

The conservation laws

Very important in fluid dynamics are the three **conservation laws**, also known as the **Navier Stokes equations**. Let's see how they are derived, and how we can write them.

1 Two ways to describe fluids

1.1 Describing fluids

When describing fluids, we usually examine a small piece of fluid, called a **fluid packet**. There are, however, two ways to represent a packet.

In the **Lagrangian description**, the fluid properties (like density, temperature, and such) are attached to the packet. On top of that, the packet has a position vector \mathbf{x} to indicate its position.

In the **Eulerian description**, the fluid properties are attached to the position vector. So, if we have a position vector \mathbf{x} , then we know the properties of the fluid packet \mathbf{x} points to.

In solid mechanics, the Lagrangian description is more convenient. In fluid mechanics, however, we usually use the Eulerian description.

1.2 Transforming between descriptions

There are several conservation laws. There are **conservation of mass**, **conservation of momentum** and **conservation of energy**. In the Lagrangian description, these three laws are described by the equations

$$\frac{dm}{dt} = 0, \quad \frac{d}{dt} \left(\frac{d\mathbf{x}}{dt} \right) = \mathbf{F} \quad \text{and} \quad \frac{dE}{dt} = \frac{dQ}{dt} + \frac{dW}{dt}. \quad (1.1)$$

m denotes mass, \mathbf{F} the external forces, E the energy, Q the added heat and W the work done.

These laws aren't the same in the Eulerian description. This is because in the Lagrangian description, the position vector \mathbf{x} changes with time. To find out how to transform these laws, we examine some property $\Phi(t, x^1, x^2, x^3)$. The time derivative of this property is given by

$$\frac{d\Phi}{dt} = \frac{\partial\Phi}{\partial t} + \frac{\partial\Phi}{\partial x^1} \frac{dx^1}{dt} + \frac{\partial\Phi}{\partial x^2} \frac{dx^2}{dt} + \frac{\partial\Phi}{\partial x^3} \frac{dx^3}{dt} = \frac{\partial\Phi}{\partial t} + u^i \frac{\partial\Phi}{\partial x^i} = \frac{\partial\Phi}{\partial t} + \mathbf{u} \cdot (\nabla\Phi). \quad (1.2)$$

So we see that, to transform the laws to the Eulerian description, we have to use

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla. \quad (1.3)$$

This derivative D/Dt is called the **Lagrangian time derivative** or the **material time derivative**. (Do not confuse it with the rate of strain tensor D .)

Now we know what to do with the time-derivative of a normal scalar. But what about time-derivatives of integrals? We often integrate over volumes. However, a volume in Lagrangian space (consisting of a lot of fluid packets) can change shape. This is rather troubling. Luckily, there is the **Reynolds transport theorem**. It states that, for some variable ϕ , we have

$$\frac{d}{dt} \int_V \Phi dV = \int_V \left(\frac{D\Phi}{Dt} + \Phi \nabla \cdot \mathbf{u} \right) dV. \quad (1.4)$$

Now we also know what to do with time-derivatives of integrals.

2 Conservation of mass

2.1 Derivation of the equation

With what we've just learned, let's derive the conservation laws. First, we examine conservation of mass. It states that the total mass doesn't change. To find the total mass, we integrate over the density. This gives us

$$\frac{dm}{dt} = \frac{d}{dt} \int_V \rho dV = \int_V \left(\frac{D\rho}{Dt} + \rho \nabla \mathbf{u} \right) dV = 0. \quad (2.1)$$

This must hold for every volume. Therefore, the part within the integral must be zero. So,

$$\frac{D\rho}{Dt} + \nabla \mathbf{u} = \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot (\nabla \rho) + \rho \nabla \mathbf{u} = 0. \quad (2.2)$$

This can further be simplified to

$$\frac{\partial \rho}{\partial t} + \nabla(\rho \mathbf{u}) = 0. \quad (2.3)$$

This equation is known as the **conservation of mass equation**.

3 Conservation of linear momentum

3.1 Derivation of the equation

Now let's consider the law of linear momentum. The time rate of change of linear momentum in a direction \mathbf{u}^i must equal the force applied in that direction. We thus have

$$\frac{d}{dt} \int_V (\rho u^i) dV = \int_V \left(\frac{D(\rho u^i)}{Dt} + \rho u^i \nabla \mathbf{u} \right) dV = \int_V \left(\frac{\partial(\rho u^i)}{\partial t} + \mathbf{u} \cdot (\nabla(\rho u^i)) + \rho u^i \nabla \mathbf{u} \right) dV = \int_V f^i dV, \quad (3.1)$$

