Velocity gradients and stress tensors

It's time to look at an actual constitutive model: the stress tensor. How can we relate it to a physical phenomenon like velocity?

1 Indifferent vectors and tensors

1.1 Configuration and source variable notation

It's time to turn our attention to some less abstract stuff. First let's examine configuration variables (like position and velocity). Suppose we have some point P. We can indicate the position of P by a vector $\mathbf{x}$. Similarly, the velocity of P is denoted by $\mathbf{v} = d\mathbf{x}/dt = \dot{\mathbf{x}}$. These vectors (and all other vectors belonging to configuration variables) live in the **primal space**. We can also add a coordinate system to our primal space. We can then write a position as $\mathbf{x} = x^i \mathbf{e_i}$ and a velocity as $\mathbf{v} = v^i \mathbf{e_i}$.

Source variables (like forces $\mathbf{f}$ and stresses τ) are, however, something different. They live in the **dual space**. For that reason, we can't represent them by vectors. But instead, we use **covectors**. Luckily, these covectors are quite similar to vectors. Only the indices have changed position. We should, for example, write the force covector as $\mathbf{f} = f_i \mathbf{e^i}$.

1.2 Links between primal and dual space

The question remains, how do we go from the primal space to the dual space? To do that, we use tensors. An example is the **stress tensor** σ. This tensor relates a normal vector $\mathbf{n}$ (in the primal space) to a stress covector $\mathbf{f}$ (in the dual space). It does this according to

$$\mathbf{f} = \sigma \mathbf{n}. \tag{1.1}$$

When we use coordinate systems, we can represent σ by a matrix. In this case we can also write that

$$f_i = \sigma_{ij} n^j. \tag{1.2}$$

1.3 Indifferent vectors

The relation that $\mathbf{f} = \sigma \mathbf{n}$ does not depend on the coordinate system we use. In other words, if we change coordinate systems, it should still be satisfied. That is, as long as the distances remain the same. (In stretched space strange things occur.) So let's examine such a (non-stretched) transformation. In fact, let's examine the worst transformation possible. Let's consider

$$\mathbf{x}^* = Q(t)(\mathbf{x} - \mathbf{z}) + \mathbf{y}(t), \tag{1.3}$$

where $\mathbf{x}$ denotes the initial vector, and $\mathbf{x}^*$ denotes the transformed vector. What does this transformation do? Well, first it moves the origin of the coordinate system to $\mathbf{z}$. Then the orthogonal **transformation matrix** $Q(t)$, which might even vary with time, causes things to rotate about point $\mathbf{z}$. (It is important that $Q(t)$ is orthogonal. In non-orthogonal transformations, distances don't remain the same.) Finally, we move the whole system by a vector $\mathbf{y}(t)$.

Yes, it's a scary transformation. Now let's make a definition. We say that any vector $\mathbf{a}$ which satisfies

$$\mathbf{a}^* = Q(t)\mathbf{a} \tag{1.4}$$

is called an **indifferent vector**. A normal position vector $\mathbf{x}$ is not indifferent, since $\mathbf{x}^* \neq Q(t)\mathbf{x}$. However, a difference vector $\mathbf{a} = \mathbf{x_2} - \mathbf{x_1}$ is indifferent. To see why, we could insert it into our transformation (1.3).

This would eventually give us

$$\mathbf{x_2^*} - \mathbf{x_1^*} = Q(t)(\mathbf{x_2} - \mathbf{x_2}) \qquad \Rightarrow \qquad \mathbf{a}^* = Q(t)\mathbf{a}. \tag{1.5}$$

(The vectors **y** and **z** cancel out.) However, the velocity vector **v** is not indifferent. To see why, we could differentiate equation (1.3) with respect to time t. This would then give us

$$\mathbf{v}^* = \dot{\mathbf{x}}^* = Q\mathbf{v} + \dot{Q}(\mathbf{x} - \mathbf{z}) + \dot{\mathbf{y}}. \tag{1.6}$$

(For simplicity we have written $Q(t)$ as Q.) So $\mathbf{v}^* \neq Q\mathbf{v}$.

1.4 Indifferent tensors

Tensors that map indifferent vectors onto indifferent vectors are called **indifferent tensors**. So if **b** and **a** are indifferent vectors, and $\mathbf{b} = A\mathbf{a}$, then the tensor A is indifferent. An indifferent tensor A also has a corresponding indifferent cotensor A^* satisfying $\mathbf{b}^* = A^*\mathbf{a}^*$. Of course there is a relation between A and A^*. To find this relation, we use

$$\mathbf{b}^* = Q\mathbf{b} = QA\mathbf{a} = QAQ^T\mathbf{a}^*. \tag{1.7}$$

Note that, in the last step, we have used $\mathbf{a} = Q^T\mathbf{a}^*$. Because Q is orthogonal, it satisfies $Q^T = Q^{-1}$. From the above equation follows that

$$A^* = QAQ^T. \tag{1.8}$$

2 Velocity properties

2.1 A vector gradient

Let's examine a vector field **a**. We now define the **gradient** $\nabla\mathbf{a}$ of a vector field **a**. It is the tensor which converts a change in position **x** (written as $d\mathbf{x}$) to a change in the vector **a** (denoted by $d\mathbf{a}$). So we have $d\mathbf{a} = \nabla\mathbf{a}\, d\mathbf{x}$.

