 system. And this continues for $N = 2, 3, \ldots$.

Now what's the use of this? Well, it will turn out that the system type will say something about the steady-state error of the system. And that's what we'll look at in the upcoming paragraph.

2.4 Effecting the steady-state error of the system

We are going to investigate the steady-state error of a basic feedback system. To do that, we must first find the error function $E(s)$. The transfer function is given by equation (2.5). The error function $E(s)$ will then be

$$E(s) = \frac{Y(s)}{G(s)} = \frac{U(s)}{1 + G(s)}. \tag{2.7}$$

To find the steady-state error, we can apply the final value theorem. It states that

$$e_{ss} = \lim_{s \to 0} \frac{sU(s)}{1 + G(s)}. \tag{2.8}$$

From equation (2.6) we can see that

$$\lim_{s \to 0} G(s) = \lim_{s \to 0} K/s^N. \tag{2.9}$$

Now let's suppose that the input function is $U(s) = 1/s^q$, for some value q. We then find that

$$e_{ss} = \lim_{s \to 0} \frac{1}{s^{q-1} + Ks^{q-1-N}}. \tag{2.10}$$

Great! Now what does it mean?

- If $q = 0$ (and we thus have a unit impulse function as input), then $s^{q-1} \to \infty$. This means $e_{ss} = 0$, regardless of the system.
- If $q = 1$ (meaning the input is a unit step function), then $s^{q-1} \to 1$. We can't say anything about the steady-state yet. So we have to examine Ks^{q-1-N}. If $N = q - 1 = 0$, then $Ks^{q-1-N} = K$. This means that there will be a steady-state error. In fact, we find that $e_{ss} = 1/(1 + K)$. If, however, $N > 0$, we find that $Ks^{q-1-N} \to \infty$. This means that $e_{ss} = 0$. So there is no steady-state error.
- If $q \geq 2$, then $s^{q-1} \to 0$. So we again have to examine Ks^{q-1-N}. If $N > q - 1$, we find that $Ks^{q-1-N} \to \infty$. This means that $e_{ss} = 0$. So there is no steady-state error. If, however, $N > q - 1$, then $e_{ss} \to \infty$. The system thus can't follow the input. Finally, if $N = q - 1$, then the steady-state error is $e_{ss} = 1/K$.

Let's summarize the results. If $N < q - 1$, there is an infinite steady-state error. If $N = q - 1$, there is a finite steady-state error. And if $N > q - 1$, there is no steady-state error. So to get rid of our steady-state error, we must increase N.

By the way, there is a very common mistake here. To find the type of the system, you should not examine the transfer function $F(s)$ of the system. Instead, investigate the feedforward transfer function $G(s)$. Only then will you find the right conclusions.