Again, this must hold for every volume. So we can get rid of the integral. We can then also rewrite it to

$$\frac{\partial(\rho u^i)}{\partial t} + u^i \mathbf{u} \cdot (\nabla \rho) + \rho \mathbf{u} \cdot (\nabla u^i) + \rho u^i \nabla \mathbf{u} = \frac{\partial(\rho u^i)}{\partial t} + \nabla(\rho u^i \mathbf{u}) = \mathbf{f}. \quad (3.2)$$

Working this out further, we get

$$\rho \frac{\partial u^i}{\partial t} + u^i \frac{\partial \rho}{\partial t} + u^i \nabla(\rho \mathbf{u}) + \rho \mathbf{u} \cdot \nabla u^i = f_i. \quad (3.3)$$

Thanks to conservation of mass, the two middle terms cancel. We remain with

$$\rho \frac{\partial u^i}{\partial t} + \rho \mathbf{u} \cdot \nabla u^i = f_i. \quad (3.4)$$

This equation is known as the **conservation of linear momentum equation**.

3.2 The force time split up

The equation of the previous paragraph looked quite simple. However, we haven't looked at the force per unit volume \mathbf{f} yet. (Or equivalently, at its component f_i .) So let's do that now.

The force per unit volume is made up out of a body force (per unit volume \mathbf{f}_b and a surface force (per unit area) \mathbf{f}_s . This surface force has to be transformed to a force per unit volume as well, if we want to

apply it in our equation. By using the stress tensor, we find that $\mathbf{f}_s = \sigma \mathbf{n} = (-pI + \tau)\mathbf{n}$. It can then be shown that the force per unit volume due to \mathbf{f}_s is given by $-\nabla p + \nabla \tau$. This eventually gives us

$$\rho \frac{\partial u^i}{\partial t} + \rho \mathbf{u} \cdot \nabla u^i = f_{b_i} - \frac{\partial p}{\partial x^i} + \frac{\partial \tau_{ij}}{\partial x^i}. \quad (3.5)$$

Note that we have used the summation convention in the last term. By the way, we have

$$\tau_{ij} = \lambda(\nabla \mathbf{u})_{ij} + \mu \left(\frac{\partial u^i}{\partial x^j} + \frac{\partial u^j}{\partial x^i} \right), \quad (3.6)$$

where δ_{ij} is the **Kronecker delta**.

3.3 Variants of the equation

An **incompressible flow** is a flow in which fluid elements don't change in volume. This implies that $\nabla \mathbf{u} = 0$. It is often also assumed that $\rho = \rho_\infty = \text{constant}$ as well. If we combine these data with equations (3.5) and (3.6), we eventually find that

$$\rho_\infty \frac{\partial u^i}{\partial t} + \rho_\infty \mathbf{u} \cdot \nabla u^i = f_{b_i} - \frac{\partial p}{\partial x^i} + \nabla^2 u^i. \quad (3.7)$$

The above equation represents only one component of the linear momentum equation. We could put the above equation back into its general vector form. Let's do that, and rewrite it a bit, by bringing in the vorticity vector ξ . (Okay, we rewrite it a lot.) We then get

$$\rho_\infty \frac{\partial \mathbf{u}}{\partial t} + \nabla \left(\frac{1}{2} \rho_\infty \mathbf{u}^2 \right) = \mathbf{f}_b - \nabla p + \mathbf{u} \times \xi + \mu \nabla^2 \mathbf{u}. \quad (3.8)$$

This is called the **Lamb-Gromeka equation**. Now let's assume that the flow is **steady** ($\partial \mathbf{u} / \partial t = 0$), there is **conservative body force** \mathbf{f}_b (there is a function ϕ satisfying $\mathbf{f}_b = \nabla \phi$) and the flow is **inviscid** ($\mu = 0$). The above equation then simplifies to

$$\nabla H(\mathbf{x}) = \rho_\infty \mathbf{u} \times \xi, \quad \text{where} \quad H(\mathbf{x}) = \nabla \left(\frac{1}{2} \rho_\infty \mathbf{u}^2 + p - \phi \right). \quad (3.9)$$

Now let's examine a curve C in space, along which $H(\mathbf{x})$ remains constant, and thus $\mathbf{u} \times \xi = \mathbf{0}$. (For example, a curve along which $\mathbf{u} = \mathbf{0}$, $\xi = \mathbf{0}$, or where \mathbf{u} and ξ are parallel.) In this case, along the curve C , the above equation simplifies to

$$\frac{1}{2} \rho_\infty \mathbf{u}^2 + p - \phi = \text{constant}. \quad (3.10)$$

You might know this equation. (In case you don't, it's called **Bernoulli's equation**.)