If we use a 3D Cartesian coordinate system, we could write this in matrix form, according to

$$\begin{bmatrix} \delta a_1 \\ \delta a_2 \\ \delta a_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial a_1}{\partial x_1} & \frac{\partial a_1}{\partial x_2} & \frac{\partial a_1}{\partial x_3} \\ \frac{\partial a_2}{\partial x_1} & \frac{\partial a_2}{\partial x_2} & \frac{\partial a_2}{\partial x_3} \\ \frac{\partial a_3}{\partial x_1} & \frac{\partial a_3}{\partial x_2} & \frac{\partial a_3}{\partial x_3} \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \end{bmatrix}. \tag{2.1}$$

2.2 The velocity gradient and the rate of strain

The **velocity gradient** is denoted by $\nabla\mathbf{v}$. It satisfies

$$d\mathbf{v} = \nabla\mathbf{v}\, d\mathbf{x} \qquad \text{and} \qquad d\mathbf{v}^* = \nabla\mathbf{v}^* d\mathbf{x}^*. \tag{2.2}$$

We could ask ourselves, is $\nabla\mathbf{v}$ an indifferent tensor? Well, it can be shown that

$$\nabla\mathbf{v}^* = Q\ \nabla\mathbf{v}\ Q^T + \dot{Q}Q^T. \tag{2.3}$$

Since $\nabla\mathbf{v}^* \neq Q\ \nabla\mathbf{v}\ Q^T$, the velocity gradient is not an indifferent tensor.

Now let's examine the **rate of strain** tensor D. It is defined as

$$D = \frac{1}{2}\left(\nabla\mathbf{v} + (\nabla\mathbf{v})^T\right). \tag{2.4}$$

In other words, the components of D are

$$D_{ij} = \frac{1}{2}\left(\frac{\partial u^i}{\partial x^j} + \frac{\partial u^j}{\partial x^i}\right). \tag{2.5}$$

The diagonal entries of D are called the **normal strain rates**. The non-diagonal entries are the **shear strain rates**. By the way, it can be shown that $\nabla D^* = Q\ \nabla D\ Q^T$. So the rate of strain tensor is indifferent.

2.3 Rotation and vorticity

Similar to the rate of strain tensor, we can define the **rate of rotation tensor** Ω. It is given by

$$\Omega = \frac{1}{2}\left(\nabla \mathbf{v} - (\nabla \mathbf{v})^T\right), \qquad \text{or} \qquad \omega_{ij} = \frac{1}{2}\left(\frac{\partial u^i}{\partial x^j} - \frac{\partial u^j}{\partial x^i}\right). \tag{2.6}$$

We can now also define the **rotation vector** ω as

$$\omega = \omega_{32}\mathbf{e_1} + \omega_{13}\mathbf{e_2} + \omega_{21}\mathbf{e_3} = \begin{bmatrix} \frac{1}{2}\left(\frac{\partial u^3}{\partial x^2} - \frac{\partial u^2}{\partial x^3}\right) \\ \frac{1}{2}\left(\frac{\partial u^1}{\partial x^3} - \frac{\partial u^3}{\partial x^1}\right) \\ \frac{1}{2}\left(\frac{\partial u^2}{\partial x^1} - \frac{\partial u^1}{\partial x^2}\right) \end{bmatrix}^T. \tag{2.7}$$

Finally, the **vorticity vector** ξ is defined as $\xi = 2\omega = \text{curl } \mathbf{u}$.

3 The stress tensor

3.1 The total stress tensor

Now we turn our attention to the important stress tensor. It can be shown that the stress tensor σ is indifferent. (The equation $\sigma^* = Q\sigma Q^T$ holds.) We can split σ up in two parts, being

$$\sigma = -pI + \tau, \tag{3.1}$$

where I is the **identity tensor**. (The identity tensor satisfies $I^* = QIQ^T = QQ^T = I$.) The part $-pI$ represents the stresses due to **compression** of the fluid. (In fact, p is the pressure, so $-p$ is the compression.) The part τ is due to viscous stresses.

If we write σ like this, we call σ the **total stress tensor**. τ is the **extra stress tensor** and $-pI$ is the **extra deviatoric part**. There is a rule stating that adding and subtracting indifferent tensors also gives indifferent tensors. Since both σ and I are indifferent, also τ must be indifferent.

3.2 The extra stress tensor

We know from experience that viscous stresses depend on the velocity $\mathbf{v}$ of the fluid. So we need to find a constitutive model that relates τ to something related to velocity. We can not relate τ directly to $\mathbf{v}$, since $\mathbf{v}$ is not indifferent. The velocity gradient $\nabla \mathbf{v}$ is not indifferent either. However, the rate of strain tensor D is indifferent. For that reason, we generally say that $g(\tau) = f(D)$, with f and g functions. These functions depend on the type of fluid.

Let's look at the stress-strain behaviour of the fluid. If it is isotropic (the same in all directions) and linear, then τ only depends on two parameters. In this case, the so-called **linear isotropic (Newtonian) stress-strain relation** applies. It states that

$$\tau = \lambda(\text{div } \mathbf{u})I + 2\mu D. \tag{3.2}$$

The variable μ is the **viscosity**. This viscosity depends on the composition of the gas, the pressure p and the temperature T. Luckily, for $T < 3000K$, the dependence on pressure is negligible. In that case, **Sutherland's formula** holds. It states that

$$\frac{CT^{3/2}}{T+S}, \tag{3.3}$$

where $C = 1.458 \cdot 10^{-6} Pa\, s\, K^{1/2}$ and $S = 110.4K$.

So, we eventually find that the stress tensor can be represented by

$$\sigma = -pI + \lambda(\text{div } \mathbf{u})I + 2\mu D. \tag{3.4}$$