4 Conservation of energy

4.1 The basic conservation of energy equation

It is time to examine **conservation of energy**. To find a relation for energy, we examine a small fluid packet. This packet has a **kinetic energy per unit mass** of $\frac{1}{2} \mathbf{u}^2$ and an **internal energy per unit mass** of e . So the total energy per unit mass is

$$E = \frac{1}{2} \mathbf{u}^2 + e. \quad (4.1)$$

Now let's examine a very small volume. The total energy is given by mE . The change of energy depends on the heat created Q and the work done W . So we have

$$\frac{d(mE)}{dt} = \frac{dQ}{dt} + \frac{dW}{dt} \quad (4.2)$$

We know that $dm/dt = 0$. Also, $m = \rho V$. This turns the above equation into

$$\rho V \frac{dE}{dt} = \frac{dQ}{dt} + \frac{dW}{dt} \quad \Rightarrow \quad \rho \frac{dE}{dt} = \frac{dq}{dt} + \frac{dw}{dt}, \quad (4.3)$$

where q is the heat added per unit volume and w is the work done per unit volume. There is just one slight problem. The above equation is in the Lagrangian description. To transform it to the Eulerian description, we need to use the material derivative. We thus have

$$\rho \frac{DE}{Dt} = \frac{Dq}{Dt} + \frac{Dw}{Dt}. \quad (4.4)$$

4.2 The work done

Let's take a closer look at the work W done. We know that work is force times distance. So $dW = \mathbf{f} \cdot d\mathbf{x}$. It follows that

$$dW/dt = \mathbf{f} \cdot \frac{d\mathbf{x}}{dt} = \mathbf{f} \cdot \mathbf{v}. \quad (4.5)$$

The work, however, is done at the boundary of the volume. It is caused by surface forces $\mathbf{f}_s = \sigma \mathbf{n}$. If we transform these surface forces to something having to do with volumes, we eventually find that

$$\frac{Dw}{Dt} = \nabla \cdot (\sigma \mathbf{u}). \quad (4.6)$$

4.3 The heat added

Now let's turn our attention to the heat added. This is a bit more difficult, since there are two ways of adding heat. We can add a certain amount of heat per unit mass q_m over the entire volume. The heat per unit volume added then becomes ρq_m . (Note that q_m is a scalar.)

We can, however, also add heat per unit area \mathbf{q}_s , also known as **heat flow**. (This time \mathbf{q}_s is a vector.) This heat flow is likely to depend on the temperature gradient ∇T . A temperature difference is, however, a configuration variable, and is defined along a line. Heat flow is a source variable (it changes the configuration) and is defined through a surface. We thus need a constitutive model connecting the two. The most common method is **Fourier's law**. It states that

$$\dot{\mathbf{q}}_s = -\kappa \nabla T, \quad (4.7)$$

where κ is a tensor. (The minus sign is present, because heat flow is defined to flow along negative temperature gradients.) If the heat flow properties are isotropic (the same in all directions), then $\kappa = \bar{\kappa} I$, with $\bar{\kappa}$ a constant. The relation $\dot{\mathbf{q}}_s = -\bar{\kappa} \nabla T$ then also holds.

The heat flow per unit area needs to be converted to a quantity per unit volume. Once we have done that, we can find that

$$\frac{Dq}{dt} = \rho \dot{q}_m + \nabla \cdot (\kappa \nabla T). \quad (4.8)$$

4.4 The final energy equation

Let's put all the terms we found together. We then get that

$$\rho \frac{DE}{Dt} = \rho \dot{q}_m + \nabla \cdot (\kappa \nabla T - p \mathbf{u} + \tau \mathbf{u}). \quad (4.9)$$

With quite some manipulation, we could also rewrite this equation to

$$\rho \frac{De}{Dt} = -p \nabla \cdot \mathbf{u} + \nabla \cdot (\kappa \nabla T) + \Phi. \quad (4.10)$$

The variable Φ is known as the **Rayleigh dissipation function** and is given by

$$\Phi = \nabla \cdot (\tau \mathbf{u}) - \mathbf{u} \cdot (\nabla \tau). \quad (4.11)$$

4.5 Fixing the gaps

So, now we've derived all three conservation laws. A bit of counting, however, would show that we have more unknowns than equations. So we need a few more equations. One of these equations relates internal energy and temperature. It states that

$$e = c_V T, \quad (4.12)$$

where c_V is the **specific heat at constant volume**. The other equation is the famous **equation of state** for ideal gases, and states that

$$p = \rho R T, \quad (4.13)$$

with R being the **specific gas constant**. Now we have the same amount of equations as unknowns. So we should be able to solve it.